Efficient algorithms for tensor scaling, quantum marginals, and moment polytopes

Cole Franks (Rutgers)

joint work with

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Michael Walter, Avi Wigderson
Overview

**Simple, efficient** algorithm for *approximate* membership for broad class of polytopes

- known as *moment polytopes* in math and physics
- can have exponentially many facets and vertices
- capture many natural problems across computer science, mathematics, and physics

In particular, *one-body quantum marginal problem.*
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Polytopes in the tensor scaling framework

- **Marginals**: marginals of joint probability distributions with constrained supports

- **Row and column sums**: of row/column reweightings of a fixed nonnegative matrix

- **Horn polytope**: spectra of symmetric matrices $A, B, C$ with $A + B = C$
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• Quantum marginals, moment polytopes, and tensor scaling
  • History, special cases, and applications
  • Our results
  • Algorithm
  • Analysis
  • Open problems
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Quantum marginals, moment polytopes, and tensor scaling
Quantum marginals

If Alice, Bob, and Carol’s qubits are jointly in a pure quantum state, the one-body marginals are $2 \times 2$ PSD matrices $\rho^{(A)}, \rho^{(B)}, \rho^{(C)}$.

One body quantum marginal problem, $d = 3$

**Input:** PSD matrices $P_A, P_B, P_C$

**Output:** Whether there is a pure state with marginals $P_A, P_B, P_C$

**Fact:** the answer depends only on $\text{spec}(P_A), \text{spec}(P_B), \text{spec}(P_C)$.
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Now consider $d$ parties each holding $n$-dimensional quantum system.

**$d$-tensors: denoted $(\mathbb{C}^n)^\otimes d$**

$n \times \cdots \times n$ arrays of complex numbers

$d$

$d = 3, n = 2$:

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\begin{pmatrix}
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0 & 1 & 0
\end{pmatrix} 
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Flattening and marginals

Three different ways of viewing a $n \times n \times n$ tensor as an $n \times n^2$ matrix.

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Consider a set $X$ of $d$-tensors (for example $X = (\mathbb{C}^n)^\otimes d$);

$$\Delta(X) = \{\text{tuples of spectra of marginals of elements of } X \text{ with trace 1}\}$$

e.g. $((.75, .25), (.5, .5), (.9, .1))$ for 3 qubits

**Amazing fact [Mumford, Ness ‘84]**

For many natural $X$, including those in this talk, $\Delta(X)$ is a convex polytope!

called the moment polytope; can be defined in much greater generality

**Central question:** Moment polytope membership

**Input:** list of spectra $p$, arithmetic circuit parametrizing $X$

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Quantum marginal problem, restated

Moment polytope membership for $\mathbb{X} = (\mathbb{C}^n)^\otimes d$!

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Method: tensor scaling

Given $X$, can we locally change basis to obtain specific marginals?

- Changing basis on the $i^{th}$ vector space by $g_i$ changes flattening by
  \[ M^{(i)} \leftarrow g_i M^{(i)} \]

- **Scaling**: Simultaneous local basis change $(g_1, \ldots, g_d) \cdot X := g \cdot X$

**Question**: Tensor scaling

**Input**: $p$ list of spectra, tensor $X$

**Output**: whether there exists scalings of $X$ with spectra of marginals approaching $p$

- say $G \cdot X$: set of all scalings of $X$, $\overline{G \cdot X}$ its closure

**Tensor scaling** is moment polytope membership for $X = \overline{G \cdot X}$!

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Applications of tensor scaling

**Matrix scaling:** reweighting rows and columns
- deterministically approximating permanent

**Operator scaling:** tensor scaling with two marginals
- noncommutative rational identity testing
- Forster’s radial isotropic position
- computing the Brascamp-Lieb constant in analysis
- Horn’s problem on eigenvalues of sums of matrices

**Uniform tensor scaling:** target marginals $l_n/n$
- Nullcone problem in invariant theory: do all invariants vanish on $X$?
- equivalence under SLOCC to locally maximally mixed state

**One body quantum marginal problem:** tensor scaling for random $X$
- The Kronecker polytope in representation theory

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- entanglement polytopes: comparing different types of entanglement
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Approximate versions of all these questions

Approximate tensor scalings:
Produce scaling of input $X$ with marginals $\varepsilon$-close to input $p$

Approximate quantum marginals:
Produce tensor with marginals $\varepsilon$-close to input $p$

Approximate moment polytope membership:
Correctly output one of

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Produce scaling of input $X$ with marginals $\varepsilon$-close to input $p$

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**Theorem (BFGOWW ’18, Tensor scaling)**

There is a randomized $\poly(n^d, \langle X \rangle + \langle p \rangle, 1/\epsilon)$-time algorithm for approximate tensor scaling on input $X, p, \epsilon$ with success probability $1/2$.

The algorithm requires $O\left( dn^{2d \langle X \rangle + \langle p \rangle + d \log d n \epsilon^2} \right)$ iterations, each dominated by computing a Cholesky decomposition of some $n \times n$ matrix.

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Actual membership oracle?

**Theorem (BFGOWW ’18)**

If the spectra of marginals of $X$ are $\varepsilon = \exp(-O(dn^{d+1}\langle p \rangle))$-close to $p$, then $p \in \Delta(X)$.

Unfortunately, doesn’t result in poly time algorithm for membership! Need $\text{poly}(\log(1/\varepsilon))$. 
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Algorithm
Vague tensor scaling algorithm

Algorithm

Input: $X, p$ with integer coordinates, $\varepsilon$.
Output: scaling of $X$ s.t. spectra of marginals $\varepsilon$-close to $p$, or OUTSIDE POLYTOPE

Repeat $T$ times:
  • If done, output $X$.
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    (ignoring damage done to other marginals!)

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Not this simple.
## Vague tensor scaling algorithm

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## Actual tensor scaling algorithm

### Algorithm

**Input:** $X, p$ with integer coordinates, $\epsilon$.

**Output:** scaling $Y$ s.t. spectra of marginals $\epsilon$-close to $p$, or OUTSIDE POLYTOPE.

- Choose $g_0$ randomly, set $Y = g_0 \cdot X$.
- Repeat $T$ times:
  - If done, output $Y$.
  - Else, scale in single factor to CAREFULLY FIX worst marginal of $Y$.
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### Theorem

For $T \geq O \left( dn^{2d} \frac{\langle Y \rangle + \langle p \rangle + d \log dn}{\epsilon^2} \right)$, this is algorithm succeeds with probability at least $1/2$. 

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Easy to fix an individual marginal;
Solve simple matrix factorization problem for $g_i$

$$g_i \rho^{(i)} g_i^\dagger = \text{diag}(p_i)$$

WARNING: not every choice works.

The correct way is to fix the marginal using only lower triangular scalings $g_i$
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Typical analysis

• Define potential function $f$ with $f(g_0) = 1$
• Show $f$ decreases by $\Omega(\epsilon^2)$ in each iteration of the algorithm if marginals off by at least $\epsilon$
• Establish lower bound for $f$

However, triangular scaling algorithm can fail without randomization!

Must establish lower bounds for $f$ that hold w.h.p over randomization
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Our lower bounds are objects arising from representation theory

**Definition (Highest weights of type $p$)**

Homogeneous polynomials on $(\mathbb{C}^n)^\otimes d$ that are eigenfunctions for the action of the lower triangular matrices;

$$P(g \cdot Y) = \text{scalar}(g, p)P(Y)$$

$\text{scalar}(g, p)$ is just the eigenvalue.

**Potential function:** $f_Y(g) = \log \frac{\|g \cdot Y\|}{\text{scalar}(g, p)}$

There's a fairly easy weak duality here: very roughly,

$$\inf_g f_Y(g) \geq \sup_{P \text{ highest weight}} P(Y)$$
Lower bounds: highest weights

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Last ingredient: moment polytope and highest weights

$p \in \Delta(G \cdot X)$

Want for random $Y$ $\inf_g f_Y(g) > \frac{1}{\text{poly}}$

"potential"
Last ingredient: moment polytope and highest weights

\[ p \in \Delta(G \cdot X) \]

Figure 3.1: Borland–Dennis polytope. The solution of the one-body \( n \)-representability problem for three fermions with local dimension six, as given by the Borland–Dennis inequalities (3.1). The vertex \((1, 1, 1)\) corresponds to a single Slater determinant.

Figure 3.2: Three-qubit polytope. The solution of the one-body quantum marginal problem for pure states of three qubits, as given by the polygonal inequalities (3.2) for \( n = 3 \).

Figure 3.3: Bravyi’s polytope, corresponding to his solution (3.3) of the one-body quantum marginal problem for two qubits and global spectrum \( AB = (0.6, 0.3, 0.1, 0) \).

$\inf_g f_Y(g) > \frac{1}{\text{poly}}$

$\sup_{\mathcal{P}_{\text{hwv}}} P(Y) > \frac{1}{\text{poly}}$

$\text{Ness–Mumford'84}$
$+ \text{Derksen'01}$
$+ \text{Lemma}$

for random \( Y \)
Last ingredient: moment polytope and highest weights

\[ p \in \Delta(G \cdot X) \]

Want for random \( Y \)

\[ \inf_g f_Y(g) > \frac{1}{\text{poly}} \]

\( \text{"potential"} \)

\[ \text{weak duality} \]

\[ \sup_{P \text{ hwv}} P(Y) > \frac{1}{\text{poly}} \]

\( Ness – Mumford '84 \)
\( + Derksen '01 \)
\( + Lemma \)
for random \( Y \)
Open problems
Open problems

- Is the tensor scaling decision problem in $\text{NP}$? Is it in $\text{coNP}$?
- Is it in $\text{RP}$? A poly log($1/\varepsilon$) algorithm would prove it! In $\text{P}$?
- Can tensor scaling be done in poly log($1/\varepsilon$) for a random tensor? Would put quantum marginal problem in $\text{RP}$!
- Approximately scale for other group actions, without alternating minimization (in progress).
- Obtain similar algorithms for multi-body quantum marginals.
- Develop separation oracles for moment polytopes.
Open problems

- Is the tensor scaling decision problem in NP? Is it in coNP?
- Is it in RP? A poly log(1/ε) algorithm would prove it! In P?
- Can tensor scaling be done in poly log(1/ε) for a random tensor? Would put quantum marginal problem in RP!
- Approximately scale for other group actions, without alternating minimization (in progress).
- Obtain similar algorithms for multi-body quantum marginals.
- Develop separation oracles for moment polytopes.
Open problems

- Is the tensor scaling decision problem in $\text{NP}$? Is it in $\text{coNP}$?
- Is it in $\text{RP}$? A poly $\log(1/\varepsilon)$ algorithm would prove it! In $\text{P}$?
- Can tensor scaling be done in poly $\log(1/\varepsilon)$ for a random tensor? Would put quantum marginal problem in $\text{RP}$!
- Approximately scale for other group actions, without alternating minimization (in progress).
- Obtain similar algorithms for multi-body quantum marginals.
- Develop separation oracles for moment polytopes.
Open problems

• Is the tensor scaling decision problem in $\textbf{NP}$? Is it in $\textbf{coNP}$?
• Is it in $\textbf{RP}$? A poly log($1/\varepsilon$) algorithm would prove it! In $\textbf{P}$?
• Can tensor scaling be done in poly log($1/\varepsilon$) for a random tensor? Would put quantum marginal problem in $\textbf{RP}$!
• Approximately scale for other group actions, without alternating minimization (in progress).
  • Obtain similar algorithms for *multi-body* quantum marginals.
  • Develop separation oracles for moment polytopes.
Open problems

- Is the tensor scaling decision problem in $\textbf{NP}$? Is it in $\textbf{coNP}$?
- Is it in $\textbf{RP}$? A poly log$(1/\varepsilon)$ algorithm would prove it! In $\textbf{P}$?
- Can tensor scaling be done in poly log$(1/\varepsilon)$ for a random tensor? Would put quantum marginal problem in $\textbf{RP}$!
- Approximately scale for other group actions, without alternating minimization (in progress).
- Obtain similar algorithms for multibody quantum marginals.
- Develop separation oracles for moment polytopes.
Open problems

- Is the tensor scaling decision problem in $\textbf{NP}$? Is it in $\textbf{coNP}$?
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- Approximately scale for other group actions, without alternating minimization (in progress).
- Obtain similar algorithms for multi-body quantum marginals.
- Develop separation oracles for moment polytopes.
Thank you!
$p \in \Delta(G \cdot X)$
Moment polytope and representation theory

$p \in \Delta(G \cdot X)$

Want for

\[ Y = g_0 \cdot X \quad \text{random} \]

\[ \inf_{g} f_{Y,p}(g) > \frac{1}{\text{poly}} \]
Moment polytope and representation theory

\[ p \in \Delta(G \cdot X) \]

\[ Q \not= 0 \text{ on } G \cdot X, \; \deg Q \leq 2^{\text{poly}} \]

\[ Q \in HWV_p \]

Want for
\[ Y = g_0 \cdot X \]
random

\[ \inf_{g} f_{Y,p}(g) > \frac{1}{\text{poly}} \]
Moment polytope and representation theory

\[ p \in \Delta(\overline{G \cdot X}) \]

\[ \text{Ness–Mumford}'84 \quad + \text{Derksen}'01 \]

\[ Q \not\equiv 0 \text{ on } \overline{G \cdot X}, \deg Q \leq 2^\text{poly} \]

\[ Q \in \text{HWV}_p \]

\[ \text{probability} > .5 \]

\[ Q(Y) \neq 0 \]

\[ \inf_g f_{Y,p}(g) > \frac{1}{\text{poly}} \]
Moment polytope and representation theory

\[ p \in \Delta(G \cdot X) \]

\[ Q \not\equiv 0 \text{ on } G \cdot X, \deg Q \leq 2^{\text{poly}} \]

\[ Q \in \text{HWV}_p \]

\[ Q(Y) \neq 0 \]

\[ \inf_g f_{Y,p}(g) > \frac{1}{\text{poly}} \]

\[ \sup_{P \in \text{HWV}_p} \frac{1}{\deg P} \frac{|P(Y)|}{\|P\|} > \frac{1}{\text{poly}} \]

Want for

\[ Y = g_0 \cdot X \text{ random} \]

Lemma

Probability > .5

Ness–Mumford'84 + Derksen'01
Moment polytope and representation theory

\[ p \in \Delta(\overline{G \cdot X}) \]

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Want for

**Lemma**

\[ Q \not\equiv 0 \text{ on } \overline{G \cdot X}, \deg Q \leq 2^{\text{poly}} \]

\[ Q \in HWV_p \]

**probability** \( > 0.5 \)

\[ Q(Y) \neq 0 \]

\[ \inf_g f_{Y,p}(g) > \frac{1}{\text{poly}} \leftrightarrow \text{weak duality} \]

\[ \sup_{P \in HWV_p} \frac{1}{\deg P} \frac{|P(Y)|}{\|P\|} > \frac{1}{\text{poly}} \]