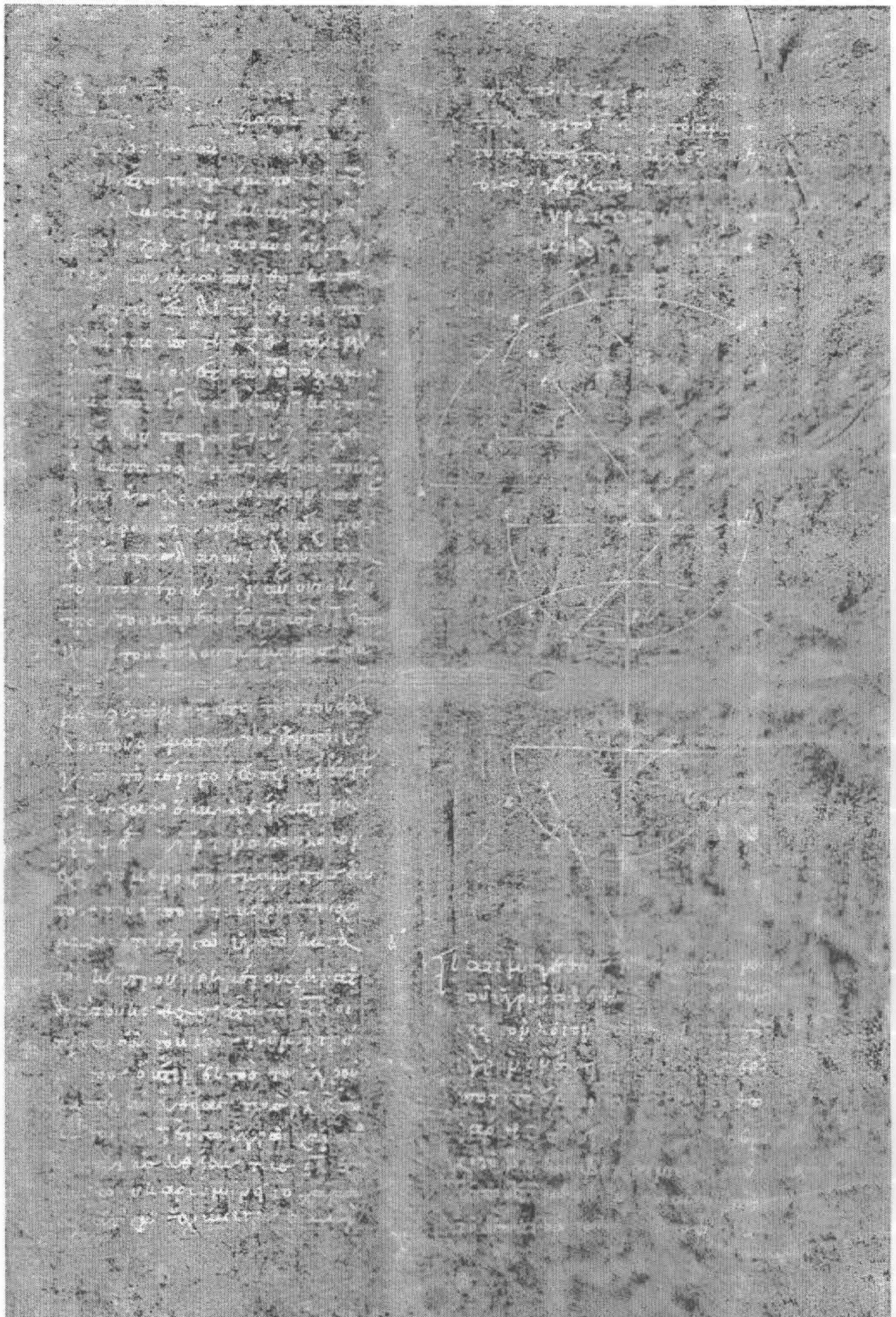
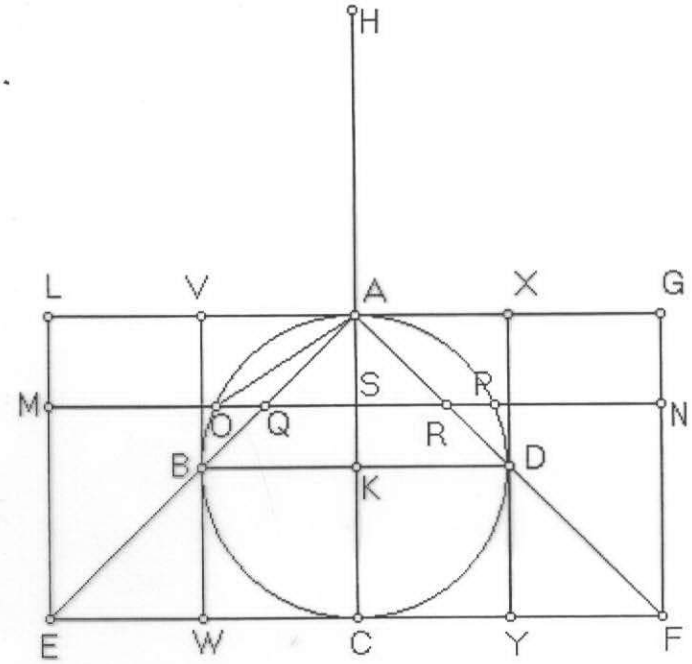
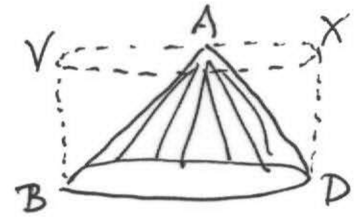
The Archimedes Palimpsest





Euclid: vol(cone) = $\frac{1}{3}$ vol(cylinder)



$\text{Vol(Sphere)} = \frac{4}{3}\pi r^3$

$\text{Vol(Cone)} = 1/3 \pi r^3$

From The *Method* of Archimedes (the translation is that of T.L. Heath):

Proposition 2.

We can investigate by the [mechanical] method the propositions that Vol(Sphere) = 4 · vol(Cone)

1. *Any sphere is (in respect of solid content) four times the cone with base equal to a great circle of the sphere and height equal to its radius; and*
2. *the cylinder with base equal to a great circle of the sphere and height equal to the diameter is 1½ times the sphere.*

1. Let ABCD be a great circle of a sphere, and AC, BD diameters at right angles to one another.

   Let a circle be drawn about BD as diameter and in a plane perpendicular to AC, and on this circle as base let a cone be described with A as vertex. Let the surface of this cone be produced and then cut by a plane through C parallel to its base; the section will be a circle on EF as diameter. On this circle as base let a cylinder be erected with height and axis AC, and produce CA to H, making AH equal to CA.

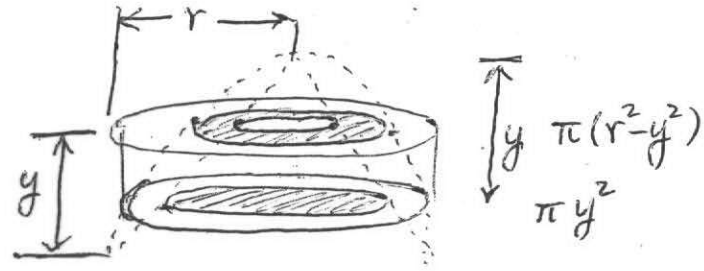
   Let CH be regarded as the bar of a balance, A being its middle point.

   Draw any straight line MN in the plane of the circle ABCD and parallel to BD. Let MN meet the circle in O,P, the diameter AC in S, and the straight lines AE, AF in Q, R respectively. Join AO.

   Through MN draw a plane at right angles to AC; this plane will cut the cylinder in a circle with diameter MN, the sphere in a circle with diameter OP, and the cone in a circle with diameter QR.

   Now, since MS = AC, and QS = AS,

$$
\begin{aligned}
MS \cdot SQ &= CA \cdot AS \\
&= AO^2 \\
&= OS^2 + SQ^2
\end{aligned}
$$



And, since HA = AC,

$$
\begin{aligned}
HA : AS &= CA : AS \\
&= MS : SQ \\
&= MS^2 : MS.SQ \\
&= MS^2 : (OS^2 + SQ^2), \text{ from above,} \\
&= MN^2 : (OP^2 + QR^2) \\
&= (\text{circle, diam. } MN) : (\text{circle, diam. } OP + \text{circle, diam. } QR).
\end{aligned}
$$

That is,

HA : AS = (circle in cylinder) : (circle in sphere + circle in cone).

Therefore the circle in the cylinder, placed where it is, is in equilibrium, about A, with the circle in the sphere together with the circle in the cone, if both latter circles are placed with their centres of gravity at H.

Similarly for the three corresponding sections made by a plane perpendicular to AC and passing through any other straight line in the parallelogram LF parallel to EF.

If we deal in the same way with all the sets of three circles in which planes perpendicular to AC cut the cylinder, the sphere, and the cone, and which make up those solids respectively, it follows that the cylinder, in the place where it is, will be in equilibrium about A with the sphere and the cone together, when both are placed with their centres of gravity at H.

Therefore, since K is the centre of gravity of the cylinder,

HA : AK = (cylinder) : (sphere + cone AEF).

But HA = 2·AK;

Therefore $\boxed{\text{cylinder = 2 (sphere + cone AEF).}}$

Now cylinder = 3 (cone AEF); [Eucl. XII 10]

Therefore cone AEF = 2 (sphere).

But, since EF = 2·BD,

Cone AEF = 8 (cone ABD);

Therefore $\boxed{\text{sphere = 4 (cone ABD).}}$

2. Through B, D draw VBW, XDY parallel to AC;

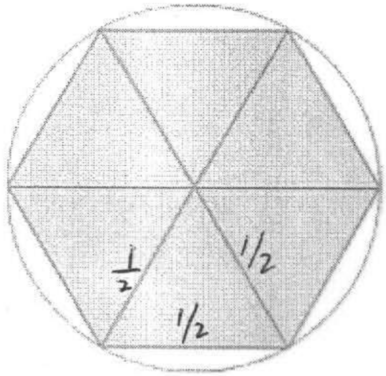
and imagine a cylinder which has AC for axis and the circles on VX, WY as diameters for bases.

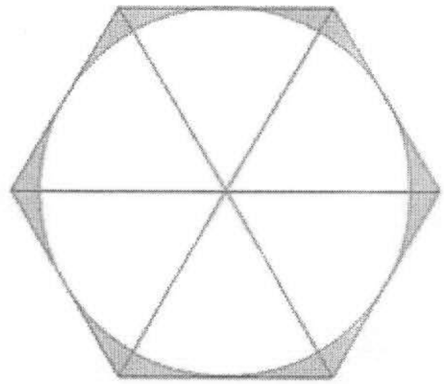
$$
\begin{aligned}
\text{Then cylinder } VY &= 2 \text{ (cylinder VD)} \\
&= 6 \text{ (cone ABD)[Eucl XII 10]} \\
&= 3/2 \text{ (sphere), from above.}
\end{aligned}
$$

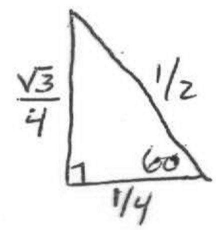
Q.E.D.

"From this theorem, to the effect that a sphere is four times as great as the cone with a great circle of the sphere as base and with height equal to the radius of the sphere, I conceived the notion that the surface of any sphere is four times as great as a great circle in it; for, judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius."

On the Measurement of the circle - Archimedes



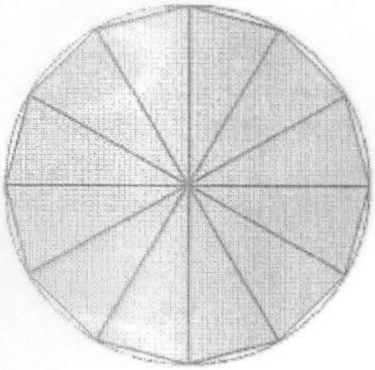


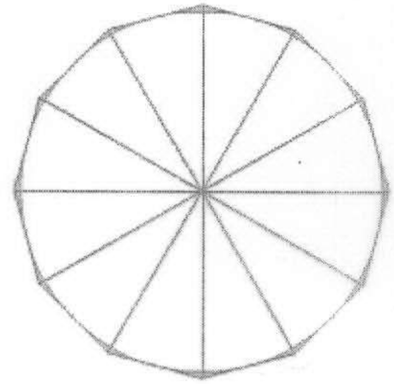


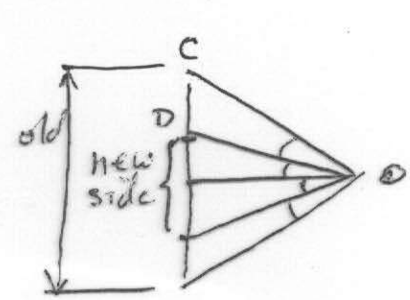
6-Sided Polygon
inscribed perimeter = 3.0
circumscribed perimeter = 3.4641

$\sqrt{3} \approx \frac{1351}{780} = 1.7320512$

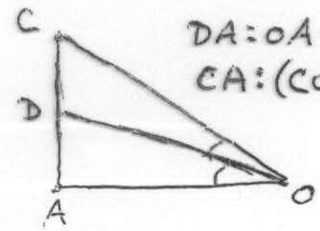
1.7320508



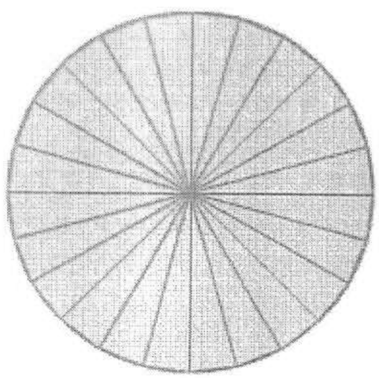




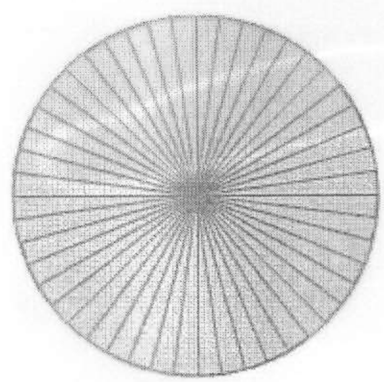
12-Sided Polygon
inscribed perimeter = 3.1058
circumscribed perimeter = 3.2154



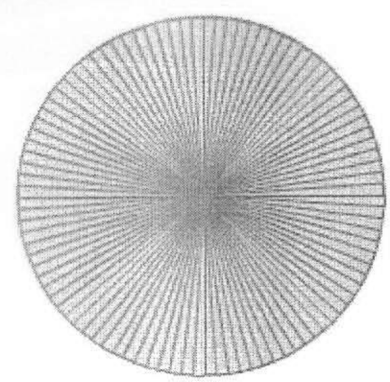
DA : OA = CA : (CO + OA)



24-Sided Polygon
inscribed perimeter = 3.1326
circumscribed perimeter = 3.1597



48-Sided Polygon
inscribed perimeter = 3.1394
circumscribed perimeter = 3.1461



96-Sided Polygon
inscribed perimeter = 3.1410
circumscribed perimeter = 3.1427

$3\frac{10}{71} < \pi < 3\frac{1}{7}$

First Research - expository paper

- **Archimedes,The Sand-Reckoner (Arenarius), ch. 1 (sects. 1-20) ©**
- translated by Henry Mendell (Cal. State U., L.A.)

Archimedes born 287 BC

Return to Vignettes of Ancient Mathematics

go to Introduction

go to Ch. 2

go to Ch. 3

go to Ch. 4

· King of Syracuse, 270-215 BC
· Relative of Archimedes
· Asked Archimedes about Gold Crown (Eureka!)

Note: the numbering is from Heiberg's text (revised Stamatis), on which the translation is based.

Syracuse

[1] Some people believe, King Gelon, that the number of sand is infinite in multitude. I mean not only of the sand in Syracuse and the rest of Sicily, but also of the sand in the whole inhabited land as well as the uninhabited. There are some who do not suppose that it is infinite, and yet that there is no number that has been named which is so large as to exceed its multitude.

Note: Archimedes speaks of the number of the sand and not of the grains of sand. He does not use a word meaning 'grain of sand'. In deference to this, I shall treat 'sand' as a mass term (some sand), but allow that one can speak of the number of sand, meaning, of course, the number of the grains of sand.

[2] It is clear that if those who hold this opinion should conceive of a volume composed of the sand as large as would be the volume of the earth when all the seas in it and hollows of the earth were filled up in height equal to the highest mountains, they would not know, many times over, any number that can be expressed exceeding the number of it.

[3] I will attempt to prove to you through geometrical demonstrations, which you will follow, that some of the numbers named by us and published in the writings addressed to Zeuxippus exceed not only the number of sand having a magnitude equal to the earth filled up, just as we said, but also the number of the sand having magnitude equal to the world. world = Kosmos = universe

Note: the book to Zeuxippus (lost) would have been the formal presentation of the system, while the *Sand-Reckoner* is the popularization.

[4] You grasp that the world is called by most astronomers the sphere whose center is the center of the earth and whose line from the center is equal to the straight-line between the center of the sun and the center of the earth, since you have heard these things in the proofs written by the astronomers. But Aristarchus of Samos produced writings of certain hypotheses in which it follows from the suppositions that the world is many times what is now claimed.

310-230 BC

Note: this claim is very odd and has not been adequately noticed by commentators. One would think that the whole world is the sphere of the fixed stars and everything within and that the sun is lower than the fixed stars, as Aristotle argues, and not the cosmology of Anaximander, who does place the sun as the outermost object. Instead, Archimedes seems to place the sun as the outermost, since 'world' (kosmos) should encompass everything, and he is aiming to give as large a universe as possible on each of the two rival theories. The issue is complicated by the fact that Hippolytus (3rd. cent. C.E.) preserves two versions of Archimedes' own dimensions of the universe:

	earth to sun	earth to outermost in list
Version 1	5581,6195 stadia	2,4826,4780 stadia to zodiac
Version 2	1,2160,4451	2,2269,2711 stadia to Saturn
Sand Reckoner	< 100,0000,0000	< 100,0000,0000,0000 (stadia to zodiac on Aristarchus' theory)

stadium ≈ 200 yards ≈ 1/8 mile

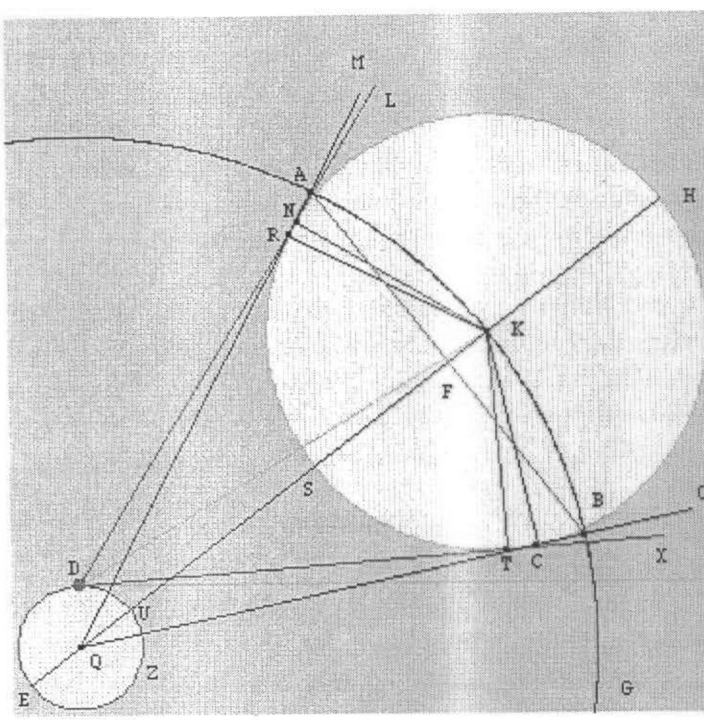
← 7 million miles to sun
15 " " "
at most a billion miles

Source: O. Neugebauer, *Hist. Anc. Math. Astron.*, 648-9.

Hence, Archimedes does not consider the sun to be at the edge of the world. Nonetheless, his value for the distance of the sun is much larger than the transmitted value for the size of the world. (For the placement of comma in numbers, cf. notes to Ch. 4).

[5] For he supposes that the fixed stars and the sun remain motionless, while the earth revolves about the sun on the circumference of a circle which is placed on the middle road, but that the sphere of the fixed stars, which is placed about the same center as the sun, is so large in magnitude that the circle on which he supposes the earth to revolve has the sort of proportion to the distance of the fixed stars that the center of the sphere has to the surface. !

This is our principal source for this view and is one of the grounds for modern interest in the treatise. Aristarchus only extant treatise, *On the Sizes and the Distances of the Sun and the Moon*, gives not a hint of such an hypothesis.



(diagram 15) And diameter EQU is smaller than diameter SH, since circle DEZ is smaller than circle SH. Therefore, both QU, KS are smaller than a hundredth part of QK. [1/2EU + 1/2SH] (diagram 16) Thus QK to US has a ratio smaller than 100 to 99. [US = QK – (KS + QU, while a < b/n ⇒ b-a : b > n-1 : n ⇒ b : (b-a) < n : (n-1))] (diagram 17) And since QK is not smaller than QR, but SU is smaller than DT, therefore QR to DT would have a ratio smaller than 100 to 99. [a Ž c & b < d ⇒ a : b > c : d] [21] (diagram 18) Since, given that QKR, DKT are right-angled triangles, sides KR, KT are equal, while QR, DT are unequal with QR larger, the angle enclosed by DT, DK to the angle enclosed by QR, QK has a ratio larger than QK to DK, but smaller than QR to DT. For if in two right-angled triangles one pair of sides about the right angle are equal and the others are unequal, the larger of the angles at the unequal sides to the smaller has a ratio larger than the larger of the lines subtending the right angle to the smaller, but smaller than the larger of the lines at the right angle to the smaller. [22] (diagram 19) Thus the angle enclosed by DL, DX to the angle at most 350,000 miles

[8] First that the perimeter of the earth is about 300,0000 stadia and not larger, although some have attempted to demonstrate it , as you too follow them, as being about 30,0000. But since I am exceeding this and posit the magnitude of the earth as ten-times what was believed by the earlier astronomers, I suppose the perimeter of it to be about 300,0000 and not larger.

35,000 miles

Aristotle reports 40,0000 stadia, and Eratosthenes (contemporary of Archimedes) calculated 25,0000 or 25,2000 stadia. It is an old problem that the length of the stadium varies in different locals, so that it is a separate problem to know what these values are in actual distance.

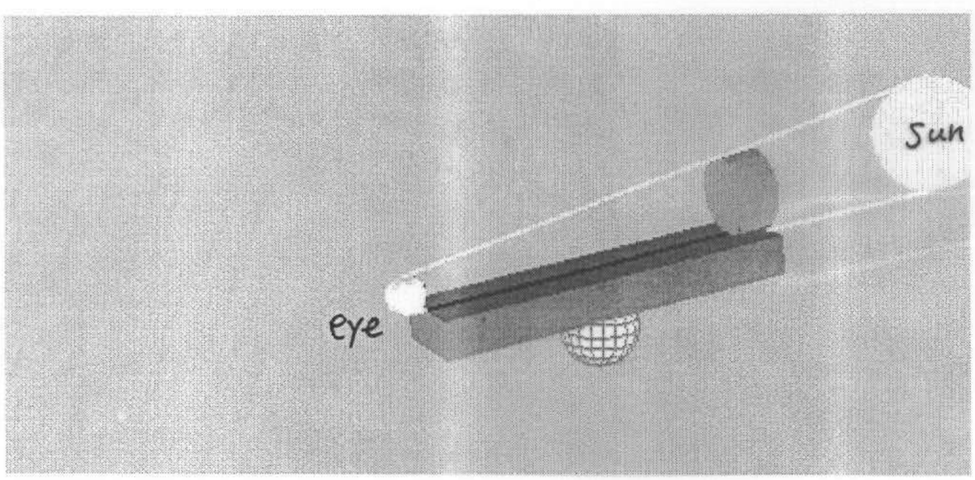
After this, I suppose that the diameter of the earth is larger than the diameter of the moon and that the diameter of the sun is larger than the diameter of the earth, and assume the same things in like manner as most of the earlier astronomers.

[9] After these, I suppose that the diameter of the sun is about thirty-times the diameter of the moon and not larger, although of earlier astronomers Eudoxus declared about nine-times, Pheidias, my father, about twelve-times, and Aristarchus has attempted to prove that the diameter of the sun is more than eighteen-times the diameter of the moon and smaller than twenty-times. But I will also exceed this amount, so that what is proposed be indisputably proved, and suppose that the diameter of the sun is about thirty-times the diameter of the moon and not larger.

For Aristarchus, cf. his *On the Sizes and the Distances of the Sun and the Moon.*

polygon with 1000 sides

[10] In addition to these, I suppose that the diameter of the sun is larger than the side of the chiliagon inscribed in the largest circle of those in the world. I suppose this given that Aristarchus has found the sun appears about one seven hundred and twentieth of the circle of the zodia, but having examined it in the following manner I attempted with instruments to get the angle into which the sun fits and which has its vertex at the eye. [11] And so it is not easy to get precision since neither the eye nor the hands nor the instruments through which we must get it are trustworthy at declaring precision. For the present it is not timely to lengthen our discussion about these things, especially since these sorts of things have been explained many times. For the demonstration of the proposed claim, it suffices for me to get an angle which is no larger than the angle into which the sun fits and which has its vertex at the eye, and again to get another angle which is not smaller than the angle into which the sun fits and which has its vertex at the eye.



measures the diameter of the eye to get angle of sun more than $\frac{1}{2}$ degree "$\frac{1}{200}$ of right angle"