Efficient algorithms for tensor scaling, quantum marginals, and moment polytopes

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joint work with

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Overview

Simple, efficient algorithm for *approximate* membership for broad class of polytopes

- known as moment polytopes in math and physics
- can have exponentially many facets and vertices
- capture many natural problems accross computer science, mathematics, and physics

In particular, one-body quantum marginal problem.

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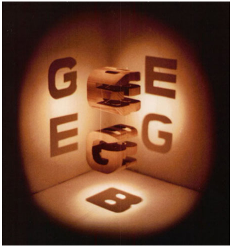
Polytopes in the tensor scaling framework

- **Marginals:** marginals of joint probability distributions with constrained supports



- **Row and column sums:** of row/column reweightings of a fixed nonnegative matrix
- **Horn polytope:** spectra of symmetric matrices A, B, C with $A + B = C$

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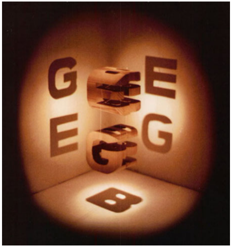

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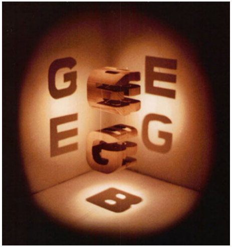

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- Quantum marginals, moment polytopes, and tensor scaling
- History, special cases, and applications
- Our results
- Algorithm
- Analysis
- Open problems

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Quantum marginals, moment polytopes, and tensor scaling

Quantum marginals

If Alice, Bob, and Carol's qubits are jointly in a pure quantum state, the one-body marginals are 2×2 PSD matrices $\rho^{(A)}, \rho^{(B)}, \rho^{(C)}$

One body quantum marginal problem, $d = 3$

Input: PSD matrices P_A, P_B, P_C

Output: Whether there is a pure state with marginals P_A, P_B, P_C

Fact: the answer depends only on $\mathrm{spec}(P_A), \mathrm{spec}(P_B), \mathrm{spec}(P_C)$.

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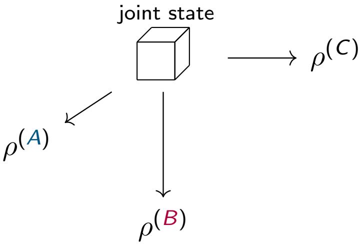

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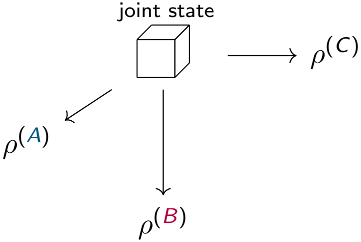

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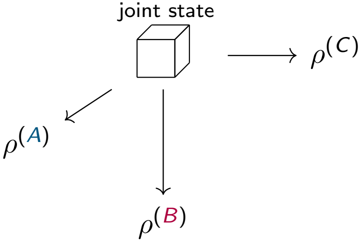

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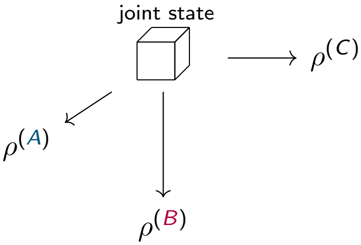

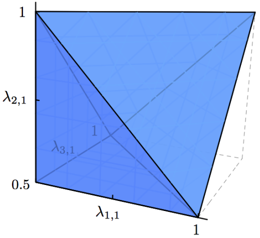

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Now consider d parties each holding n-dimensional quantum system.

d-tensors: denoted $(\mathbb{C}^n)^{\otimes d}$

$\underbrace{n \times \cdots \times n}_{d}$ arrays of complex numbers

$d = 3, n = 2$:

$\in (\mathbb{C}^2)^{\otimes 3}$

AKA $|000\rangle + |111\rangle$

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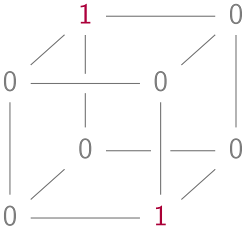

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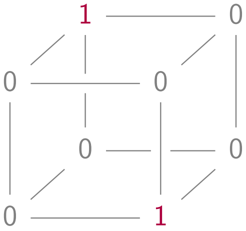

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Flattening and marginals

Three different ways of viewing a $n \times n \times n$ tensor as an $n \times n^2$ matrix.

Flattening

i^{th} *flattening* $M^{(i)}$ of X is an $n \times n^{d-1}$ matrix

rows = slices of X orthogonal to the i^{th} direction.

Marginal

i^{th} marginal of X is the $n_i \times n_i$ matrix

$$\rho^{(i)} = M^{(i)} \left(M^{(i)} \right)^{\dagger}$$

$\rho^{(i)}$ AKA partial trace of $|X\rangle \langle X|$ onto i^{th} subsystem

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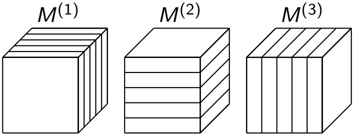

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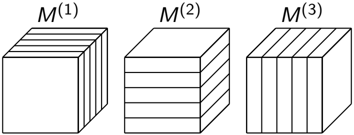

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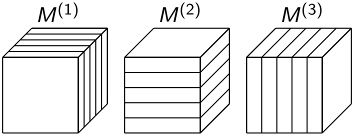

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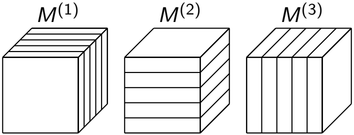

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Moment polytopes

Consider a set $\mathbb{X}$ of d-tensors (for example $\mathbb{X} = (\mathbb{C}^n)^{\otimes d}$);

$$\Delta(\mathbb{X}) = \{\text{tuples of spectra of marginals of elements of } \mathbb{X} \text{ with trace } 1\}$$

e.g. $((.75, .25), (.5, .5), (.9, .1))$ for 3 qubits

Amazing fact [Mumford, Ness '84]

For many natural $\mathbb{X}$, including those in this talk, $\Delta(\mathbb{X})$ is a convex polytope!

called the moment polytope; can be defined in much greater generality

Central question: Moment polytope membership

Input: list of spectra $\boldsymbol{p}$, arithmetic circuit parametrizing $\mathbb{X}$

Output: Whether $\boldsymbol{p}$ is in moment polytope $\Delta(\mathbb{X})$

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Quantum marginal problem, restated

Moment polytope membership for $\mathbb{X} = (\mathbb{C}^n)^{\otimes d}$!

Quantum marginal problem

Input: p list of spectra

Output: Whether p is in the moment polytope $\Delta(\mathbb{X})$ for $\mathbb{X} = (\mathbb{C}^n)^{\otimes d}$.

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Input: $\boldsymbol{p}$ list of spectra

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Method: tensor scaling

Given X, can we locally change basis to obtain specific marginals?

- Changing basis on the i^{th} vector space by g_i changes flattening by

$$M^{(i)} \leftarrow g_i M^{(i)}$$

- **Scaling:** Simultaneous local basis change $(g_1, \ldots, g_d) \cdot X := g \cdot X$

Question: Tensor scaling

Input: $\boldsymbol{p}$ list of spectra, tensor X

Output: whether there exists scalings of X with spectra of marginals approaching $\boldsymbol{p}$

- say $G \cdot X$: set of all scalings of X, $\overline{G \cdot X}$ its *closure*

Tensor scaling is moment polytope membership for $\mathbb{X} = \overline{G \cdot X}$!

Fact: **Quantum marginal** problem for $\boldsymbol{p}$ $\iff$ **tensor scaling** problem for $\boldsymbol{p}$ and *random* X!

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History, special cases, and applications

Applications of tensor scaling

Matrix scaling: reweighting rows and columns

- deterministically approximating permanent

Operator scaling: tensor scaling with two marginals

- noncommutative rational identity testing
- Forster's radial isotropic position
- computing the Brascamp-Lieb constant in analysis
- Horn's problem on eigenvalues of sums of matrices

Uniform tensor scaling: target marginals I_n/n

- Nullcone problem in invariant theory: do all invariants vanish on X?
- equivalence under SLOCC to locally maximally mixed state

One body quantum marginal problem: tensor scaling for random X

- The *Kronecker polytope* in representation theory

Nonuniform tensor scaling:

- *entanglement polytopes:* comparing different types of entanglement
- *Computing quantum functionals:* information theoretic value of a state

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- *entanglement polytopes:* comparing different types of entanglement
- *Computing quantum functionals:* information theoretic value of a state

Applications of tensor scaling

Matrix scaling: reweighting rows and columns

- deterministically approximating permanent

Operator scaling: tensor scaling with two marginals

- noncommutative rational identity testing
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Approximate versions of all these questions

Approximate tensor scalings:

Produce scaling of input X with marginals ε-close to input $\boldsymbol{p}$

Approximate quantum marginals:

Produce tensor with marginals ε-close to input $\boldsymbol{p}$

Approximate moment polytope membership:

Correctly output one of

$$\boldsymbol{p} \in \Delta(\mathbb{X}) + B(\varepsilon) \text{ or } \boldsymbol{p} \in \Delta(\mathbb{X})^c + B(\varepsilon).$$

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Approximate scaling algorithms: nonuniform vs uniform

	uniform	**nonuniform**
Matrix scaling	[LSW '98] poly $\log(1/\varepsilon)$	[LSW '98] poly $\log(1/\varepsilon)$
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Tensor scaling	[BGOWW '17] poly$(1/\varepsilon)$	this work: poly$(1/\varepsilon)$

One body quantum marginal problem (nonuniform)

• [Higuchi,Sudbery'02] qubits; [Klaychko'04] polytope, [WDGC13] algebraic algorithm, [BCMW'17]: membership in $\mathbf{NP} \cap \mathbf{coNP}$

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Our results

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Theorem (BFGOWW '18, Tensor scaling)

There is a randomized $\mathrm{poly}(n^d, \langle X\rangle + \langle \boldsymbol{p}\rangle, 1/\varepsilon)$*-time algorithm for approximate tensor scaling on input* $X, \boldsymbol{p}, \varepsilon$ *with success probability* $1/2$.

The algorithm requires $O\left(dn^{2d}\frac{\langle X\rangle+\langle \boldsymbol{p}\rangle+d\log dn}{\varepsilon^2}\right)$ iterations, each dominated by computing a Cholesky decomposition of some $n \times n$ matrix.

Corollary (BFGOWW '18, Quantum marginals)

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There is a randomized algorithm for approximate moment polytope membership running in time $\mathrm{poly}(n^d, \langle \textit{parameterization of } \mathbb{X}\rangle, \boldsymbol{p}, 1/\varepsilon)$

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Theorem (BFGOWW '18)

If the spectra of marginals of X are $\varepsilon = \exp(-O(dn^{d+1})\langle \boldsymbol{p} \rangle)$-close to $\boldsymbol{p}$, then $\boldsymbol{p} \in \Delta(\mathbb{X})$.

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Algorithm

Vague tensor scaling algorithm

Algorithm

Input: $X, \boldsymbol{p}$ with integer coordinates, ε.

Output: scaling of X s.t. spectra of marginals ε-close to $\boldsymbol{p}$, or OUTSIDE POLYTOPE

- Repeat T times:
  - If done, output X.
  - Else, change basis in a single vector space to FIX the worst marginal of X.
    (*ignoring damage done to other marginals!*)
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Not this simple.

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Actual tensor scaling algorithm

Algorithm

Input: $X, \boldsymbol{p}$ with integer coordinates, ε.

Output: scaling Y s.t. spectra of marginals ε-close to $\boldsymbol{p}$, or OUTSIDE POLYTOPE

- **Choose** g_0 **randomly, set** $Y = g_0 \cdot X$**.**
- Repeat T times:
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Theorem

For $T \geq O\left(dn^{2d}\frac{\langle Y\rangle + \langle \boldsymbol{p}\rangle + d\log dn}{\varepsilon^2}\right)$, this is algorithm succeeds with probability at least $1/2$.

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Easy to fix an individual marginal;

Solve simple matrix factorization problem for g_i

$$g_i \rho^{(i)} g_i^\dagger = \mathrm{diag}(\boldsymbol{p}_i)$$

WARNING: not every choice works.

The correct way is to fix the marginal using only lower triangular scalings g_i

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Typical analysis

- Define potential function f with $f(g_0) = 1$
- Show f decreases by $\Omega(\varepsilon^2)$ in each iteration of the algorithm if marginals off by at least ε
- Establish lower bound for f

However, triangular scaling algorithm can fail without randomization!
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Lower bounds: highest weights

Our lower bounds are objects arising from representation theory

Definition (Highest weights of type $\boldsymbol{p}$)

Homogeneous polynomials on $((\mathbb{C}^n)^{\otimes d})$ that are eigenfunctions for the action of the lower triangular matrices;

$$P(g \cdot Y) = scalar(g, \boldsymbol{p}) P(Y)$$

$scalar(g, \boldsymbol{p})$ is just the *eigenvalue*.

$$\textbf{Potential function: } f_Y(g) = \log \frac{\|g \cdot Y\|}{scalar(g, \boldsymbol{p})}$$

There's a fairly easy *weak duality* here: very roughly,

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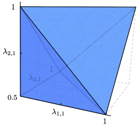
Last ingredient: moment polytope and highest weights

$\boldsymbol{p} \in \Delta(\overline{G \cdot X})$



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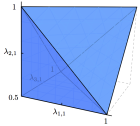

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"*potential*"

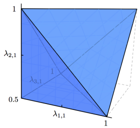
$$\inf_g \; f_Y(g) > \frac{1}{\text{poly}}$$

Ness–Mumford'84
+*Derksen*'01
+*Lemma*
for random Y

$$\sup_{P \text{ hwv}} P(Y) > \frac{1}{\text{poly}}$$

Last ingredient: moment polytope and highest weights

$\boldsymbol{p} \in \Delta(\overline{G \cdot X})$



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$\Uparrow$ weak duality

Open problems

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- Is the tensor scaling decision problem in **NP**? Is it in **coNP**?
- Is it in **RP**? A $\mathrm{poly}\log(1/\varepsilon)$ algorithm would prove it! In **P**?
- Can tensor scaling be done in $\mathrm{poly}\log(1/\varepsilon)$ for a random tensor? Would put quantum marginal problem in **RP**!
- Approximately scale for other group actions, without alternating minimization (in progress).
- Obtain similar algorithms for *multi-body* quantum marginals.
- Develop separation oracles for moment polytopes.

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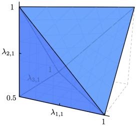
Thank you!

Moment polytope and representation theory

$\boldsymbol{p} \in \Delta(\overline{G \cdot X})$



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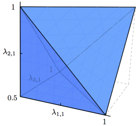

Want for

$$Y = \underbrace{g_0}_{\textit{random}} \cdot X$$

$$\inf_{g} \overbrace{f_{Y,\boldsymbol{p}}(g)}^{\text{"potential"}} > \frac{1}{\text{poly}}$$

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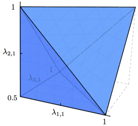

$$\xLeftrightarrow{\substack{\textit{Ness-Mumford}\text{`84} \\ +\textit{Derksen}\text{`01}}}$$

$$Q \not\equiv 0 \text{ on } \overline{G \cdot X}, \deg Q \leq 2^{\text{poly}}$$
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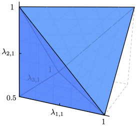

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$\Big\Downarrow$ *probability* $> .5$

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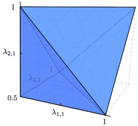

Ness–Mumford'84 + *Derksen*'01

$\Longleftrightarrow$

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$\Big\Downarrow$ *Lemma*

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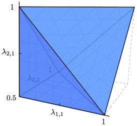

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$$\xLeftarrow[\textit{weak duality}]{}$$

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