MATH 552 NOTES – LECTURE 11

Let R be a commutative noetherian ring.
The *radical* of an ideal I is $\sqrt{I} = \{r \in R : \text{for some } r^n \in I\}$
An ideal I is *primary* if whenever $ab \in I, a \notin I$, some $b^N \in I$.

Lemma 1. *If I is a primary ideal of R then $P = \sqrt{I}$ is a prime ideal. (We say I is P-primary.)*

Proof. If $ab \in P$ then some $(ab)^n = a^n b^n$ is in I. If $a \notin P$ then $a^n \notin I$ and so some $b^N \in I$, and $b \in P$. As I is primary, this implies that $b^N \in I$ and hence $b \in \sqrt{I} = P$. □

Corollary 2. *If $\sqrt{I} = m$ is a maximal ideal then I is m-primary.*

An ideal I is *irreducible* if $I = J \cap K$ implies $I = J$ or $I = K$.

Lemma 3. *Every ideal I is a finite intersection of irreducible ideals.*

Proof. Suppose not. Because R is noetherian, there is a maximal counterexample I. As I is reducible, $I = J_1 \cap J_2$, and the J_i are irreducible. Contradiction! □

Lemma 4. *If 0 (the zero ideal) is irreducible, it is a primary ideal.*

Proof. Set $\operatorname{ann}(I) = \{r \in R : rI = 0\}$. Suppose $ab = 0$ and $a, b \neq 0$. Look at

$$\operatorname{ann}(b) \subseteq \operatorname{ann}(b^2) \subseteq \cdots \operatorname{ann}(b^n) \subseteq \cdots$$

and choose n so $\operatorname{ann}(b^n) = \operatorname{ann}(b^{n+1})$. Then $aR \cap b^n R = 0$. *(why?)* Since 0 is irreducible, $aR = 0$ or $b^n = 0$. □

Passing from R to R/I, this implies:

Corollary 5. *Every irreducible ideal I is primary.*

Theorem 6 (H, Thm 8.3.6)**.** *Every ideal of R has a primary decomposition: $I = \cap_{i=1}^n Q_i$, with all Q_i primary.*

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Proof. Pick I maximal with no primary decomposition. Choose r, $a \notin I$ so $ar \in I$ but $r^n \notin I$ for all n. (I is not primary.) Set $J_n = \{b \in R : r^n b \in I\}$; fix n so $J_n = J_{n+1}$, and set $D = I + J_n$.

I claim that $I = J_n \cap D$. Clearly $I \subseteq J_n \cap D$. If $x \in J_n \cap D$ then $x = a + r^n y$ and $r^n x \in I$ so $y \in J_{2n} = J_n$. This implies that $J_n \cap D \subseteq I$.

By the maximality of I, both J_n and D have primary decompositions, so I has a primary decomposition. Contradiction. □

Examples: 1) If $R = \mathbb{Z}$, $I = (60)$ is $(4) \cap (3) \cap (5)$

2) If $R = k[x, y]$, $I = (x^2, xy)$, $I = (x) \cap Q$, where $Q = m^2 = (x^2, xy, y^2)$. Note that I is not a primary ideal.

The primary decomposition is not unique: we also have $I = (x) \cap (x^2, y)$ in Example 2. However, the set $\mathrm{Ass}(I)$ of associated primes is unique:

Theorem 7. *Let $I = \cap_{i=1}^n Q_i$ be a primary decomposition, and set $P_i = \sqrt{Q_i}$. Then the P_i are independent of the choice of primary decomposition.*

Proof. We may assume that $I = 0$ (why?), and show that the P_i are exactly the prime ideals occurring in the set S of ideals $\{\mathrm{ann}(x) : x \in R\}$. Clearly if some $P = \mathrm{ann}(x)$ is prime it is in S.

Let $0 = \cap Q_i$ be a primary decomposition, and set $P_i = \sqrt{Q_i}$, $J_i = \bigcap_{j \neq i} Q_i$. Since $P_i = \sqrt{\mathrm{ann}(x)}$ for any nonzero $x \in J_i$, we have $\mathrm{ann}(x) \subseteq P_i$.

As Q_i is P_i-primary, some $P_i^m \subseteq Q_i$; let m be smallest. If $x \in J_i P_i^{m-1}$ then for any $x \in J_i P^{m-1}$ we have $\mathrm{ann}(x) = P_i$. □