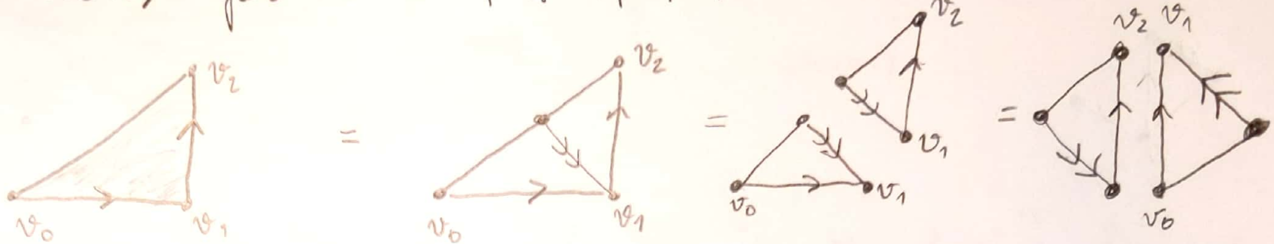
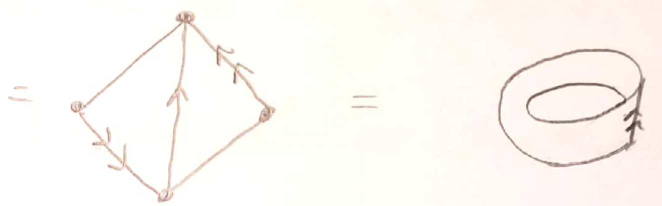
Solutions for Ch 2.1, # 1, 2, 8, 17.

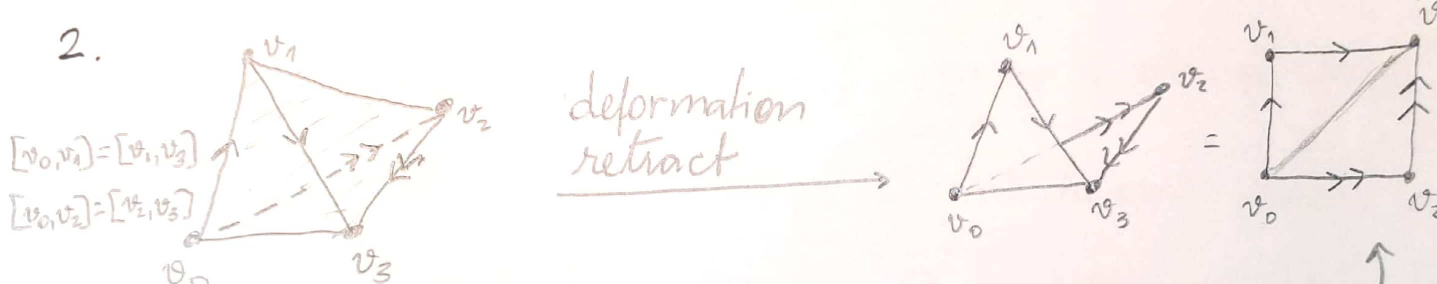
1.





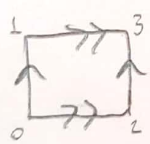
mobius strip.

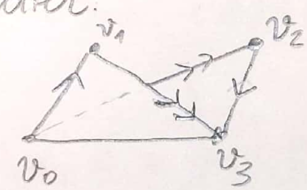
2.



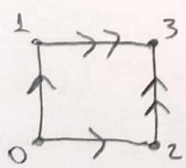
Klein bottle

For the others, the idea is to start with the picture ↑ in mind and think about Δ^3 later.

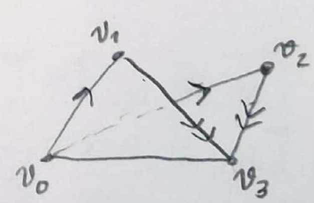
Torus 

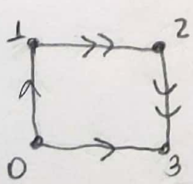


so $[v_0,v_1] \sim [v_2,v_3]$
$[v_0,v_2] \sim [v_1,v_3]$

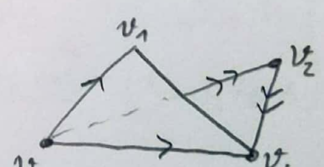
S^2 

so $[v_0,v_2] \sim [v_0,v_1]$
$[v_1,v_3] \sim [v_2,v_3]$

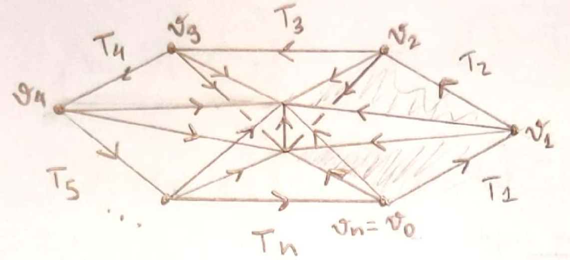


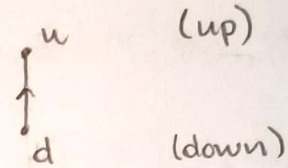
$\mathbb{RP}^2$ 

so $[v_0,v_1] \sim [v_0,v_3]$
$[v_1,v_2] \sim [v_2,v_3]$



8.





Naming like this, we got $T_i = [v_{i-1}, v_i, d, u]$ for $i = 1, \dots, n$.

And there is also the relation $[v_{i-1}, v_i, d] \sim [v_i, v_{i+1}, u]$.

$C_3 : T_1, \dots, T_n$ (n tetrahedras)

$C_2 : [v_{i-1}, v_i, d] \sim [v_i, v_{i+1}, u]$ (n up & down faces)

$[v_i, d, u]$ (n lateral faces)

2n faces

$C_1 : [v_0, v_1], [v_{i-1}, d] \sim [v_i, u]$ (n+2 edges)

$[d, u]$

$C_0 : [v_0] = [v_1] = \dots = [v_n], [d] = [u].$ (2 points)

$$\partial_3(T_i) = \big([v_i, d, u] - [v_{i-1}, d, u]\big) + \big([v_{i-1}, v_i, d] - [v_{i-1}, v_i, u]\big)$$

$$\begin{aligned}\partial_3(T_1 + \dots + T_n) = {} & [v_0, d, u] - [v_1, d, u] + [v_0, v_1, d] - [v_0, v_1, u] \\ & + [v_1, d, u] - [v_2, d, u] + [v_1, v_2, d] - [v_1, v_2, u] + \dots \\ & + \dots - [v_{n-1}, d, u] + [v_{n-2}, v_{n-1}, d] - [v_{n-1}, v_n, u] \\ & + [v_{n-1}, d, u] - [v_0, d, u] + [v_{n-1}, v_0, d] - [v_{n-1}, v_0, u] \\ & = 0\end{aligned}$$

$$\partial_2([v_{i-1}, v_i, d]) = [v_i, d] - [v_{i-1}, d] + [v_{i-1}, v_i]$$

$$\|$$

$$\partial_2([v_i, v_{i+1}, u]) = [v_{i+1}, u] - [v_i, u] + [v_i, v_{i+1}]$$

$$\partial_2([v_i, d, u]) = [d, u] - [v_i, u] + [v_i, d] = [d,u] - [v_i, u] + [v_{i+1}, u]$$

$$\partial_1([v_{i-1}, v_i]) = [v_i] - [v_{i-1}] = 0$$

$$\partial_2([v_i, u]) = [u] - [v_i]$$

$$\partial_1([d, u]) = [u] - [d] = 0$$

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$$\left.\begin{aligned} \ker(\partial_0) &= \langle [u], [v_i] \rangle \\ \mathrm{Im}(\partial_1) &= \langle [u] - [v_i] \rangle \end{aligned}\right\} \; H_0(X) = \mathbb{Z}$$

$$\ker(\partial_1) = \langle [v_0, v_1], \; [d, u] - [v_i, u] + [v_{i+1}, u] \rangle$$

$$\left.\begin{aligned} &\left\{\begin{aligned} \partial_2([v_0, v_1, u]) &= [v_0, v_1] + (-[v_0, u] + [v_1, u]) \\ &\;\vdots \\ \partial_2([v_{n-1}, v_0, u]) &= [v_0, v_1] + (-[v_{n-1}, u] + [v_0, u]) \end{aligned}\right. \\ &\left\{\begin{aligned} \partial_2([v_0, d, u]) &= [d, u] + (-[v_0, u] + [v_1, u]) \\ &\;\vdots \\ \partial_2([v_{n-1}, d, u]) &= [d, u] + (-[v_{n-1}, u] + [v_0, u]) \end{aligned}\right. \end{aligned}\right\} \; \mathrm{Im}(\partial_2) \cong \mathbb{Z} \times \mathbb{Z} \times n\mathbb{Z}$$

which yields

$$H_1(X) = \frac{\mathbb{Z}}{n\mathbb{Z}}$$

$\ker(\partial_2) = \mathrm{Im}(\partial_3)$ since

$$\partial_3(T_i) = \big([v_i, d, u] - [v_{i-1}, d, u]\big) + \big([v_{i-1}, v_i, d] - [v_{i-1}, v_i, d]\big), \text{ so } H_2(X) = 0$$

and $\ker(\partial_3) = \langle T_1 + \dots + T_n \rangle = \mathbb{Z}$.

17. a) Compute $H_n(X,A)$ when $X = S^2$ or $S^1 \times S^1$ and A is a finite set of points.

Solution: Let $A = \{p_1, \dots, p_k\}$ be a finite set of points.

then $\tilde{H}_0(A) = \dfrac{\mathbb{Z}^k}{\mathbb{Z}} = \mathbb{Z}^{k-1}$ and $\tilde{H}_n(A) = 0$ for all $n > 0$.

Consider $X = S^2$, so $\tilde{H}_2(X) = \mathbb{Z}$ and $\tilde{H}_n(X) = 0$ for $n \neq 2$.

Now for the long exact sequence,

$$\tilde{H}_2(A) \to \tilde{H}_2(X) \to H_2(X,A) \to \tilde{H}_1(A) \to \tilde{H}_1(X) \to H_1(X,A) \to \tilde{H}_0(A) \to \tilde{H}_0(X) \to H_0(X,A) \to 0$$

$$0 \to \mathbb{Z} \to H_2(X,A) \to 0$$

$$0 \to H_1(X,A) \to \mathbb{Z}^{k-1} \to 0$$

$$0 \to H_0(X,A) \to 0$$

so $H_2(X,A) = \mathbb{Z}$, $H_1(X,A) = \mathbb{Z}^{k-1}$, $H_0(X,A) = 0$.

Consider $X = S^1 \times S^1$, so doing the same yields

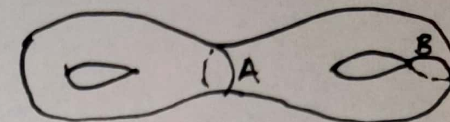
$$0 \to \mathbb{Z} \to H_2(X,A) \to 0 \to \mathbb{Z}^2 \to H_1(X,A) \to \mathbb{Z}^{k-1} \to 0 \to H_0(X,A) \to 0$$

$H_2(X,A) = \mathbb{Z}$

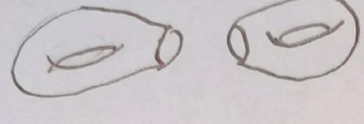
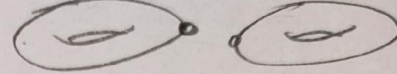
$H_1(X,A) = \mathbb{Z}^2 \oplus \mathbb{Z}^{k-1} = \mathbb{Z}^{k+1}$

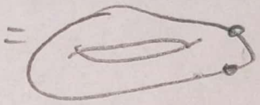
$H_0(X,A) = 0$

$\mathbb{Z}^{k-1}$ is free, so it splits.

ⓑ Compute $H_n(X,A)$ and $H_n(X,B)$ for 

Observe that

$X - A =$  $=$  two torus without 1 point

$X - B =$  $=$  one torus without 2 points.

so we just use item ⓐ for

$$H_n(X,A) \cong \tilde{H}_n(X/A) \cong \tilde{H}_n(T^2-\{p\}) \oplus \tilde{H}_n(T^2-\{p\})$$

$$= \begin{cases} \mathbb{Z}\oplus\mathbb{Z}, & n=2 \\ \mathbb{Z}^2\oplus\mathbb{Z}^2, & n=1 \\ 0, & \text{otherwise} \end{cases}$$

$$H_n(X,B) = \tilde{H}_n(X \setminus B) = H_n(T^2-\{p_1,p_2\}) = \begin{cases} \mathbb{Z}, & n=2 \\ \mathbb{Z}^3, & n=1 \\ 0, & \text{otherwise} \end{cases}$$