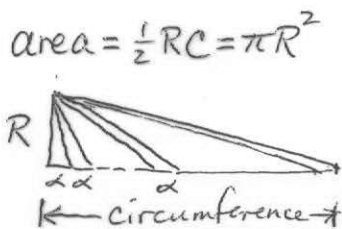
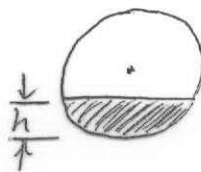


# Indivisibles and Area

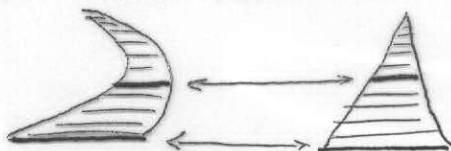
① Kepler 1615 (Wine Bottles)



How much wine is left in the barrel?

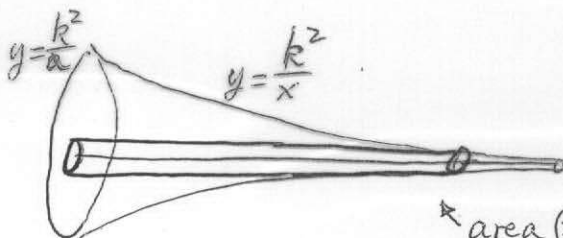


② Bonaventura Cavalieri 1635 and 1647



- Archimedes method of indivisibles
- Cavalieri's Principle in Nine Chapters (China ~200BC Han Dyn.)

③ Evangelista Torricelli 1643



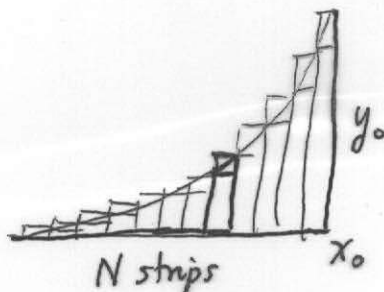
- surface area of shell  $x=a$  constant, area of circle of radius  $k\sqrt{2}$
- volume = (height)  $A = \frac{k^2}{a}(2\pi k^2)$

area  $(2\pi x) \left(\frac{k^2}{x}\right) = 2\pi k^2$

④ Fermat 1636 letter to Roberval (& vice versa)

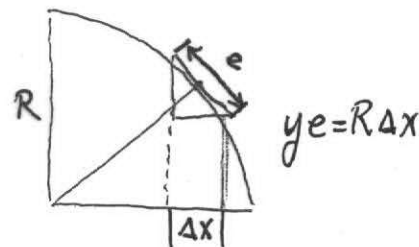
Area under  $y = ax^n$

$$A = \frac{x_0 y_0}{n+1} = \frac{a}{n+1} x_0^{n+1}$$



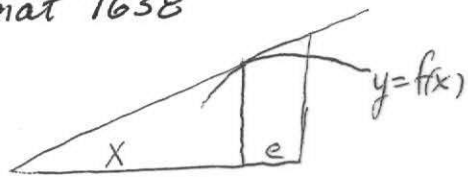
⑤ Blaise Pascal 1657  
Area under sine curve

$$\int_{\alpha}^{\beta} r \sin(\theta) d(\theta) = r[r \cos(\alpha) - r \cos(\beta)]$$

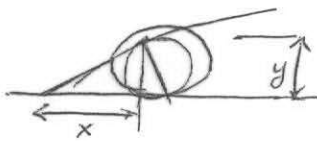


# Differential Triangles

Fermat 1638



Descartes  
1637



Van Heuraet  
1659

$$y = \frac{ds}{dx} = \frac{AC}{AB} = \frac{PS}{PR}$$

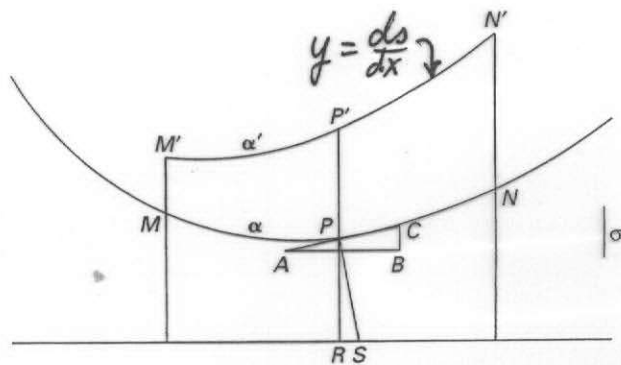
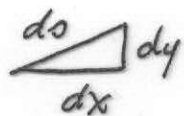


FIGURE 11.11 Van Heuraet's rectification of a curve

$$\sigma \cdot (\text{length of } MN) = \int_a^b z dx = \int_a^b \sigma \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx,$$

James Gregory  
1668

$$\int_0^x y dx = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

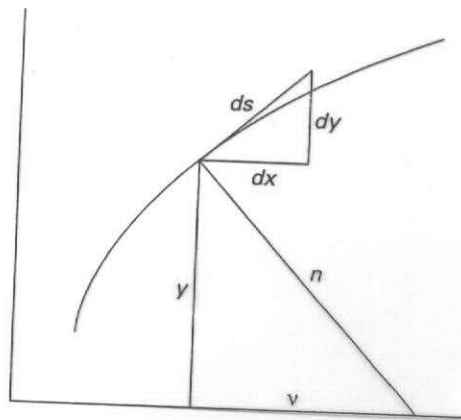


FIGURE 11.12 Gregory's differential triangle

## James Gregory (1638-1685)

Grew up, home-schooled and lived in Scotland. In 1663-1667 travelled to Flanders, Paris, Padua and Florence.

1667: wrote *Vera Circuli et Hyperbolae Quadratura*, in which he established the basic ideas of an infinitesimal analysis.

Using it, he showed how the areas of the circle and hyperbola could be obtained in the form of infinite power series,

1668: Gregory determined the power series expansions of the sine, cosine and tangent. He also established that

$$\int \sec x \, dx = \log(\sec x + \tan x)$$

solving a long standing problem in the construction of nautical tables.

In 1671, after seeing Barrow's book, Gregory established Taylor's Theorem (1715).



## Grégoire de Saint-Vincent (1584-1667) Brussels, Belgium

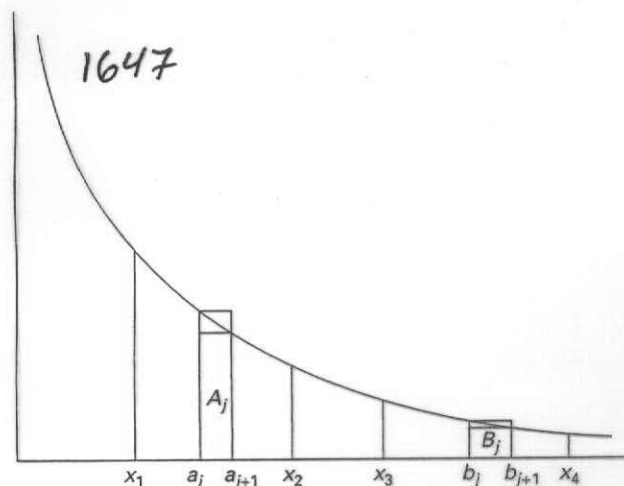


FIGURE 11.10 Gregory of St. Vincent's area under the hyperbola  $xy = 1$



1668 Nicolas Mercator calculated

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

used Fermat-Roberval formulas for area

# John Wallis (1616-1703)

He went to Cambridge to become a (medical) doctor. He stayed at Cambridge until 1644, when he was forced to resign for religious reasons.

Between 1643 and 1689 he served as chief cryptographer for the English Parliament against King Charles I (executed 1649). He was rewarded with a position at Oxford, for life. Charles II made him Royal Cryptographer (1661). Wallis rejected as absurd the now usual idea of a negative number as being less than nothing, but accepted the view that it is something greater than infinity (and is credited with introducing the symbol  $\infty$  for infinity.)



### Author of mathematics books:

- Arithmetica Infinitorum* (1655) \*
- Mathesis Universalis* (1657)
- Tractatus de Sectionibus Conicis* (1659)
- Mechanica, sive Tractatus de Motu* (1669-71, 3 parts)

Studied the area enclosed between the curve  $y = x^m$

Used indivisibles to compute  $\int_0^1 x^{p/2} dx$ . Stated "interpolation" rule for  $\int_0^1 x^{-n} dx$

 Built table of values for  $\int_0^1 (1-x^{1/p})^n dx = \binom{p+n}{n}$

With  $p$  fixed, adjacent entries have ratio  $\frac{\binom{p+n}{n}}{\binom{p+n-1}{n-1}} = \frac{p+n}{n}$



Interpolate to  $p = 1/2$ : adjacent entries  $\frac{1/2+n}{n} = \frac{2n+1}{2n}$

$$\frac{\pi}{2} = \frac{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \cdot 10^2}{3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 \cdot 11^2} \dots \text{(which is now known as the Wallis product).}$$

an approximation, the area of the semicircle  $\int_0^1 \sqrt{1-x^2} dx$  which is  $\frac{1}{2}\pi$

1	$3/2$	$15/8$	$105/48 \dots$	integer values of $n$
$\square$	$4/3 \square$	$8/5 \square$	$\dots$	half-integer values of $n$

$$\frac{\square}{1} > \frac{3/2}{\square} > \frac{4/3 \square}{3/2} \quad \left( \frac{105}{48} \right) < \left( \frac{8/5 \square}{15/8} \right) \text{ yields } \frac{64 \square^2}{75} > \frac{7 \cdot 5 \cdot 5 \cdot 3 \cdot 3}{6 \cdot 6 \cdot 4 \cdot 4 \cdot 2}$$

$$\frac{4}{3} \square^2 < \frac{3 \cdot 3}{2 \cdot 2}$$

## Isaac Barrow (1630-1677)

Age 13, admitted to Cambridge but lost his scholarship because his father had Royalist views. Went to Oxford for a year until the Siege of Oxford (Cromwell vs. Royalists) 1646. Privately tutored, back at Cambridge

Age 25, left for Paris, Italy, Turkey on a 3-year travel award. Returned as Charles II was being restored and given chair of Greek (as well as a professor of Geometry at Gresham College, London).

Spring 1664-Spring 1665: Gave 15 lectures on geometry at Cambridge entitled *Lectiones Geometricae*.

[Fall 1664-1666: Isaac Newton worked out his paper on fluxions.]

In 1669 Barrow resigned from the Lucasian Chair and did no further mathematics. This allowed Newton to take over the chair. In 1670, Barrow was appointed as Royal Chaplain to Charles II



Fall 1664 (Lecture X) (published 1670)

Given curve  $y = f(x)$ , define  $y = g(x)$  by

$$\text{area} = R \cdot g(x) = \int_A^D f(x) dx$$

Define  $T$  by  $DT = R \frac{DF}{DE} = R \frac{g(x)}{f(x)}$

Then tangent line is  $FT$ , ie, slope

$$m = \frac{DF}{DT} = R \cdot DE \text{ or } \boxed{g'(x) = \frac{1}{R} f(x)}$$

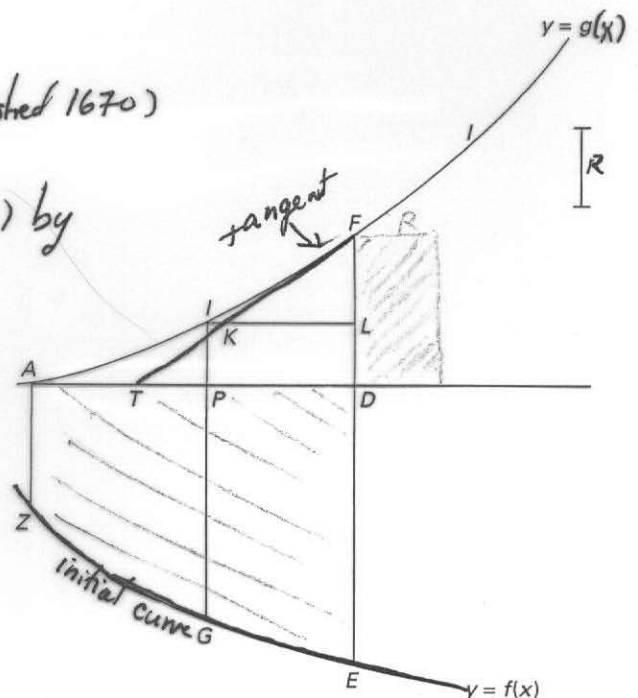


FIGURE 11.13 Barrow's version of the fundamental theorem



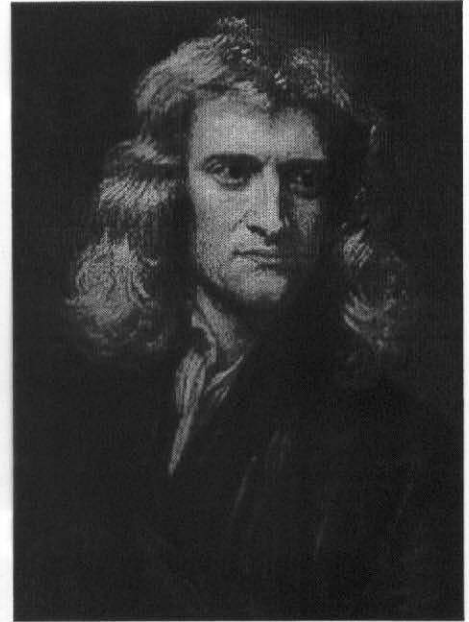
Isaac Newton (1643\*-1727)

Removed from school at age 17 in Lincolnshire.

1661: work-study student at Cambridge University,  
read Barrow's edition of Euclid's *Elements*. 1665 grad  
1665-1667 Cambridge closed for the plague.

Newton's *De Methodis Serierum et Fluxionum* was written in 1671  
but Newton failed to get it published and it did not appear in print  
until John Colson produced an English translation in 1736.

*Philosophiæ Naturalis Principia Mathematica*, published in 1687.



# A N A L Y S I S

Per Quantitatum

SERIES, FLUXIONES,

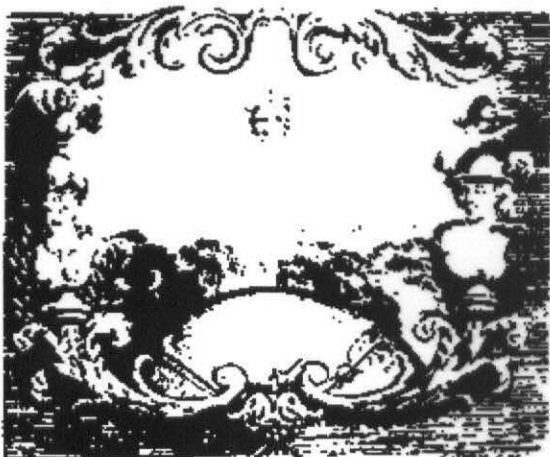
A C

D I F F E R E N T I A S :

C U M

*Enumerationem Linearum*

TERTII ORDINIS.



L O N D I N I :

Ex Officina P. ROSSIAE. Anno M.DCC.XI.

# PHILOSOPHIÆ NATURALIS P R I N C I P I A M A T H E M A T I C A .

---

Auctore J. S. NEWTON, Trin. Coll. Cantab. Soc. Matheseos  
Professore Lucasiano, & Societatis Regalis Sodali.

---

IMPRIMATUR.  
S. P E P Y S, Reg. Soc. P R Æ S E S .

Julii 5. 1686.

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L O N D I N I ,

Jussu Societatis Regiæ ac Typis Josephi Streater. Prostat apud  
plures Bibliopolas. Anno MDCLXXXVII.

# Gottfried Leibniz (1646-1716)

Born in Leipzig Germany (father was Philosophy Professor)  
After graduation, became an employee of Elector of Mainz.

Diplomacy re Louis IV of France: take Egypt and leave Germany alone. Napoleon's failed invasion of Egypt in 1798 can be seen as an unwitting implementation of Leibniz's plan.

In 1672, the French government invited Leibniz to Paris for diplomatic discussion – he stayed until 1676 and discovered Differential Calculus based on differential triangles.

In 1676, Leibniz accepted a position from the Duke of Hanover  
He remained there for the last 40 years of his life.



## ACTA ERUDITORUM

ANNO M DCLXXXIV

publicata,

at

SERENISSIMO FRATRUM PARTI,

DN. JOHANNI

GEORGIO IV,

Electoatus Saxonici Hæredi,

&

DN. FRIDERICO

AUGUSTO,

Ducibus Saxonæ &c. &c. &c.

PRINCIPIBUS JUVENTUTIS

dicata,

*Cum S. Cæsarea Majestatis & Potentissimi  
Electoris Saxonie Privilegio.*

L I P S I Æ.

Prostat apud J. GROSSUM & J. F. CHETTSCHUM.

Typis CHRISTOPHORI GANTHERI.

Anno M DCLXXXIV.

Here follow the two letters from Newton to Oldenburg; they are taken from *The correspondence of Isaac Newton*, ed. H. W. Turnbull (Cambridge University Press, New York, 1959), vol. 1.

LETTER OF JUNE 13, 1676

Most worthy Sir,

Though the modesty of Mr. Leibniz, in the extracts from his letter which you have lately sent me, pays great tribute to our countrymen for a certain theory of infinite series, about which there now begins to be some talk, yet I have no doubt that he has discovered not only a method for reducing any quantities whatever to such series, as he asserts, but also various shortened forms, perhaps like our own, if not even better. Since, however, he very much wants to know what has been discovered in this subject by the English, and since I myself fell upon this theory some years ago, I have sent you some of those things which occurred to me in order to satisfy his wishes, at any rate in part.

Fractions are reduced to infinite series by division; and radical quantities by extraction of the roots, by carrying out those operations in the symbols just as

they are commonly carried out in decimal numbers. These are the foundations of these reductions: but extractions of roots are much shortened by this theorem,

$$(P + PQ)^{m/n} = P^{m/n} + \frac{m}{n} AQ + \frac{m-n}{2n} BQ + \frac{m-2n}{3n} CQ + \frac{m-3n}{4n} DQ + \text{etc.},$$

where  $P + PQ$  signifies the quantity whose root or even any power, or the root of a power, is to be found;  $P$  signifies the first term of that quantity,  $Q$  the remaining terms divided by the first, and  $m/n$  the numerical index of the power of  $P + PQ$ , whether that power is integral or (so to speak) fractional, whether positive or negative. For as analysts, instead of  $aa$ ,  $aaa$ , etc., are accustomed to write  $a^2$ ,  $a^3$ , etc., so instead of  $\sqrt{a}$ ,  $\sqrt{a^3}$ ,  $\sqrt{c:a^5}$ , etc. I write  $a^{\frac{1}{2}}$ ,  $a^{\frac{3}{2}}$ ,  $a^{\frac{5}{2}}$ , and instead of  $1/a$ ,  $1/aa$ ,  $1/a^3$ , I write  $a^{-1}$ ,  $a^{-2}$ ,  $a^{-3}$ .<sup>1</sup> And so for

$$\frac{aa}{\sqrt{c:(a^3 + bbx)}}$$

I write  $aa(a^3 + bbx)^{-\frac{1}{2}}$ , and for

$$\frac{aab}{\sqrt{c:\{(a^3 + bbx)(a^3 + bbx)\}}}$$

I write  $aab(a^3 + bbx)^{-1}$ : in which last case, if  $(a^3 + bbx)^{-\frac{1}{2}}$  is supposed to be  $(P + PQ)^{m/n}$  in the Rule, then  $P$  will be equal to  $a^3$ ,  $Q$  to  $bbx/a^3$ ,  $m$  to  $-2$ , and  $n$  to 3. Finally, for the terms found in the quotient in the course of the working I employ  $A$ ,  $B$ ,  $C$ ,  $D$ , etc., namely,  $A$  for the first term,  $P^{m/n}$ ;  $B$  for the second term,  $(m/n)AQ$ ; and so on. For the rest, the use of the rule will appear from the examples.

Example 1.

$$\sqrt{(c^2 + x^2)} \quad \text{or} \quad (c^2 + x^2)^{\frac{1}{2}} = c + \frac{x^2}{2c} - \frac{x^4}{8c^3} + \frac{x^6}{16c^5} - \frac{5x^8}{128c^7} + \frac{7x^{10}}{256c^9} + \text{etc.}$$

For in this case,  $P = c^2$ ,  $Q = x^2/c^2$ ,  $m = 1$ ,  $n = 2$ ,  $A (= P^{m/n} = (c^2)^{\frac{1}{2}}) = c$ ,  $B (= (m/n)AQ) = x^2/2c$ ,  $C (= \frac{m-n}{2n} BQ) = -\frac{x^4}{8c^3}$ ; and so on.

<sup>1</sup> Newton had learned this method of broken and negative exponents from Wallis, but the idea goes back as far as Oresme and Chuquet; see Selection II.2. Through the influence of Wallis and Newton the method was gradually adopted by other mathematicians. The notation  $\sqrt{c:( )}$  indicates the cube root.