Fundamentals of Plane Geometry involving Straight-Lines

Ὅροι.

α=1 α΄. Σημεῖόν ἐστιν, οὗ μέρος οὐθέν.

β=2 β΄. Γραμμὴ δὲ μῆκος ἀπλατές.

γ=3 γ΄. Γραμμῆς δὲ πέρατα σημεῖα.

δ=4 δ΄. Εὐθεῖα γραμμή ἐστιν, ἥτις ἐξ ἴσου τοῖς ἐφ᾿ ἑαυτῆς σημείοις κεῖται.

ε=5 ε΄. Ἐπιφάνεια δέ ἐστιν, ὃ μῆκος καὶ πλάτος μόνον ἔχει.

φ=6 ϛ΄. Ἐπιφανείας δὲ πέρατα γραμμαί.

ζ=7 ζ΄. Ἐπίπεδος ἐπιφάνειά ἐστιν, ἥτις ἐξ ἴσου ταῖς ἐφ᾿ ἑαυτῆς εὐθείαις κεῖται.

η=8 η΄. Ἐπίπεδος δὲ γωνία ἐστὶν ἡ ἐν ἐπιπέδῳ δύο γραμμῶν ἁπτομένων ἀλλήλων καὶ μὴ ἐπ᾿ εὐθείας κειμένων πρὸς ἀλλήλας τῶν γραμμῶν κλίσις.

θ=9 θ΄. Ὅταν δὲ αἱ περιέχουσαι τὴν γωνίαν γραμμαὶ εὐθεῖαι ὦσιν, εὐθύγραμμος καλεῖται ἡ γωνία.

ι=10 ι΄. Ὅταν δὲ εὐθεῖα ἐπ᾿ εὐθεῖαν σταθεῖσα τὰς ἐφεξῆς γωνίας ἴσας ἀλλήλαις ποιῇ, ὀρθὴ ἑκατέρα τῶν ἴσων γωνιῶν ἐστι, καὶ ἡ ἐφεστηκυῖα εὐθεῖα κάθετος καλεῖται, ἐφ᾿ ἣν ἐφέστηκεν.

ια=11 ια΄. Ἀμβλεῖα γωνία ἐστὶν ἡ μείζων ὀρθῆς.

ιβ΄. Ὀξεῖα δὲ ἡ ἐλάσσων ὀρθῆς.

ιγ΄. Ὅρος ἐστίν, ὅ τινός ἐστι πέρας.

ιδ΄. Σχῆμά ἐστι τὸ ὑπό τινος ἤ τινων ὅρων περιεχόμενον.

ιε΄. Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἣ καλεῖται περιφέρεια], πρὸς ἣν ἀφ᾿ ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν.

ιϛ΄. Κέντρον δὲ τοῦ κύκλου τὸ σημεῖον καλεῖται.

ιζ΄. Διάμετρος δὲ τοῦ κύκλου ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ᾿ ἑκάτερα τὰ μέρη ὑπὸ τῆς τοῦ κύκλου περιφερείας, ἥτις καὶ δίχα τέμνει τὸν κύκλον.

ιη΄. Ἡμικύκλιον δέ ἐστι τὸ περιεχόμενον σχῆμα ὑπό τε τῆς διαμέτρου καὶ τῆς ἀπολαμβανομένης ὑπ᾿ αὐτῆς περιφερείας. κέντρον δὲ τοῦ ἡμικυκλίου τὸ αὐτό, ὃ καὶ τοῦ κύκλου ἐστίν.

ιθ΄. Σχήματα εὐθύγραμμά ἐστι τὰ ὑπὸ εὐθειῶν περιεχόμενα, τρίπλευρα μὲν τὰ ὑπὸ τριῶν, τετράπλευρα δὲ τὰ ὑπὸ τεσσάρων, πολύπλευρα δὲ τὰ ὑπὸ πλειόνων ἢ τεσσάρων εὐθειῶν περιεχόμενα.

χ=20 κ΄. Τῶν δὲ τριπλεύρων σχημάτων ἰσόπλευρον μὲν τρίγωνόν ἐστι τὸ τὰς τρεῖς ἴσας ἔχον πλευράς, ἰσοσκελὲς δὲ τὸ τὰς δύο μόνας ἴσας ἔχον πλευράς, σκαληνὸν δὲ τὸ τὰς τρεῖς ἀνίσους ἔχον πλευράς.

κα΄ Ἔτι δὲ τῶν τριπλεύρων σχημάτων ὀρθογώνιον μὲν τρίγωνόν ἐστι τὸ ἔχον ὀρθὴν γωνίαν, ἀμβλυγώνιον δὲ τὸ ἔχον ἀμβλεῖαν γωνίαν, ὀξυγώνιον δὲ τὸ τὰς τρεῖς ὀξείας ἔχον γωνίας.

Definitions

"lines"

straight

1. A point is that of which there is no part.
2. And a line is a length without breadth.
3. And the extremities of a line are points.
4. A straight-line is (any) one which lies evenly with points on itself.
5. And a surface is that which has length and breadth only.
6. And the extremities of a surface are lines. 
7. A plane surface is (any) one which lies evenly with the straight-lines on itself. in a plane
8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line.
9. And when the lines containing the angle are straight then the angle is called rectilinear.
10. And when a straight-line stood upon (another) straight-line makes adjacent angles (which are) equal to one another, each of the equal angles is a right-angle, and the former straight-line is called a perpendicular to that upon which it stands.
11. An obtuse angle is one greater than a right-angle.
12. And an acute angle (is) one less than a right-angle.
13. A boundary is that which is the extremity of something.
14. A figure is that which is contained by some boundary or boundaries.
15. A circle is a plane figure contained by a single line [which is called a circumference], (such that) all of the straight-lines radiating towards [the circumference] from one point amongst those lying inside the figure are equal to one another.
16. And the point is called the center of the circle.
17. And a diameter of the circle is any straight-line, being drawn through the center, and terminated in each direction by the circumference of the circle. (And) any such (straight-line) also cuts the circle in half.†
18. And a semi-circle is the figure contained by the diameter and the circumference cuts off by it. And the center of the semi-circle is the same (point) as (the center of) the circle.

polygons

19. Rectilinear figures are those (figures) contained by straight-lines: trilateral figures being those contained by three straight-lines, quadrilateral by four, and multilateral by more than four.
20. And of the trilateral figures: an equilateral triangle is that having three equal sides, an isosceles (triangle) that having only two equal sides, and a scalene (triangle) that having three unequal sides.

κβ′. Τῶν δὲ τετραπλεύρων σχημάτων τετράγωνον μέν ἐστιν, ὃ ἰσόπλευρόν τέ ἐστι καὶ ὀρθογώνιον, ἑτερόμηκες δέ, ὃ ὀρθογώνιον μέν, οὐκ ἰσόπλευρον δέ, ῥόμβος δέ, ὃ ἰσόπλευρον μέν, οὐκ ὀρθογώνιον δέ, ῥομβοειδὲς δὲ τὸ τὰς ἀπεναντίον πλευράς τε καὶ γωνίας ἴσας ἀλλήλαις ἔχον, ὃ οὔτε ἰσόπλευρόν ἐστιν οὔτε ὀρθογώνιον· τὰ δὲ παρὰ ταῦτα τετράπλευρα τραπέζια καλείσθω.

κγ′. Παράλληλοί εἰσιν εὐθεῖαι, αἵτινες ἐν τῷ αὐτῷ ἐπιπέδῳ οὖσαι καὶ ἐκβαλλόμεναι εἰς ἄπειρον ἐφ᾽ ἑκάτερα τὰ μέρη ἐπὶ μηδέτερα συμπίπτουσιν ἀλλήλαις.

21. And further of the trilateral figures: a right-angled triangle is that having a right-angle, an obtuse-angled (triangle) that having an obtuse angle, and an acute-angled (triangle) that having three acute angles.

22. And of the quadrilateral figures: a square is that which is right-angled and equilateral, a rectangle that which is right-angled but not equilateral, a rhombus that which is equilateral but not right-angled, and a rhomboid that having opposite sides and angles equal to one another which is neither right-angled nor equilateral. And let quadrilateral figures besides these be called trapezia.

23. Parallel lines are straight-lines which, being in the same plane, and being produced to infinity in each direction, meet with one another in neither (of these directions).

† This should really be counted as a postulate, rather than as part of a definition.

Αἰτήματα.

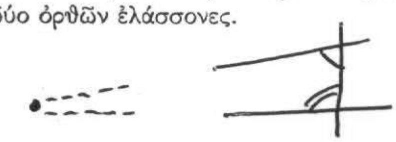
α′. Ἠιτήσθω ἀπὸ παντὸς σημείου ἐπὶ πᾶν σημεῖον εὐθεῖαν γραμμὴν ἀγαγεῖν.

β′. Καὶ πεπερασμένην εὐθεῖαν κατὰ τὸ συνεχὲς ἐπ᾽ εὐθείας ἐκβαλεῖν.

γ′. Καὶ παντὶ κέντρῳ καὶ διαστήματι κύκλον γράφεσθαι.

δ′. Καὶ πάσας τὰς ὀρθὰς γωνίας ἴσας ἀλλήλαις εἶναι.

ε′. Καὶ ἐὰν εἰς δύο εὐθείας εὐθεῖα ἐμπίπτουσα τὰς ἐντὸς καὶ ἐπὶ τὰ αὐτὰ μέρη γωνίας δύο ὀρθῶν ἐλάσσονας ποιῇ, ἐκβαλλομένας τὰς δύο εὐθείας ἐπ᾽ ἄπειρον συμπίπτειν, ἐφ᾽ ἃ μέρη εἰσὶν αἱ τῶν δύο ὀρθῶν ἐλάσσονες.



Postulates

1. Let it have been postulated† to draw a straight-line from any point to any point.

2. And to produce a finite straight-line continuously in a straight-line.

3. And to draw a circle with any center and radius.

4. And that all right-angles are equal to one another.

5. And that if a straight-line falling across two (other) straight-lines makes internal angles on the same side (of itself whose sum is) less than two right-angles, then the two (other) straight-lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right-angles (and do not meet on the other side).‡

† The Greek present perfect tense indicates a past action with present significance. Hence, the 3rd-person present perfect imperative Ἠιτήσθω could be translated as "let it be postulated", in the sense "let it stand as postulated", but not "let the postulate be now brought forward". The literal translation "let it have been postulated" sounds awkward in English, but more accurately captures the meaning of the Greek.

‡ This postulate effectively specifies that we are dealing with the geometry of *flat*, rather than curved, space.

Κοιναὶ ἔννοιαι.

α′. Τὰ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα.

β′. Καὶ ἐὰν ἴσοις ἴσα προστεθῇ, τὰ ὅλα ἐστὶν ἴσα.

γ′. Καὶ ἐὰν ἀπὸ ἴσων ἴσα ἀφαιρεθῇ, τὰ καταλειπόμενά ἐστιν ἴσα.

δ′. Καὶ τὰ ἐφαρμόζοντα ἐπ᾽ ἀλλήλα ἴσα ἀλλήλοις ἐστίν.

ε′. Καὶ τὸ ὅλον τοῦ μέρους μεῖζόν [ἐστιν].

Common Notions (What is "equal"?)

1. Things equal to the same thing are also equal to one another.

2. And if equal things are added to equal things then the wholes are equal.

3. And if equal things are subtracted from equal things then the remainders are equal.†

4. And things coinciding with one another are equal to one another.

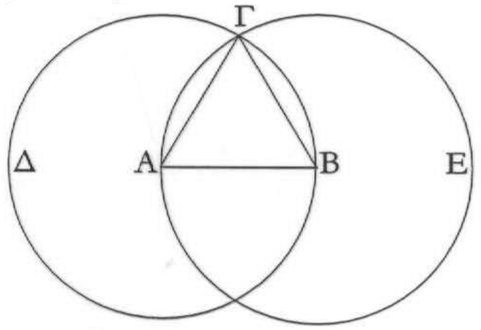
5. And the whole [is] greater than the part.

† As an obvious extension of C.N.s 2 & 3—if equal things are added or subtracted from the two sides of an inequality then the inequality remains

an inequality of the same type.

α'.

Ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τρίγωνον ἰσόπλευρον συστήσασθαι.



Ἔστω ἡ δοθεῖσα εὐθεῖα πεπερασμένη ἡ ΑΒ.

Δεῖ δὴ ἐπὶ τῆς ΑΒ εὐθείας τρίγωνον ἰσόπλευρον συστήσασθαι.

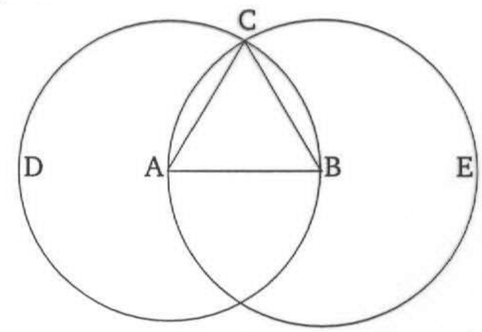
Κέντρῳ μὲν τῷ Α διαστήματι δὲ τῷ ΑΒ κύκλος γεγράφθω ὁ ΒΓΔ, καὶ πάλιν κέντρῳ μὲν τῷ Β διαστήματι δὲ τῷ ΒΑ κύκλος γεγράφθω ὁ ΑΓΕ, καὶ ἀπὸ τοῦ Γ σημείου, καθ' ὃ τέμνουσιν ἀλλήλους οἱ κύκλοι, ἐπί τὰ Α, Β σημεῖα ἐπεζεύχθωσαν εὐθεῖαι αἱ ΓΑ, ΓΒ.

Καὶ ἐπεὶ τὸ Α σημεῖον κέντρον ἐστὶ τοῦ ΓΔΒ κύκλου, ἴση ἐστὶν ἡ ΑΓ τῇ ΑΒ· πάλιν, ἐπεὶ τὸ Β σημεῖον κέντρον ἐστὶ τοῦ ΓΑΕ κύκλου, ἴση ἐστὶν ἡ ΒΓ τῇ ΒΑ. ἐδείχθη δὲ καὶ ἡ ΓΑ τῇ ΑΒ ἴση· ἑκατέρα ἄρα τῶν ΓΑ, ΓΒ τῇ ΑΒ ἐστιν ἴση. τὰ δὲ τῷ αὐτῷ ἴσα καὶ ἀλλήλοις ἐστὶν ἴσα· καὶ ἡ ΓΑ ἄρα τῇ ΓΒ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΓΑ, ΑΒ, ΒΓ ἴσαι ἀλλήλαις εἰσίν.

Ἰσόπλευρον ἄρα ἐστὶ τὸ ΑΒΓ τρίγωνον. καὶ συνέσταται ἐπὶ τῆς δοθείσης εὐθείας πεπερασμένης τῆς ΑΒ. ὅπερ ἔδει ποιῆσαι.

Proposition 1

To construct an equilateral triangle on a given finite straight-line.



Let AB be the given finite straight-line.

So it is required to construct an equilateral triangle on the straight-line AB.

Let the circle BCD with center A and radius AB have been drawn [Post. 3], and again let the circle ACE with center B and radius BA have been drawn [Post. 3]. And let the straight-lines CA and CB have been joined from the point C, where the circles cut one another,† to the points A and B (respectively) [Post. 1].

And since the point A is the center of the circle CDB, AC is equal to AB [Def. 1.15]. Again, since the point B is the center of the circle CAE, BC is equal to BA [Def. 1.15]. But CA was also shown (to be) equal to AB. Thus, CA and CB are each equal to AB. But things equal to the same thing are also equal to one another [C.N. 1]. Thus, CA is also equal to CB. Thus, the three (straight-lines) CA, AB, and BC are equal to one another.

Thus, the triangle ABC is equilateral, and has been constructed on the given finite straight-line AB. (Which is) the very thing it was required to do.

† The assumption that the circles do indeed cut one another should be counted as an additional postulate. There is also an implicit assumption that two straight-lines cannot share a common segment.

β'.

Πρὸς τῷ δοθέντι σημείῳ τῇ δοθείσῃ εὐθείᾳ ἴσην εὐθεῖαν θέσθαι.

Ἔστω τὸ μὲν δοθὲν σημεῖον τὸ Α, ἡ δὲ δοθεῖσα εὐθεῖα ἡ ΒΓ· δεῖ δὴ πρὸς τῷ Α σημείῳ τῇ δοθείσῃ εὐθείᾳ τῇ ΒΓ ἴσην εὐθεῖαν θέσθαι.

Ἐπεζεύχθω γὰρ ἀπὸ τοῦ Α σημείου ἐπί τὸ Β σημεῖον εὐθεῖα ἡ ΑΒ, καὶ συνεστάτω ἐπ' αὐτῆς τρίγωνον ἰσόπλευρον τὸ ΔΑΒ, καὶ ἐκβεβλήσθωσαν ἐπ' εὐθείας ταῖς ΔΑ, ΔΒ

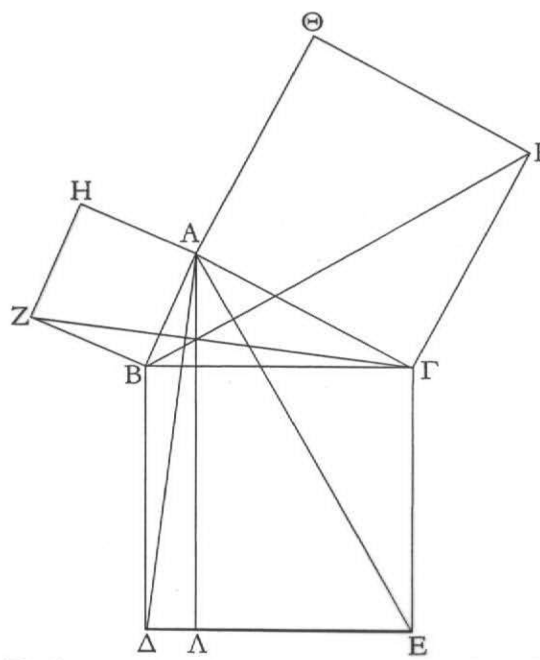
Proposition 2†

To place a straight-line equal to a given straight-line at a given point (as an extremity).

Let A be the given point, and BC the given straight-line. So it is required to place a straight-line at point A equal to the given straight-line BC.

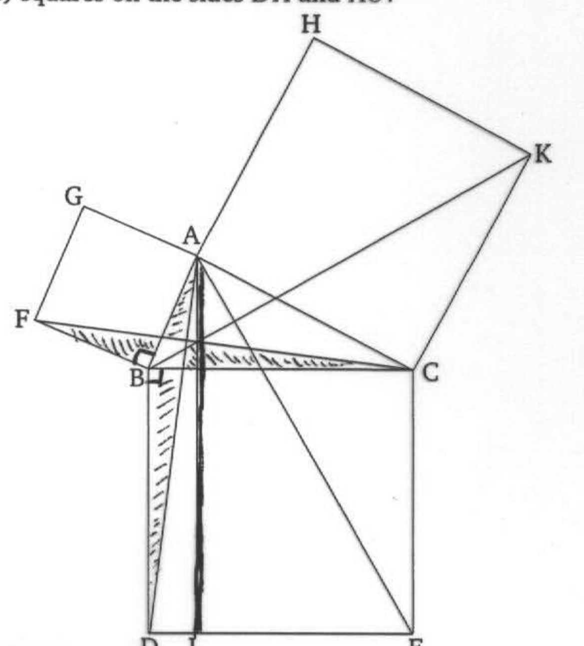
For let the straight-line AB have been joined from point A to point B [Post. 1], and let the equilateral triangle DAB have been been constructed upon it [Prop. 1.1].

τρίγωνον τῷ ΖΒΓ τριγώνῳ ἐστὶν ἴσον· καί [ἐστι] τοῦ μὲν ΑΒΔ τριγώνου διπλάσιον τὸ ΒΛ παραλληλόγραμμον· βάσιν τε γὰρ τὴν αὐτὴν ἔχουσι τὴν ΒΔ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ΒΔ, ΑΛ· τοῦ δὲ ΖΒΓ τριγώνου διπλάσιον τὸ ΗΒ τετράγωνον· βάσιν τε γὰρ πάλιν τὴν αὐτὴν ἔχουσι τὴν ΖΒ καὶ ἐν ταῖς αὐταῖς εἰσι παραλλήλοις ταῖς ΖΒ, ΗΓ. [τὰ δὲ τῶν ἴσων διπλάσια ἴσα ἀλλήλοις ἐστίν·] ἴσον ἄρα ἐστὶ καὶ τὸ ΒΛ παραλληλόγραμμον τῷ ΗΒ τετραγώνῳ. ὁμοίως δὴ ἐπιζευγνυμένων τῶν ΑΕ, ΒΚ δειχθήσεται καὶ τὸ ΓΛ παραλληλόγραμμον ἴσον τῷ ΘΓ τετραγώνῳ· ὅλον ἄρα τὸ ΒΔΕΓ τετράγωνον δυσὶ τοῖς ΗΒ, ΘΓ τετραγώνοις ἴσον ἐστίν. καί ἐστι τὸ μὲν ΒΔΕΓ τετράγωνον ἀπὸ τῆς ΒΓ ἀναγραφέν, τὰ δὲ ΗΒ, ΘΓ ἀπὸ τῶν ΒΑ, ΑΓ. τὸ ἄρα ἀπὸ τῆς ΒΓ πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν ΒΑ, ΑΓ πλευρῶν τετραγώνοις.



Ἐν ἄρα τοῖς ὀρθογωνίοις τριγώνοις τὸ ἀπὸ τῆς τὴν ὀρθὴν γωνίαν ὑποτεινούσης πλευρᾶς τετράγωνον ἴσον ἐστὶ τοῖς ἀπὸ τῶν τὴν ὀρθὴν [γωνίαν] περιεχουσῶν πλευρῶν τετραγώνοις· ὅπερ ἔδει δεῖξαι.

two (straight-lines) CB, BF,[†] respectively. And angle DBA (is) equal to angle FBC. Thus, the base AD [is] equal to the base FC, and the triangle ABD is equal to the triangle FBC [Prop. 1.4]. And parallelogram BL [is] double (the area) of triangle ABD. For they have the same base, BD, and are between the same parallels, BD and AL [Prop. 1.41]. And square GB is double (the area) of triangle FBC. For again they have the same base, FB, and are between the same parallels, FB and GC [Prop. 1.41]. [And the doubles of equal things are equal to one another.][‡] Thus, the parallelogram BL is also equal to the square GB. So, similarly, AE and BK being joined, the parallelogram CL can be shown (to be) equal to the square HC. Thus, the whole square $BDEC$ is equal to the (sum of the) two squares GB and HC. And the square $BDEC$ is described on BC, and the (squares) GB and HC on BA and AC (respectively). Thus, the square on the side BC is equal to the (sum of the) squares on the sides BA and AC.



Thus, in right-angled triangles, the square on the side subtending the right-angle is equal to the (sum of the) squares on the sides surrounding the right-[angle]. (Which is) the very thing it was required to show.

[†] The Greek text has "FB, BC", which is obviously a mistake.

[‡] This is an additional common notion.

Fundamentals of Plane Geometry
Involving Circles

τὸ ΑΕΒ τμῆμα ἐπὶ τὸ ΓΖΔ· ἐφαρμόσει ἄρα, καὶ ἴσον αὐτῷ ἔσται.

Τὰ ἄρα ἐπὶ ἴσων εὐθειῶν ὅμοια τμήματα κύκλων ἴσα ἀλλήλοις ἐστίν· ὅπερ ἔδει δεῖξαι.

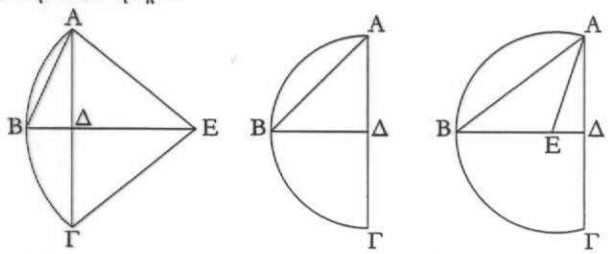
thing is impossible [Prop. 3.10]. Thus, if the straight-line AB is applied to CD, the segment AEB cannot not also coincide with CFD. Thus, it will coincide, and will be equal to it [C.N. 4].

Thus, similar segments of circles on equal straight-lines are equal to one another. (Which is) the very thing it was required to show.

† Both this possibility, and the previous one, are precluded by Prop. 3.23.

κε'.

Κύκλου τμήματος δοθέντος προσαναγράψαι τὸν κύκλον, οὗπέρ ἐστι τμῆμα.



Ἔστω τὸ δοθὲν τμῆμα κύκλου τὸ ΑΒΓ· δεῖ δὴ τοῦ ΑΒΓ τμήματος προσαναγράψαι τὸν κύκλον, οὗπέρ ἐστι τμῆμα.

Τετμήσθω γὰρ ἡ ΑΓ δίχα κατὰ τὸ Δ, καὶ ἤχθω ἀπὸ τοῦ Δ σημείου τῇ ΑΓ πρὸς ὀρθὰς ἡ ΔΒ, καὶ ἐπεζεύχθω ἡ ΑΒ· ἡ ὑπὸ ΑΒΔ γωνία ἄρα τῆς ὑπὸ ΒΑΔ ἤτοι μείζων ἐστὶν ἢ ἴση ἢ ἐλάττων.

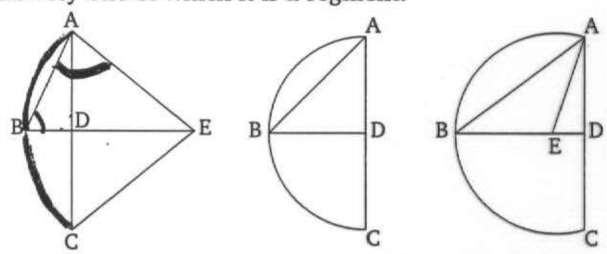
Ἔστω πρότερον μείζων, καὶ συνεστάτω πρὸς τῇ ΒΑ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Α τῇ ὑπὸ ΑΒΔ γωνίᾳ ἴση ἡ ὑπὸ ΒΑΕ, καὶ διήχθω ἡ ΔΒ ἐπὶ τὸ Ε, καὶ ἐπεζεύχθω ἡ ΕΓ. ἐπεὶ οὖν ἴση ἐστὶν ἡ ὑπὸ ΑΒΕ γωνία τῇ ὑπὸ ΒΑΕ, ἴση ἄρα ἐστὶ καὶ ἡ ΕΒ εὐθεῖα τῇ ΕΑ. καὶ ἐπεὶ ἴση ἐστὶν ἡ ΑΔ τῇ ΔΓ, κοινὴ δὲ ἡ ΔΕ, δύο δὴ αἱ ΑΔ, ΔΕ δύο ταῖς ΓΔ, ΔΕ ἴσαι εἰσὶν ἑκατέρα ἑκατέρᾳ· καὶ γωνία ἡ ὑπὸ ΑΔΕ γωνίᾳ τῇ ὑπὸ ΓΔΕ ἐστιν ἴση· ὀρθὴ γὰρ ἑκατέρα· βάσις ἄρα ἡ ΑΕ βάσει τῇ ΓΕ ἐστιν ἴση. ἀλλὰ ἡ ΑΕ τῇ ΒΕ ἐδείχθη ἴση· καὶ ἡ ΒΕ ἄρα τῇ ΓΕ ἐστιν ἴση· αἱ τρεῖς ἄρα αἱ ΑΕ, ΕΒ, ΕΓ ἴσαι ἀλλήλαις εἰσίν· ὁ ἄρα κέντρῳ τῷ Ε διαστήματι δὲ ἑνὶ τῶν ΑΕ, ΕΒ, ΕΓ κύκλος γραφόμενος ἥξει καὶ διὰ τῶν λοιπῶν σημείων καὶ ἔσται προσαναγεγραμμένος. κύκλου ἄρα τμήματος δοθέντος προσαναγέγραπται ὁ κύκλος. καὶ δῆλον, ὡς τὸ ΑΒΓ τμῆμα ἔλαττόν ἐστιν ἡμικυκλίου διὰ τὸ τὸ Ε κέντρον ἐκτὸς αὐτοῦ τυγχάνειν.

Ὁμοίως [δὲ] κἂν ᾖ ἡ ὑπὸ ΑΒΔ γωνία ἴση τῇ ὑπὸ ΒΑΔ, τῆς ΑΔ ἴσης γενομένης ἑκατέρᾳ τῶν ΒΔ, ΔΓ αἱ τρεῖς αἱ ΔΑ, ΔΒ, ΔΓ ἴσαι ἀλλήλαις ἔσονται, καὶ ἔσται τὸ Δ κέντρον τοῦ προσαναπεπληρωμένου κύκλου, καὶ δηλαδὴ ἔσται τὸ ΑΒΓ ἡμικύκλιον.

Ἐὰν δὲ ἡ ὑπὸ ΑΒΔ ἐλάττων ᾖ τῆς ὑπὸ ΒΑΔ, καὶ συστησώμεθα πρὸς τῇ ΒΑ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ

Proposition 25

For a given segment of a circle, to complete the circle, the very one of which it is a segment.



Let ABC be the given segment of a circle. So it is required to complete the circle for segment ABC, the very one of which it is a segment.

For let AC have been cut in half at (point) D [Prop. 1.10], and let DB have been drawn from point D, at right-angles to AC [Prop. 1.11]. And let AB have been joined. Thus, angle ABD is surely either greater than, equal to, or less than (angle) BAD.

First of all, let it be greater. And let (angle) BAE, equal to angle ABD, have been constructed on the straight-line BA, at the point A on it [Prop. 1.23]. And let DB have been drawn through to E, and let EC have been joined. Therefore, since angle ABE is equal to BAE, the straight-line EB is thus also equal to EA [Prop. 1.6]. And since AD is equal to DC, and DE (is) common, the two (straight-lines) AD, DE are equal to the two (straight-lines) CD, DE, respectively. And angle ADE is equal to angle CDE. For each (is) a right-angle. Thus, the base AE is equal to the base CE [Prop. 1.4]. But, AE was shown (to be) equal to BE. Thus, BE is also equal to CE. Thus, the three (straight-lines) AE, EB, and EC are equal to one another. Thus, if a circle is drawn with center E, and radius one of AE, EB, or EC, it will also go through the remaining points (of the segment), and the (associated circle) will have been completed [Prop. 3.9]. Thus, a circle has been completed from the given segment of a circle. And (it is) clear that the segment ABC is less than a semi-circle, because the center E happens to lie outside it. Case 1 φερείας περιεχομένη ἐλάττων ἐστὶν ὀρθῆς.

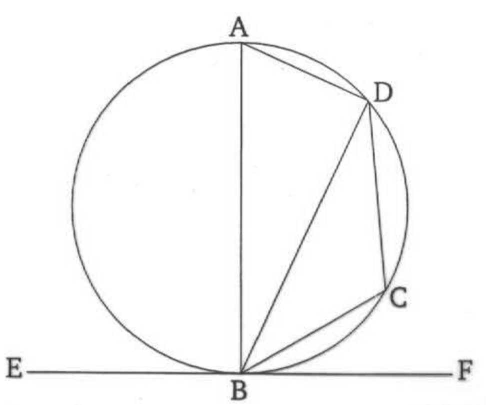
Ἐν κύκλῳ ἄρα ἡ μὲν ἐν τῷ ἡμικυκλίῳ γωνία ὀρθή ἐστιν, ἡ δὲ ἐν τῷ μείζονι τμήματι ἐλάττων ὀρθῆς, ἡ δὲ ἐν τῷ ἐλάττονι [τμήματι] μείζων ὀρθῆς· καὶ ἔτι ἡ μὲν τοῦ μείζονος τμήματος [γωνία] μείζων [ἐστὶν] ὀρθῆς, ἡ δὲ τοῦ ἐλάττονος τμήματος [γωνία] ἐλάττων ὀρθῆς· ὅπερ ἔδει δεῖξαι.

by the circumference $AD[C]$ and the straight-line AC, is less than a right-angle. And this is immediately apparent. For since the (angle contained by) the two straight-lines BA and AC is a right-angle, the (angle) contained by the circumference ABC and the straight-line AC is thus greater than a right-angle. Again, since the (angle contained by) the straight-lines AC and AF is a right-angle, the (angle) contained by the circumference $AD[C]$ and the straight-line CA is thus less than a right-angle.

Thus, in a circle, the angle in a semi-circle is a right-angle, and that in a greater segment (is) less than a right-angle, and that in a lesser [segment] (is) greater than a right-angle. And, further, the [angle] of a segment greater (than a semi-circle) [is] greater than a right-angle, and the [angle] of a segment less (than a semi-circle) is less than a right-angle. (Which is) the very thing it was required to show.

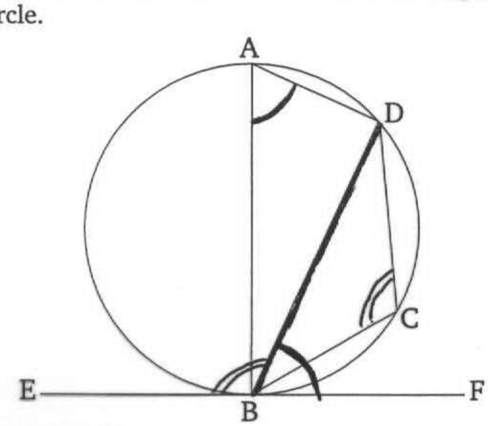
λβ'.

Ἐὰν κύκλου ἐφάπτηταί τις εὐθεῖα, ἀπὸ δὲ τῆς ἁφῆς εἰς τὸν κύκλον διαχθῇ τις εὐθεῖα τέμνουσα τὸν κύκλον, ἃς ποιεῖ γωνίας πρὸς τῇ ἐφαπτομένῃ, ἴσαι ἔσονται ταῖς ἐν τοῖς ἐναλλὰξ τοῦ κύκλου τμήμασι γωνίαις.



Proposition 32

If some straight-line touches a circle, and some (other) straight-line is drawn across, from the point of contact into the circle, cutting the circle (in two), then those angles the (straight-line) makes with the tangent will be equal to the angles in the alternate segments of the circle.



Κύκλου γὰρ τοῦ ΑΒΓΔ ἐφαπτέσθω τις εὐθεῖα ἡ ΕΖ κατὰ τὸ Β σημεῖον, καὶ ἀπὸ τοῦ Β σημείου διήχθω τις εὐθεῖα εἰς τὸν ΑΒΓΔ κύκλον τέμνουσα αὐτὸν ἡ ΒΔ. λέγω, ὅτι ἃς ποιεῖ γωνίας ἡ ΒΔ μετὰ τῆς ΕΖ ἐφαπτομένης, ἴσας ἔσονται ταῖς ἐν τοῖς ἐναλλὰξ τμήμασι τοῦ κύκλου γωνίαις, τουτέστιν, ὅτι ἡ μὲν ὑπὸ ΖΒΔ γωνία ἴση ἐστὶ τῇ ἐν τῷ ΒΑΔ τμήματι συνισταμένῃ γωνίᾳ, ἡ δὲ ὑπὸ ΕΒΔ γωνία ἴση ἐστὶ τῇ ἐν τῷ ΔΓΒ τμήματι συνισταμένῃ γωνίᾳ.

Ἤχθω γὰρ ἀπὸ τοῦ Β τῇ ΕΖ πρὸς ὀρθὰς ἡ ΒΑ, καὶ εἰλήφθω ἐπὶ τῆς ΒΔ περιφερείας τυχὸν σημεῖον τὸ Γ, καὶ ἐπεζεύχθωσαν αἱ ΑΔ, ΔΓ, ΓΒ.

Καὶ ἐπεὶ κύκλου τοῦ ΑΒΓΔ ἐφάπτεταί τις εὐθεῖα ἡ ΕΖ

For let some straight-line EF touch the circle $ABCD$ at the point B, and let some (other) straight-line BD have been drawn from point B into the circle $ABCD$, cutting it (in two). I say that the angles BD makes with the tangent EF will be equal to the angles in the alternate segments of the circle. That is to say, that angle FBD is equal to the angle constructed in segment BAD, and angle EBD is equal to the angle constructed in segment DCB.

For let BA have been drawn from B, at right-angles to EF [Prop. 1.11]. And let the point C have been taken at random on the circumference BD. And let AD, DC, Construction of Rectilinear Figures (Polygons) In and Around Circles

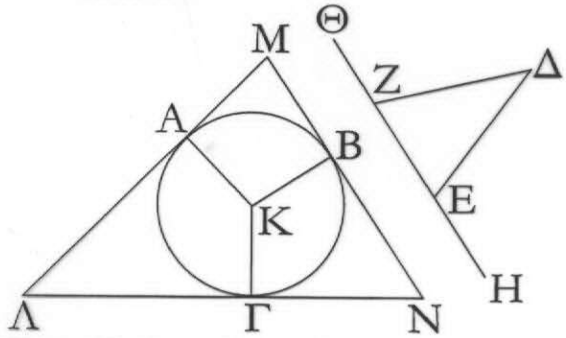
ABC].

Thus, a triangle, equiangular with the given triangle, has been inscribed in the given circle. (Which is) the very thing it was required to do.

† See the footnote to Prop. 3.34.

γ'.

Περὶ τὸν δοθέντα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον περιγράψαι.



Ἔστω ὁ δοθεὶς κύκλος ὁ ΑΒΓ, τὸ δὲ δοθὲν τρίγωνον τὸ ΔΕΖ· δεῖ δὴ περὶ τὸν ΑΒΓ κύκλον τῷ ΔΕΖ τριγώνῳ ἰσογώνιον τρίγωνον περιγράψαι.

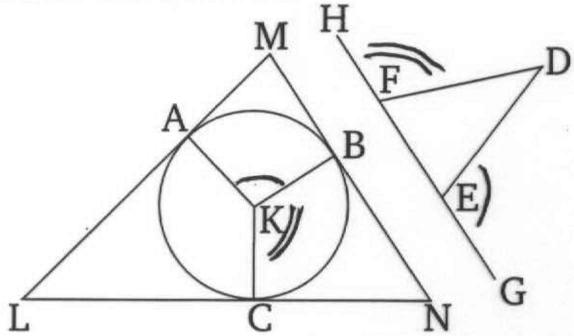
Ἐκβεβλήσθω ἡ ΕΖ ἐφ' ἑκάτερα τὰ μέρη κατὰ τὰ Η, Θ σημεῖα, καὶ εἰλήφθω τοῦ ΑΒΓ κύκλου κέντρον τὸ Κ, καὶ διήχθω, ὡς ἔτυχεν, εὐθεῖα ἡ ΚΒ, καὶ συνεστάτω πρὸς τῇ ΚΒ εὐθείᾳ καὶ τῷ πρὸς αὐτῇ σημείῳ τῷ Κ τῇ μὲν ὑπὸ ΔΕΗ γωνίᾳ ἴση ἡ ὑπὸ ΒΚΑ, τῇ δὲ ὑπὸ ΔΖΘ ἴση ἡ ὑπὸ ΒΚΓ, καὶ διὰ τῶν Α, Β, Γ σημείων ἤχθωσαν ἐφαπτόμεναι τοῦ ΑΒΓ κύκλου αἱ ΛΑΜ, ΜΒΝ, ΝΓΛ.

Καὶ ἐπεὶ ἐφάπτονται τοῦ ΑΒΓ κύκλου αἱ ΛΜ, ΜΝ, ΝΛ κατὰ τὰ Α, Β, Γ σημεῖα, ἀπὸ δὲ τοῦ Κ κέντρου ἐπὶ τὰ Α, Β, Γ σημεῖα ἐπεζευγμέναι εἰσὶν αἱ ΚΑ, ΚΒ, ΚΓ, ὀρθαὶ ἄρα εἰσὶν αἱ πρὸς τοῖς Α, Β, Γ σημείοις γωνίαι. καὶ ἐπεὶ τοῦ ΑΜΒΚ τετραπλεύρου αἱ τέσσαρες γωνίαι τέτρασιν ὀρθαῖς ἴσαι εἰσίν, ἐπειδήπερ καὶ εἰς δύο τρίγωνα διαιρεῖται τὸ ΑΜΒΚ, καί εἰσιν ὀρθαὶ αἱ ὑπὸ ΚΑΜ, ΚΒΜ γωνίαι, λοιπαὶ ἄρα αἱ ὑπὸ ΑΚΒ, ΑΜΒ δυσὶν ὀρθαῖς ἴσαι εἰσίν. εἰσὶ δὲ καὶ αἱ ὑπὸ ΔΕΗ, ΔΕΖ δυσὶν ὀρθαῖς ἴσαι· αἱ ἄρα ὑπὸ ΑΚΒ, ΑΜΒ ταῖς ὑπὸ ΔΕΗ, ΔΕΖ ἴσαι εἰσίν, ὧν ἡ ὑπὸ ΑΚΒ τῇ ὑπὸ ΔΕΗ ἐστιν ἴση· λοιπὴ ἄρα ἡ ὑπὸ ΑΜΒ λοιπῇ τῇ ὑπὸ ΔΕΖ ἐστιν ἴση. ὁμοίως δὴ δειχθήσεται, ὅτι καὶ ἡ ὑπὸ ΛΝΒ τῇ ὑπὸ ΔΖΕ ἐστιν ἴση· καὶ λοιπὴ ἄρα ἡ ὑπὸ ΜΛΝ [λοιπῇ] τῇ ὑπὸ ΕΔΖ ἐστιν ἴση. ἰσογώνιον ἄρα ἐστὶ τὸ ΛΜΝ τρίγωνον τῷ ΔΕΖ τριγώνῳ· καὶ περιγέγραπται περὶ τὸν ΑΒΓ κύκλον.

Περὶ τὸν δοθέντα ἄρα κύκλον τῷ δοθέντι τριγώνῳ ἰσογώνιον τρίγωνον περιγέγραπται· ὅπερ ἔδει ποιῆσαι.

Proposition 3

To circumscribe a triangle, equiangular with a given triangle, about a given circle.



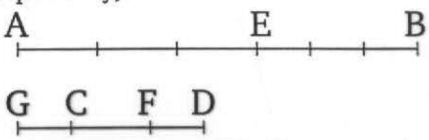
Let ABC be the given circle, and DEF the given triangle. So it is required to circumscribe a triangle, equiangular with triangle DEF, about circle ABC.

Let EF have been produced in each direction to points G and H. And let the center K of circle ABC have been found [Prop. 3.1]. And let the straight-line KB have been drawn, at random, across (ABC). And let (angle) BKA, equal to angle DEG, have been constructed on the straight-line KB at the point K on it, and (angle) BKC, equal to DFH [Prop. 1.23]. And let the (straight-lines) LAM, MBN, and NCL have been drawn through the points A, B, and C (respectively), touching the circle ABC.†

And since LM, MN, and NL touch circle ABC at points A, B, and C (respectively), and KA, KB, and KC are joined from the center K to points A, B, and C (respectively), the angles at points A, B, and C are thus right-angles [Prop. 3.18]. And since the (sum of the) four angles of quadrilateral $AMBK$ is equal to four right-angles, inasmuch as $AMBK$ (can) also (be) divided into two triangles [Prop. 1.32], and angles KAM and KBM are (both) right-angles, the (sum of the) remaining (angles), AKB and AMB, is thus equal to two right-angles. And DEG and DEF is also equal to two right-angles [Prop. 1.13]. Thus, AKB and AMB is equal to DEG and DEF, of which AKB is equal to DEG. Thus, the remainder AMB is equal to the remainder DEF. So, similarly, it can be shown that LNB is also equal to DFE. Thus, the remaining (angle) MLN is also equal to the

Proposition 5†

If a magnitude is the same multiple of a magnitude that a (part) taken away (is) of a (part) taken away (respectively) then the remainder will also be the same multiple of the remainder as that which the whole (is) of the whole (respectively).



$\frac{AB}{CD} = \frac{AE}{CF}$

$\Rightarrow \frac{EB}{FD}$ = this

For let the magnitude AB be the same multiple of the magnitude CD that the (part) taken away AE (is) of the (part) taken away CF (respectively). I say that the remainder EB will also be the same multiple of the remainder FD as that which the whole AB (is) of the whole CD (respectively).

For as many times as AE is (divisible) by CF, so many times let EB also have been made (divisible) by CG.

And since AE and EB are equal multiples of CF and GC (respectively), AE and AB are thus equal multiples of CF and GF (respectively) [Prop. 5.1]. And AE and AB are assumed (to be) equal multiples of CF and CD (respectively). Thus, AB is an equal multiple of each of GF and CD. Thus, GF (is) equal to CD. Let CF have been subtracted from both. Thus, the remainder GC is equal to the remainder FD. And since AE and EB are equal multiples of CF and GC (respectively), and GC (is) equal to DF, AE and EB are thus equal multiples of CF and FD (respectively). And AE and AB are assumed (to be) equal multiples of CF and CD (respectively). Thus, EB and AB are equal multiples of FD and CD (respectively). Thus, the remainder EB will also be the same multiple of the remainder FD as that which the whole AB (is) of the whole CD (respectively).

Thus, if a magnitude is the same multiple of a magnitude that a (part) taken away (is) of a (part) taken away (respectively) then the remainder will also be the same multiple of the remainder as that which the whole (is) of the whole (respectively). (Which is) the very thing it was required to show.

Magnitudes are in the same ratio (proportional)

$a:b = c:d$ For any m, n numbers

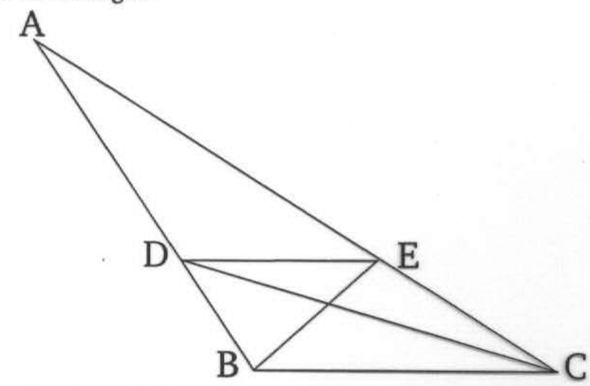
$ma > nb \Rightarrow mc > nd$

$ma < nb \Rightarrow mc < nd$

due to Eudoxus (380BC?)

Proposition 2

If some straight-line is drawn parallel to one of the sides of a triangle then it will cut the (other) sides of the triangle proportionally. And if (two of) the sides of a triangle are cut proportionally then the straight-line joining the cutting (points) will be parallel to the remaining side of the triangle.



For let DE have been drawn parallel to one of the sides BC of triangle ABC. I say that as BD is to DA, so CE (is) to EA.

Book 8 - Continued Proportion

Proposition 20

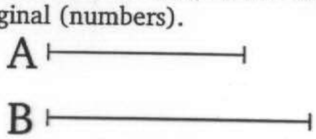
The least numbers of those (numbers) having the same ratio measure those (numbers) having the same ratio as them an equal number of times, the greater (measuring) the greater, and the lesser the lesser.

For let CD and EF be the least numbers having the same ratio as A and B (respectively). I say that CD measures A the same number of times as EF (measures) B.

If $\frac{A}{B} = \frac{CD}{EF}$ then CD divides A and $\frac{A}{CD} = \frac{B}{EF}$

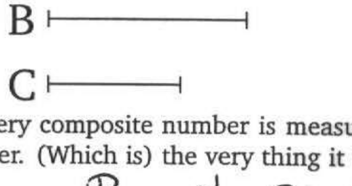
Proposition 30

If two numbers make some (number by) multiplying one another, and some prime number measures the number (so) created from them, then it will also measure one of the original (numbers).



"If p divides AB then p divides one of A and B."

If A is composite, some B measures A. If B is prime then that which was prescribed has happened. And if (B is) composite then some number will measure it. Let it (so) measure (B), and let it be C. And since C measures B, and B measures A, C thus also measures A. And if C is prime then that which was prescribed has happened. And if (C is) composite then some number will measure it. So, in this manner of continued investigation, some prime number will be found which will measure (the number preceding it, which will also measure A). And if (such a number) cannot be found then an infinite (series of) numbers, each of which is less than the preceding, will measure the number A. The very thing is impossible for numbers. Thus, some prime number will (eventually) be found which will measure the (number) preceding it, which will also measure A.



Thus, every composite number is measured by some prime number. (Which is) the very thing it was required to show.

Proposition 31 ($\lambda\alpha$)

Definitions

1. A unit is (that) according to which each existing (thing) is said (to be) one.

2. And a number (is) a multitude composed of units.† } positive

3. A number is part of a(nother) number, the lesser of the greater, when it measures the greater.‡ } divides

4. But (the lesser is) parts (of the greater) when it does not measure it.§

5. And the greater (number is) a multiple of the lesser when it is measured by the lesser.

6. An even number is one (which can be) divided in half.

7. And an odd number is one (which can)not (be) divided in half, or which differs from an even number by a unit.

8. An even-times-even number is one (which is) measured by an even number according to an even number.¶ } divisibl by 4

9. And an even-times-odd number is one (which is) measured by an even number according to an odd number.* } $\equiv 2$ mod 4

10. And an odd-times-odd number is one (which is) measured by an odd number according to an odd number.$ } odd, not prime

11. A $\boxed{\text{prime}^{\|}\text{ number}}$ is one (which is) measured by a unit alone.

12. Numbers prime to one another are those (which are) measured by a unit alone as a common measure.

13. A composite number is one (which is) measured by some number.

14. And numbers composite to one another are those (which are) measured by some number as a common measure.

15. A number is said to multiply a(nother) number when the (number being) multiplied is added (to itself) as many times as there are units in the former (number), and (thereby) some (other number) is produced.

16. And when two numbers multiplying one another make some (other number) then the (number so) created is called plane, and its sides (are) the numbers which multiply one another.

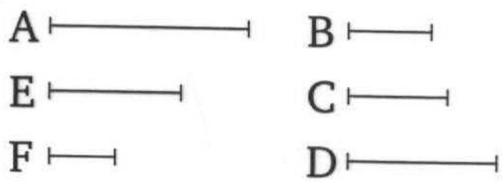
17. And when three numbers multiplying one another make some (other number) then the (number so) created is (called) solid, and its sides (are) the numbers which multiply one another.

18. A square number is an equal times an equal, or (a plane number) contained by two equal numbers.

19. And a cube (number) is an equal times an equal times an equal, or (a solid number) contained by three equal numbers.

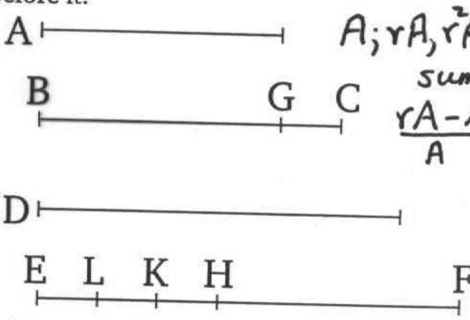
Proposition 14 $p_1p_2p_3 = A$

If a least number is measured by (some) prime numbers then it will not be measured by any other prime number except (one of) the original measuring (numbers).



Proposition 35†

If there is any multitude whatsoever of continually proportional numbers, and (numbers) equal to the first are subtracted from (both) the second and the last, then as the excess of the second (number is) to the first, so the excess of the last will be to (the sum of) all those (numbers) before it.



$A; rA, r^2A, \cdots r^nA$ sums to S

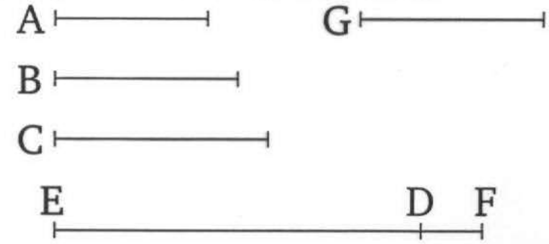
$\frac{rA-A}{A} = \frac{r^nA-A}{S}$

Let A, BC, D, EF be any multitude whatsoever of continuously proportional numbers, beginning from the least A. And let BG and FH, each equal to A, have been subtracted from BC and EF (respectively). I say that as GC is to A, so EH is to A, BC, D.

For let FK be made equal to BC, and FL to D. And since FK is equal to BC, of which FH is equal to BG, the remainder HK is thus equal to the remainder GC. And since as EF is to D, so D (is) to BC, and BC to A [Prop. 7.13], and D (is) equal to FL, and BC to FK, and A to FH, thus as EF is to FL, so LF (is) to FK, and FK to FH. By separation, as EL (is) to LF, so LK (is) to FK, and KH to FH [Props. 7.11, 7.13]. And thus as one of the leading (numbers) is to one of the following, so (the sum of) all of the leading (numbers is) to (the sum of) all of the following [Prop. 7.12]. Thus, as KH is to FH, so EL, LK, KH (are) to LF, FK, HF. And KH (is) equal to CG, and FH to A, and LF, FK, HF to D, BC, A. Thus, as CG is to A, so EH (is) to D, BC, A. Thus, as the excess of the second (number) is to the first, so the excess of the last (is) to (the sum of) all those (numbers) before it. (Which is) the very thing it was required to show.

Proposition 20

The (set of all) prime numbers is more numerous than any assigned multitude of prime numbers.



Let A, B, C be the assigned prime numbers. I say that the (set of all) primes numbers is more numerous than A, B, C.

For let the least number measured by A, B, C have been taken, and let it be DE [Prop. 7.36]. And let the unit DF have been added to DE. So EF is either prime, or not. Let it, first of all, be prime. Thus, the (set of) prime numbers A, B, C, EF, (which is) more numerous than A, B, C, has been found.

And so let EF not be prime. Thus, it is measured by some prime number [Prop. 7.31]. Let it be measured by the prime (number) G. I say that G is not the same as any of A, B, C. For, if possible, let it be (the same). And A, B, C (all) measure DE. Thus, G will also measure DE. And it also measures EF. (So) G will also measure the remainder, unit DF, (despite) being a number [Prop. 7.28]. The very thing (is) absurd. Thus, G is not the same as one of A, B, C. And it was assumed (to be) prime. Thus, the (set of) prime numbers A, B, C, G, (which is) more numerous than the assigned multitude (of prime numbers), A, B, C, has been found. (Which is) the very thing it was required to show.

Incommensurable Magnitudes

Definitions

1. Those magnitudes measured by the same measure are said (to be) commensurable, but (those) of which no (magnitude) admits to be a common measure (are said to be) incommensurable.[†]

2. (Two) straight-lines are commensurable in square[‡] when the squares on them are measured by the same area, but (are) incommensurable (in square) when no area admits to be a common measure of the squares on them.[§]

3. These things being assumed, it is proved that there exist an infinite multitude of straight-lines commensurable and incommensurable with an assigned straight-line—those (incommensurable) in length only, and those also (commensurable or incommensurable) in square.[¶] Therefore, let the assigned straight-line be called rational. And (let) the (straight-lines) commensurable with it, either in length and square, or in square only, (also be called) rational. But let the (straight-lines) incommensurable with it be called irrational.[*]

4. And let the square on the assigned straight-line be called rational. And (let areas) commensurable with it (also be called) rational. But (let areas) incommensurable with it (be called) irrational, and (let) their square-roots[$] (also be called) irrational—the sides themselves, if the (areas) are squares, and the (straight-lines) describing squares equal to them, if the (areas) are some other rectilinear (figure).[‖]

Proposition 1[†] (Eudoxus)

If, from the greater of two unequal magnitudes (which are) laid out, (a part) greater than half is subtracted, and (if from) the remainder (a part) greater than half (is subtracted), and (if) this happens continually, then some magnitude will (eventually) be left which will be less than the lesser laid out magnitude.

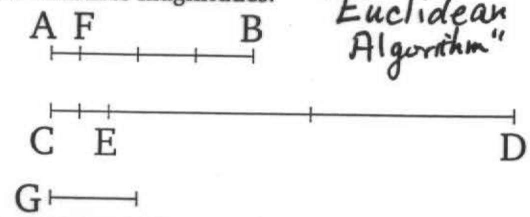
Proposition 2

If the remainder of two unequal magnitudes (which are) [laid out] never measures the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater, then the (original) magnitudes will be incommensurable.

For, AB and CD being two unequal magnitudes, and AB (being) the lesser, let the remainder never measure the (magnitude) before it, (when) the lesser (magnitude is) continually subtracted in turn from the greater. I say that the magnitudes AB and CD are incommensurable.

Proposition 3

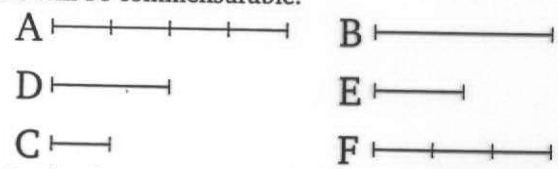
To find the greatest common measure of two given commensurable magnitudes. "Euclidean Algorithm"



Let AB and CD be the two given magnitudes, of which (let) AB (be) the lesser. So, it is required to find the greatest common measure of AB and CD.

Proposition 6

If two magnitudes have to one another the ratio which (some) number (has) to (some) number then the magnitudes will be commensurable.

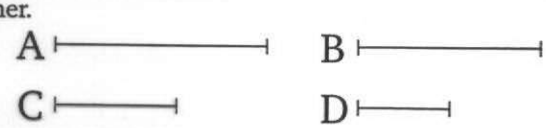


For let the two magnitudes A and B have to one another the ratio which the number D (has) to the number E. I say that the magnitudes A and B are commensurable.

Connects numbers to lengths!

Proposition 9

Squares on straight-lines (which are) commensurable in length have to one another the ratio which (some) square number (has) to (some) square number. And squares having to one another the ratio which (some) square number (has) to (some) square number will also have sides (which are) commensurable in length. But squares on straight-lines (which are) incommensurable in length do not have to one another the ratio which (some) square number (has) to (some) square number. And squares not having to one another the ratio which (some) square number (has) to (some) square number will not have sides (which are) commensurable in length either.



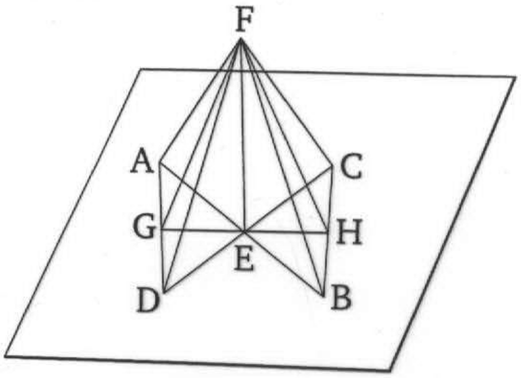
For let A and B be (straight-lines which are) commensurable in length. I say that the square on A has to the square on B the ratio which (some) square number (has) to (some) square number.

A and B are commensurable

$\iff$ A$\underset{A}{\square}$ and B$\underset{B}{\square}$ is $m^2 : n^2$

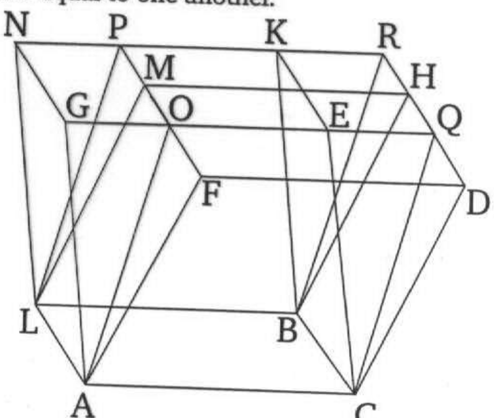
Proposition 4

If a straight-line is set up at right-angles to two straight-lines cutting one another, at the common point of section, then it will also be at right-angles to the plane (passing) through them (both).



Proposition 30

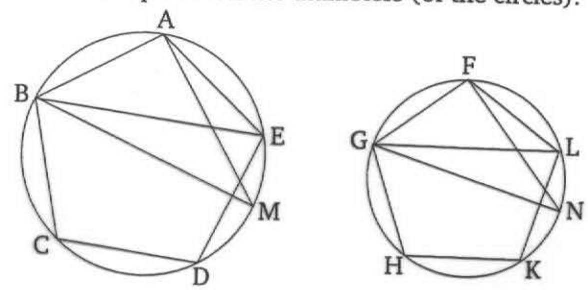
Parallelepiped solids which are on the same base, and (have) the same height, and in which the (ends of the straight-lines) standing up are not on the same straight-lines, are equal to one another.



Let the parallelepiped solids CM and CN be on the same base, AB, and (have) the same height, and let the (ends of the straight-lines) standing up in them, AF, AG,

Proposition 1

Similar polygons (inscribed) in circles are to one another as the squares on the diameters (of the circles).

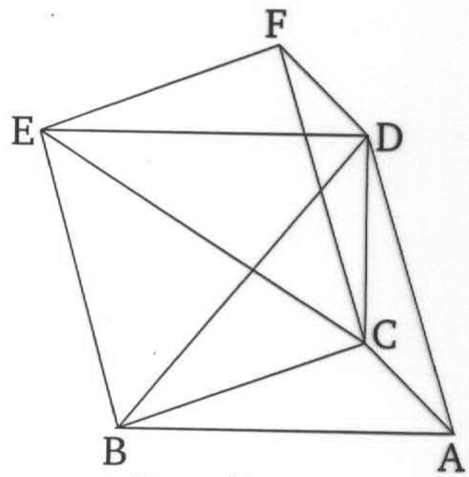


Proposition 2

Circles are to one another as the squares on (their) diameters.

Proposition 7

Any prism having a triangular base is divided into three pyramids having triangular bases (which are) equal to one another.

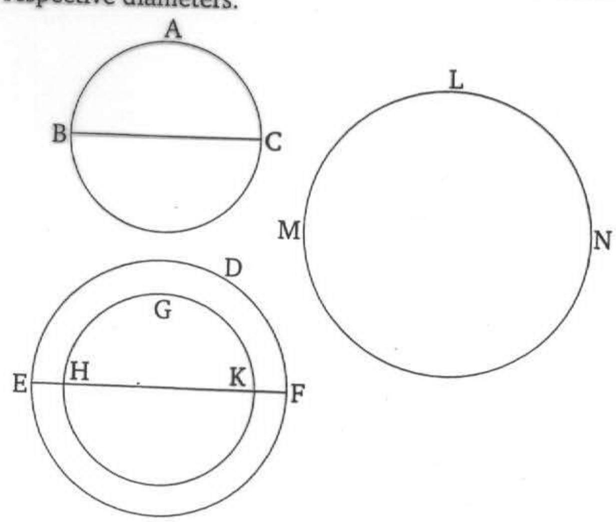


Proposition 10

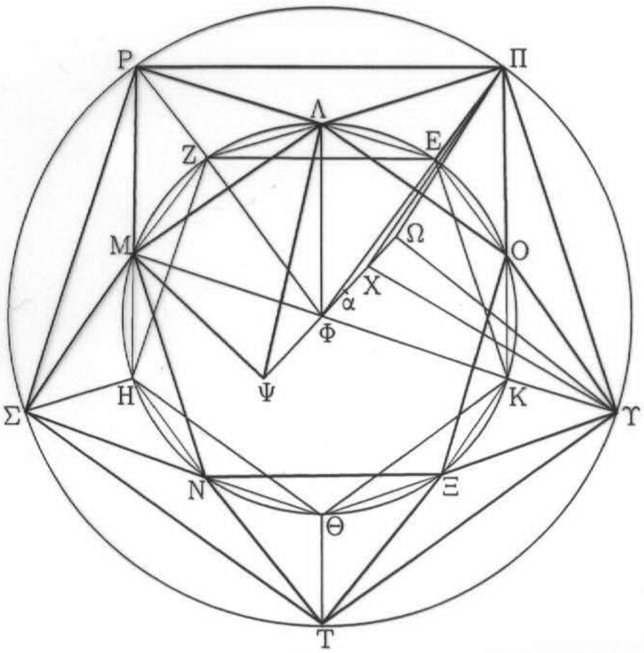
Every cone is the third part of the cylinder which has the same base as it, and an equal height.

Proposition 18

Spheres are to one another in the cubed ratio of their respective diameters.



πλευρά. ἔστι δὲ καὶ ἡ ΠΥ πενταγώνου· ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΟΥ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΠΛΡ, ΡΜΣ, ΣΝΤ, ΤΞΥ ἰσόπλευρόν ἐστιν. καὶ ἐπεὶ πενταγώνου ἐδείχθη ἑκατέρα τῶν ΠΛ, ΠΟ, ἔστι δὲ καὶ ἡ ΛΟ πενταγώνου, ἰσόπλευρον ἄρα ἐστὶ τὸ ΠΛΟ τρίγωνον. διὰ τὰ αὐτὰ δὴ καὶ ἕκαστον τῶν ΛΡΜ, ΜΣΝ, ΝΤΞ, ΞΥΟ τριγώνων ἰσόπλευρόν ἐστιν.



Εἰλήφθω τὸ κέντρον τοῦ ΕΖΗΘΚ κύκλου τὸ Φ σημεῖον· καὶ ἀπὸ τοῦ Φ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθὰς ἀνεστάτω ἡ ΦΩ, καὶ ἐκβεβλήσθω ἐπὶ τὰ ἕτερα μέρη ὡς ἡ ΦΨ, καὶ ἀφῃρήσθω ἑξαγώνου μὲν ἡ ΦΧ, δεκαγώνου δὲ ἑκατέρα τῶν ΦΨ, ΧΩ, καὶ ἐπεζεύχθωσαν αἱ ΠΩ, ΠΧ, ΥΩ, ΕΦ, ΛΦ, ΛΨ, ΨΜ.

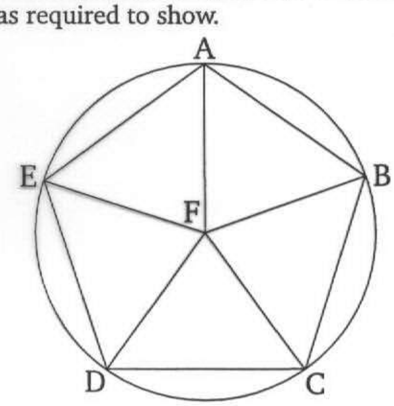
Καὶ ἐπεὶ ἑκατέρα τῶν ΦΧ, ΠΕ τῷ τοῦ κύκλου ἐπιπέδῳ πρὸς ὀρθάς ἐστιν, παράλληλος ἄρα ἐστὶν ἡ ΦΧ τῇ ΠΕ. εἰσὶ δὲ καὶ ἴσαι· καὶ αἱ ΕΦ, ΠΧ ἄρα ἴσαι τε καὶ παράλληλοί εἰσιν. ἑξαγώνου δὲ ἡ ΕΦ· ἑξαγώνου ἄρα καὶ ἡ ΠΧ. καὶ ἐπεὶ ἑξαγώνου μέν ἐστιν ἡ ΠΧ, δεκαγώνου δὲ ἡ ΧΩ, καὶ ὀρθή ἐστιν ἡ ὑπὸ ΠΧΩ γωνία, πενταγώνου ἄρα ἐστὶν ἡ ΠΩ. διὰ τὰ αὐτὰ δὴ καὶ ἡ ΥΩ πενταγώνου ἐστίν, ἐπειδήπερ,

So, I say that, beside the five aforementioned figures, no other (solid) figure can be constructed (which is) contained by equilateral and equiangular (planes), equal to one another.

For a solid angle cannot be constructed from two triangles, or indeed (two) planes (of any sort) [Def. 11.11]. And (the solid angle) of the pyramid (is constructed) from three (equiangular) triangles, and (that) of the octahedron from four (triangles), and (that) of the icosahedron from (five) triangles. And a solid angle cannot be (made) from six equilateral and equiangular triangles set up together at one point. For, since the angles of a equilateral triangle are (each) two-thirds of a right-angle, the (sum of the) six (plane) angles (containing the solid angle) will be four right-angles. The very thing (is) impossible. For every solid angle is contained by (plane angles whose sum is) less than four right-angles [Prop. 11.21]. So, for the same (reasons), a solid angle cannot be constructed from more than six plane angles (equal to two-thirds of a right-angle) either. And the (solid) angle of a cube is contained by three squares. And (a solid angle contained) by four (squares is) impossible. For, again, the (sum of the plane angles containing the solid angle) will be four right-angles. And (the solid angle) of a dodecahedron (is contained) by three equilateral and equiangular pentagons. And (a solid angle contained) by four (equiangular pentagons is) impossible. For, the angle of an equilateral pentagon being one and one-fifth of right-angle, four (such) angles will be greater (in sum) than four right-angles. The very thing (is) impossible. And, on account of the same absurdity, a solid angle cannot be constructed from any other (equiangular) polygonal figures either.

no others

Thus, beside the five aforementioned figures, no other solid figure can be constructed (which is) contained by equilateral and equiangular (planes). (Which is) the very thing it was required to show.



Lemma

It can be shown that the angle of an equilateral and equiangular pentagon is one and one-fifth of a right-angle, as follows.