Operator Scaling with Specified Marginals

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Abstract

The completely positive operators, which can be viewed as a generalization of the nonnegative matrices, are maps between spaces of linear operators arising in the study of C*-algebras. The existence of the operator analogues of doubly stochastic scalings of matrices is equivalent to a multitude of problems in computer science and mathematics, such rational identity testing in non-commuting variables, noncommutative rank of symbolic matrices, and a basic problem in invariant theory (Garg, Gurvits, Oliveira and Wigderson, *FOCS*, 2016).

We study *operator scaling with specified marginals*, which is the operator analogue of scaling matrices to specified row and column sums (or marginals). We characterize the operators which can be scaled to given marginals, much in the spirit of the Gurvits' algorithmic characterization of the operators that can be scaled to doubly stochastic (Gurvits, *Journal of Computer and System Sciences*, 2004). Our algorithm produces approximate scalings in time $\mathrm{poly}(n, m)$ whenever scalings exist. A central ingredient in our analysis is a reduction from the specified marginals setting to the doubly stochastic setting.

Operator scaling with specified marginals arises in diverse areas of study such as the Brascamp-Lieb inequalities, communication complexity, eigenvalues of sums of Hermitian matrices, and quantum information theory. Some of the known theorems in these areas, several of which had no effective proof, are straightforward consequences of our characterization theorem. For instance, we obtain a simple algorithm to find, when they exist, a tuple of Hermitian matrices with given spectra whose sum has a given spectrum. We also prove new theorems such as a generalization of Forster's theorem (Forster, *Journal of Computer and System Sciences*, 2002) concerning radial isotropic position.

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1 Introduction

Completely positive maps are linear maps between spaces of linear operators that, informally speaking, preserve positive-semidefiniteness in a strong sense. Completely positive maps generalize nonnegative matrices in some sense and arise naturally in quantum information theory and the study of C*-algebras [**J13**]. If Y is a complex inner product space, let $L(Y)$ denote the space of Hermitian operators on Y. To each completely positive map $T : L(V) \to L(W)$ is associated another completely positive operator $T^* : L(W) \to L(V)$ known as the *dual* of T. In analogy with the matrix case, say a completely positive map is *doubly stochastic* if $T(I) = I$ and $T^*(I) = I$. A *scaling* T' of a completely positive map T by a pair of invertible linear maps (g, h) is the completely positive map $T' : X \mapsto g^\dagger T(hXh^\dagger)g$. One is led to ask which completely positive maps have doubly stochastic scalings; *operator scaling* is the study of this question. In fact, several other problems such as rational identity testing in non-commuting variables, membership in the null-cone of the left-right action of $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_m(\mathbb{C})$ [**GGOW16**], and a special case of Edmonds' problem [**Gu04**] each reduce to (or are equivalent to) an approximate version of this question. In [**Gu04**], Gurvits gave two useful equivalent conditions for approximate scalability: a completely positive map T can be approximately scaled to doubly stochastic if and only if T is *rank-nondecreasing*, i.e. $\operatorname{rk} T(X) \geq \operatorname{rk} X$ for all $X \succeq 0$, or equivalently $\operatorname{cap} T > 0$ where

$$\operatorname{cap} T := \inf_{X \succeq 0, \det X = 1} \det T(X).$$

Gurvits also gave an algorithm to compute approximate scalings if either of these equivalent conditions hold. The authors of [**GGOW16**, **GGOWbl16**, **Gu04**] analyzed the same algorithm to obtain polynomial-time decision algorithms for each of the aforementioned problems.

We consider a natural generalization of doubly stochastic scalings. Say T *maps* $(A \to B, C \to D)$ if $T(A) = B$ and $T^*(C) = D$ and say $\hat{T}$ *is an* $(A \to B, C \to D)$*-scaling of* T if $\hat{T}$ is a scaling of T that maps $(A \to B, C \to D)$.

Question 1. Given positive semidefinite matrices $A, D \in L(V)$ and $B, C \in L(W)$ and a completely positive map $T : L(V) \to L(W)$, does T have an $(A \to B, C \to D)$-scaling?

We extend Gurvits' characterization of approximate scalability to the setting of Question 1. As in [**Gu04**], our existence proofs lead to algorithms that efficiently produce approximate scalings when they exist. Theorem 3.8, which closely resembles the characterization in [**Gu04**], characterizes the existence of approximate $(A \to B, C \to D)$-scalings by block-upper-triangular matrices. Theorem 3.9 extends this characterization to handle scalings in the full general-linear group with a somewhat surprising outcome - informally, a completely positive map T has approximate $(A \to B, C \to D)$-scalings if and only if a suitable random scaling of T satisfies the conditions of Theorem 3.8 with high probability. We also give an exponential time algorithm to decide if T can be scaled to map $(A \to B, C \to D)$ with arbirarily small error.

A close variant of Question 1 first appeared in [GP15], in which the authors propose $(P \to Q, I_W \to I_V)$-scalings as quantum analogues of Markov chains satisfying certain relative entropy minimality conditions. The authors of [GP15] conjectured a partial answer to Question 1, which was confirmed in [Fr16]. Our Theorem 10.20 extends the answer of [Fr16], and prove the conjecture of [GP15] apart from one small caveat.

This paper is organized as follows: in Section 2, we describe several questions that can be reduced to Question 1 and for which our results yield a number of new characterizations and algorithms. In Section 3, after providing the necessary background, we state our main results, Theorems 3.8 and 3.9. We prove Theorem 3.8 in Sections 4 through 7 and Theorem 3.9 in Section 8. In Section 9 we describe a sufficient condition called (P, Q)-indecomposability that guarantees the existence of exact scalings. Finally in Section 10 bring Theorems 3.8 and 3.9 to bear on the questions from Section 2.

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2 Applications and special cases

Here we mention a few questions that can be answered by via reduction to Question 1.

Question 2 (Matrix scaling)**.** Given a nonnegative matrix $A \in \mathrm{Mat}_{m,n}(\mathbb{R})$ and nonnegative row- and column-sum vectors $r \in \mathbb{R}^m_{\geq 0}$ and $c \in \mathbb{R}^n_{\geq 0}$, do there exist diagonal matrices X, Y such that the row (resp. column) sums of $A' = XAY$ are r (resp. c)?

It is well-known that matrix scaling can be reduced to an instance of operator scaling with specified marginals, but Gurvits' characterization does not apply to this instance unless r and c are the all-ones vectors. In 10.1, we recall the reduction from Question 2 to Question 1 and derive the classic theorem of [RS89] on the existence of such scalings as a consequence of Theorem 3.8.

Question 3 (Eigenvalues of sums of Hermitian matrices)**.** Given nonincreasing sequences α, β, γ of m real numbers, are α, β, γ the spectra of some $m \times m$ Hermitian matrices A, B, C satisfying $A + B = C$?

In [Kl98], Klyachko showed (amazingly) that the answer to Question 3 is "yes" if and only if α, β, γ satisfy a certain finite set S_m of linear inequality constraints. That is, such (α, β, γ) form a polyhedral cone. A long line of work has been devoted to describing the set S_m, which has connections to representation theory, Schubert calculus, and combinatorics [KT00, Kl98, F00]. There are even polynomial-time algorithms to test if α, β, γ satisfy S_m [MNS12]. However, no previous work has provided an algorithm to find the Hermitian matrices in question. Our reduction, which can be found in 10.2, yields an algorithmic proof of the result in [Kl98]. That is, we exhibit an algorithm that outputs a sequence of Hermitian matrices (in particular, real symmetric matrices!) $A_n + B_n = C_n$ with spectra approaching α, β, γ if α, β, γ satisfy S_m.

Question 4 (Forster's scalings)**.** Given vectors $u_1, \ldots, u_n \in \mathbb{C}^m$, nonnegative numbers $p_1, \ldots, p_n$, and a positive-semidefinite matrix Q, when does there exist an invertible linear transformation

$B : \mathbb{C}^m \to \mathbb{C}^m$ such that

$$\sum_{i=1}^{n} p_i \frac{Bu_i(Bu_i)^\dagger}{\|Bu_i\|^2} = Q?$$

Barthe [B98] answered this question completely for the case $Q = I_m$. Forster independently answered Question 4 in the positive for $p_i = 1$, u_i in general position, and $Q = \frac{n}{m} I_m$; as a consequence he was able to prove previously unattainable lower bounds in communication complexity [Fo02]. As noted in [Gu04], Forster's result is a consequence of Gurvits' characterization of doubly stochastic scalings. In 10.3 we reduce the general case of Question 4 to an instance of Question 1, and use this reduction to answer the approximate version of Question 4. For fixed $u_1, \ldots, u_n$ and Q, the admissible $(p_1, \ldots, p_n)$ form a convex polytope whose form is a natural generalization of the polytope, known as the *basis polytope*, described in [B98]. In fact, one can derive the Schur-Horn theorem on diagonals of Hermitian matrices with given spectra [H54] from our answer to Question 4.

Lastly, we hope our techniques will be of use in quantum information theory. The completely positive maps T that map $(I_V \to Q, I_W \to P)$ have a meaningful interpretation: by a fact known as *channel-state duality* [Ja74], to each completely positive map $T : L(V) \to L(W)$ is associated a unique unnormalized mixed bipartite quantum state $\rho \in L(V \otimes W)$. The operator T maps $(I_V \to Q, I_W \to P)$ if and only if $\operatorname{Tr}_V \rho = Q$ and $\operatorname{Tr}_W \rho = \overline{P}$. That is, the *local* mixed states induced by ρ are $T(I)$ and $\overline{T^*(I)}$. Operator scaling has established connections between separable quantum states and matroid theory [Gu04], so perhaps our techniques can shed further light on such relationships. We discuss this further in Section 11.

3 Preliminaries and main theorems

Before presenting the main theorems we fill in some background and justify a few assumptions we will make throughout the paper. The notation established in 3.1 and 3.2 will be summarised in 3.6.

3.1 Preliminaries

Definition 3.1 (Completely positive maps)**.** A completely positive map is a map $T : L(V) \to L(W)$ of the form

$$T : X \mapsto \sum_{i=1}^{r} A_i X A_i^\dagger,$$

where V and W are finite dimensional complex inner product spaces and $A_i : V \to W$ are linear maps called *Kraus operators* of T. Note that T preserves positive-semidefiniteness. The map $T^* : L(W) \to L(V)$ is given by

$$T^* : X \mapsto \sum_{i=1}^{r} A_i^\dagger X A_i,$$

and is the adjoint of T in the trace inner product. Recall that we say T *maps* $(A \to B, C \to D)$ if $T(A) = B$ and $T^*(C) = D$.

Definition 3.2 (Scalings of completely positive maps)**.** If $T : L(V) \to L(W)$ is a completely positive map, $g \in \mathrm{GL}(W)$ and $h \in \mathrm{GL}(V)$, we define the completely positive map $T_{g,h}$ by

$$T_{g,h} : X \mapsto g^\dagger T(hXh^\dagger) g.$$

Observe that

$$(T_{g,h})^* = T^*_{h,g}.$$

$T_{g,h}$ is called the scaling of T by (g,h).

Here G (resp. H) will be a subset of $\mathrm{GL}(W)$ (resp. $\mathrm{GL}(V)$), and often a subgroup.

Definition 3.3 (Approximate scalings)**.** Say a scaling T' of T is an ϵ-$(A \to B, C \to D)$-*scaling* if T' maps $(A \to B', C \to D')$ with $\|B - B'\| \leq \epsilon$ and $\|D - D'\| \leq \epsilon$. If $G \subset \mathrm{GL}(W)$ and $H \subset \mathrm{GL}(V)$, say T is *approximately* (G,H)*-scalable to* $(A \to B, C \to D)$ if for all $\epsilon > 0$, T has an ϵ-$(A \to B, C \to D)$-scaling T' by $(g,h) \in (G,H)$.

If A and C are invertible, approximate (resp. exact) scalability to $(A \to B, C \to D)$ is equivalent to approximate (resp. exact) scalability to $(I_V \to Q, I_W \to P)$ for $Q = A^{1/2}DA^{1/2}$ and $P = C^{1/2}BC^{1/2}$, so we mainly restrict attention to $(I_V \to Q, I_W \to P)$-scalings.

It will be handy to be able to easily move back and forth between $(I_V \to Q, I_W \to P)$-scalings and $(P \to I_W, Q \to I_V)$-scalings. The following easy lemma, which we prove in Appendix 12.2, gives us this freedom.

Lemma 3.1. *Suppose $P \in L(V)$ and $Q \in L(W)$ are positive-definite,* $\operatorname{Tr} P = \operatorname{Tr} Q = 1$*, and that (G,H,P,Q,T) is block-diagonal. The following are equivalent:*

1. *T is approximately (resp) exactly scalable to $(P \to I_V, Q \to I_V)$ by $(G_{E_\circ}, H_{F_\circ})$.*
2. *T is approximately (resp) exactly scalable to $(I_V \to I_W, Q \to P)$ by $(G_{E_\circ}, H_{F_\circ})$.*
3. *T is approximately (resp) exactly scalable to $(P \to Q, I_W \to I_V)$ by $(G_{E_\circ}, H_{F_\circ})$.*
4. *T is approximately (resp) exactly scalable to $(I_V \to Q, I_W \to P)$ by $(G_{E_\circ}, H_{F_\circ})$.*

Moreover, if T has an ϵ-$(P \to I_V, Q \to I_V)$-scaling by $(G_{E_\circ}, H_{F_\circ})$ then T has ϵ-$(I_V \to I_W, Q \to P)$, ϵ-$(I_V \to Q, I_W \to P)$, and ϵ-$(P \to Q, I_W \to I_V)$-scalings by $(G_{E_\circ}, H_{F_\circ})$.

Henceforward P (resp. Q) denotes a positive-semidefinite operator in $L(V)$ (resp. $L(W)$). We further assume $\operatorname{Tr} P = \operatorname{Tr} Q$ because $\operatorname{Tr} T(I_V) = \operatorname{Tr} T^*(I_W)$.

3.1.1 Flags

We will think of positive-semidefinite operators in terms of their spectrum and an associated sequence of subspaces called a *flag*.

Definition 3.4 (Flags)**.**

1. If V is an n-dimensional vector space, a *flag* $F_\circ$ on V is a sequence of subspaces

$$0 \subset F_{i_1} \subsetneq \cdots \subsetneq F_{i_k} \subset V$$

where $\dim i_k = i_k$.

2. The *signature* of $F_\circ$, denoted $\sigma(F_\circ)$, is the set $\{i_1, \ldots, i_k\}$ of dimensions appearing in the flag. Say a flag $F_\circ$ is *complete* if it has signature $\{0, \ldots, n\}$; else $F_\circ$ is *partial*.

3. The *standard flag* in an orthonormal basis $F = (f_1, \ldots f_n)$ of V is the complete flag

$$F_\bullet = (\{0\}, \langle f_1 \rangle, \ldots, \langle f_1, \ldots, f_{n-1} \rangle, V).$$

4. Conversely, a complete flag is the standard flag in a unique orthonormal basis up to multiplication of each basis vector by a complex number of modulus 1. In general, if $F_\circ$ is a flag, say $(f_1, \ldots, f_n)$ is an *adapted basis* for $F_\circ$ if F is orthonormal and $F_i = \langle f_1, \ldots f_i \rangle$ for $i \in \sigma(F_\circ)$. That is, $F_\circ$ is a subflag of the standard flag in F.

5. If $H \subset End(V)$ is a set of linear transformations of V, $H_{F_\circ}$ denote the set

$$\{h : hF_i \subset F_i \text{ for all } i \in \sigma(F_\circ)\}.$$

When H is a subgroup of $\mathrm{GL}(V)$, $H_{F_\circ}$ is the stabilizer subgroup of $F_\circ$ under the action of H.

Definition 3.5 (Block-upper-triangular scalings)**.** If $F_\circ$ is flag on V and h is a linear operator on V, we say h is *block-upper-triangular* (with respect to $F_\circ$) if

$$hF_i \subset F_i \text{ for all } i \in \sigma(F_\circ).$$

If $H \subset End(V)$ is a set of linear transformations of V, let

$$H_{F_\circ} := \{h \in H : h \text{ is block-upper-triangular w.r.t } F_\circ\}.$$

When H is a subgroup of $\mathrm{GL}(V)$, $H_{F_\circ}$ is the stabilizer subgroup of $F_\circ$ under the action of H.

Note that a linear transformation h is block-upper-triangular if and only if the matrix for h is block-upper-triangular with block-sizes $i_1, i_2 - i_1, \ldots, i_k - i_{k-1}, n - i_k$ in an adapted basis for $F_\circ$.

Next we discuss how to view Hermitian operators in terms of their spectra and an associated flag. $L \subset V$ is a subspace, let π_L denote the orthogonal projection to L. Observe that if $F_\circ$ is a flag and $(c_i : i \in [n])$ a sequence of positive numbers, then

$$\sum_{i \in \sigma(F_\circ)} c_i \pi_{F_i}$$

is a positive-semidefinite operator in $L(V)$.

In fact, *every* positive semidefinite operator has a unique representation of this form; this can be seen by taking the sequence F_i to be the sequence of sums of eigenspaces of A in decreasing order of eigenvalue. More precisely:

Fact 3.2 (See the survey **F00**, e.g.)**.** Let $A \in L(V)$ be positive-semidefinite. Let $\lambda(A) = (\alpha_1, \ldots, \alpha_n)$ denote the spectrum of A. Then there is a unique shortest flag, denoted $F_\circ(A)$, such that there exist $c_1 \geq 0, \ldots c_n \geq 0$ satisfying

$$\sum_{i \in \sigma(F_\circ(A))} c_i \pi_{F_i(A)} = A.$$

Further, we have $\sigma(F_\circ(A)) = \{i \in [n] : \alpha_i - \alpha_{i+1} > 0\}$ and $c_i = \alpha_i - \alpha_{i+1}$ for $i \in \sigma(F_\circ(A))$ where $\alpha_{n+1} := 0$.

Note that for any flag $F_\circ$ such that $A = \sum_{i \in \sigma(F_\circ)} c_i \pi_{F_i}$, we must have $\sigma(F_\circ) \supset \{i \in [n] : \alpha_i - \alpha_{i+1} > 0\}$. Thus, not all spectra and flags are compatible. We give a name to those flags that are compatible with given spectrum.

Definition 3.6 (α-partial flag)**.** If $(\alpha_1, \ldots, \alpha_n)$ is a non-increasing sequence of nonnegative numbers, say $F_\circ$ is an *α-partial flag* if

$$\sigma(F_\circ) \supset \{i \in [n] : \alpha_i - \alpha_{i+1} > 0\}.$$

It will be useful to have some shorthand for the difference sequence $\alpha_i - \alpha_{i+1}$.

Definition 3.7 (Difference sequences)**.** If $\alpha = (\alpha_1, \dots, \alpha_n)$ is a sequence, define $\Delta\alpha$ to be the sequence

$$\Delta\alpha_i = \alpha_i - \alpha_{i+1}.$$

Here $\alpha_{n+1} := 0$. We define

$$\sigma(\alpha) = \{i : \Delta\alpha_i \neq 0\}.$$

Note that $\sigma(\alpha) \subset \sigma(F_\circ)$ if and only if $F_\circ$ is an α-partial flag.

Definition 3.8 (Flag notation convention)**.**

1. $E_\circ$ will denote the flag $F_\circ(Q)$ and $F_\circ$ will denote the flag $F_\circ(P)$.

2. E and F will denote adapted bases for $E_\circ$ and $F_\circ$, respectively.

Note that P (resp. Q) is diagonal with nonincreasing diagonal in basis F (resp. E).

Definition 3.9 (Projections to flags)**.** For $j \in \sigma(F_\circ)$, let $\eta_j : V \to F_j$ be a partial isometry. That is, $\eta_j^\dagger \eta_j$ is π_{F_j}, the orthogonal projection to F_j. In the basis F (and basis $f_1, \dots f_j$ for F_j) we have

$$\eta_j = \begin{array}{cc} n & \\ \begin{bmatrix} 1 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & 0 & \dots & 0 \end{bmatrix} & j. \end{array}$$

Let $\nu_i : W \to E_i$ for $i \in \sigma(E_\circ)$ be the analogous partial isometries.

3.1.2 Restrictions on scalings

We must impose some restrictions on T, G, H, P, Q in order for our methods to work. Luckily, this level of generality suffices for all the applications known to the author. In particular, these restrictions will never rule out the case $G = \mathrm{GL}(W)$ and $H = \mathrm{GL}(V)$, so any reader only interested in $(\mathrm{GL}(W), \mathrm{GL}(V))$-scalings can safely skim this subsection.

Our characterization can apply in a more general setting than the one discussed here. For the sake of simplicity, we describe this more general setting in Remark 8.12 after presenting our main theorems and algorithms.

Our groups will take the form

$$G = \bigoplus_i \mathrm{GL}(W_i) \text{ and } H = \bigoplus_i \mathrm{GL}(V_i) \tag{1}$$

where $V = \bigoplus_i V_i$ and $W = \bigoplus_i W_i$ (here the direct sums are assumed to be orthogonal direct sums).

For our proof techniques to work, we must assume T respects the decompositions $V = \bigoplus_i V_i$ and $W = \bigoplus_i W_i$. That is, we require

$$T \bigoplus_i L(V_i) \subset \bigoplus_j L(W_j) \tag{2}$$

$$T^* \bigoplus_j L(W_j) \subset \bigoplus_i L(V_i). \tag{3}$$

If 2 and 3 hold, say T is *compatible with G and H*. Note that if T is compatible with G and H, then $T_{g,h}(I_V) = \oplus_i A_i$ and $T^*_{h,g}(I_W) = \oplus_j B_j$ where $A_i \in L(V_i)$, $B_j \in L(W_j)$ are positive-semidefinite operators. Say an operator $B = \oplus_j B_j$ of this form is *compatible with* G, and analogously for A and H. Observe that compatibility of $B \in L(W)$ with G depends only on $F_\circ(B)$. Further, B and G are compatible if and only if

$$F_j(B) = \bigoplus_i F_j(B) \cap W_i.$$

For this reason we say a flag $D_\circ$ is *compatible* with G if $D_j = \bigoplus_i D_j \cap W_i$ for all $j \in \sigma(D_\circ)$; we define compatibility with H analogously. Since we are interested in $(I_V \to Q, I_W \to P)$-scalability, we may assume $F_\circ(Q)$ and $F_\circ(P)$ are compatible with G and H, respectively, or else any T that is compatible with G and H is clearly neither exactly nor approximately (G,H)-scalable to $(I_V \to Q, I_W \to P)$. We summarise our assumptions in the next definition.

Definition 3.10 (Block-diagonal)**.** If there exist decompositions $V = \bigoplus_i V_i$ and $W = \bigoplus_i W_i$ such that

1. G and H satisfy 1,
2. T is compatible with G and H, and
3. $E_\circ$ and $F_\circ$ are compatible with G and H, respectively,

say $(G, H, F_\circ, E_\circ, T)$ is *block-diagonal*. For convenience, say (G, H, P, Q, T) *is block-diagonal* if $(G, H, F_\circ(P), F_\circ(Q), T)$ is block-diagonal.

Example 1. The tuple $(\mathrm{GL}(W), \mathrm{GL}(V), P, Q, T)$ is always block-diagonal.

Example 2. If Kraus operators $A_l : l \in [r]$ of T satisfy

$$A_l V \subset \{0\} \dots \{0\} \oplus W_{i(l)} \oplus \{0\} \dots \tag{4}$$

$$\text{and } A_l^\dagger W \subset \{0\} \dots \{0\} \oplus V_{j(l)} \oplus \{0\} \dots \tag{5}$$

for some $i(l)$ and $j(l)$, then T is compatible with G and H.

As we will see in Section 10, the Kraus operators of the completely positive maps arising in examples 2, 3, and 4 satisfy the containments 4 and 5.

Observation 3.3. *If (G, H, P, Q, T) is block-diagonal, then approximate or exact (G, H)-scalability of T to $(I_V \to Q, I_W \to P)$ depends only on the spectra of P and Q.*

Proof. All P and Q with fixed spectra such that (G, H, P, Q, T) is block-diagonal are conjugate by unitaries in H and G, respectively. However, for any unitaries $U \in G$ and $O \in H$, the change of variables by the transformation $g \mapsto gU, h \mapsto hO$ shows approximate (resp. exact) (G, H)-scalability to $(I_V \to Q, I_W \to P)$ is equivalent to approximate (resp. exact) (G, H)-scalability to $(I_V \to U^\dagger Q U, I_W \to O^\dagger P O)$. □

3.1.3 Extensions of Gurvits' conditions

We remind the reader of Gurvits' theorem characterizing scalability of completely positive maps to doubly stochastic.

Theorem 3.4 (Gu04)**.** *Suppose $T : L(V) \to L(V)$ is a completely positive map. The following are equivalent:*

1. $0 < \operatorname{cap}(T) := \inf_{X \succ 0} \frac{\det T(X)}{\det X}$.

2. T *is rank-nondecreasing, that is, for all* $X \succeq 0$, $\operatorname{rk} T(X) \geq \operatorname{rk} X$.

3. T *is approximately* $(\mathrm{GL}(V), \mathrm{GL}(V))$*-scalable to* $(I_V \to I_V, I_V \to I_V)$.

In order to state our main theorems, we'll need extensions of rank-nondecreasingness and capacity. To define our notion of rank-nondecreasingness, we'll define a polytope depending on $T, E_\circ, F_\circ$, and then define T, p, q to have the rank-nondecreasingness property if (p, q) is in the polytope defined by $T, E_\circ(Q), F_\circ(P)$.

Definition 3.11 (rank-nondecreasingness for specified marginals)**.** Suppose $E_\circ, F_\circ$ are given partial flags.

1. We say a pair of subspaces $(L \subset W, R \subset V)$ is T*-independent* if $L \subset (A_i R)^\perp$ for all $i \in [r]$.

2. Define $\mathcal{K}(T, E_\circ, F_\circ) \subset \mathbb{R}^{m+n}$ to be the the set of (p, q) satisfying $\sigma(p) \subset \sigma(F_\circ)$, $\sigma(q) \subset \sigma(E_\circ)$, $\sum_{i=1}^m q_i = \sum_{j=1}^n p_j := N$ and

$$\text{and} \sum_{i \in \sigma(E_\circ)} \Delta q_i \dim E_i \cap L + \sum_{j \in \sigma(F_\circ)} \Delta p_j \dim F_j \cap R \leq N. \tag{6}$$

for all T-independent pairs (L, R). Because the coefficients of the Δq_i in the above sum can take on only a finitely many values, $\mathcal{K}(T, E_\circ, F_\circ)$ is a convex polytope.

3. Say T is (P, Q)-rank-nondecreasing if $(p, q) \in \mathcal{K}(T, F_\circ(Q), F_\circ(P))$.

This definition extends the definition of rank-nondecreasingness. Rank-nondecreasingness is usually, and equivalently, defined by the nonexistence of a *shrunk subspace*, or a subspace $L \subset V$ such that $\dim \sum A_i L < \dim L$. Since

$$\left(L, \left(\sum A_i L\right)^\perp\right)$$

is a T-independent pair and all other T-independent pairs (L, R) have $R \subset (\sum A_i L)^\perp$, there is no shrunk subspace if and only if $\dim L + \dim R \leq n = \operatorname{Tr} I_V$ for all T-independent pairs (L, R). That is, T is rank-nondecreasing if and only if T is (I_V, I_V)-rank-nondecreasing, because $F_\circ(I_V) = (\{0\}, V)$.

Remark 3.5. (P, Q)-rank-nondecreasingness does not depend on the particular choice of Kraus operators for T, because T-independence of (L, R) does not depend on the choice of Kraus operators for T. This is because (L, R) is T-independent if and only if $T(\pi_L)\pi_R = 0$, where π_Z denotes the orthogonal projection to the subspace Z.

We will need a variant of the determinant that depends on additional argument which is a positive semidefinite operator.

Definition 3.12 (Relative determinant)**.** Define the *determinant of* X *relative to* P, denoted $\det(P, X)$, by

$$\det(P, X) = \prod_{j \in \sigma(F_\circ(P))} (\det \eta_j X \eta_j^\dagger)^{\Delta p_j}. \tag{7}$$

Of course, $\det(A, X)$ can be defined analogously for any positive-semidefinite operator A.

The relative determinant inherits a few of the multiplicative properties of determinant when restricted to block-upper-triangular matrices.

Lemma 3.6 (Properties of $\det(P, X)$)**.** *If $h \in \mathrm{GL}(V)_{F_\circ(P)}$, then*

$$\det(P, Xh) = \det(P, X)\det(P, h), \tag{8}$$

$$\det(P, h^\dagger X h) = \det(P, h^\dagger h)\det(P, X), \tag{9}$$

$$\text{and } \det(P, h^{-\dagger}h^{-1}) = \det(P, h^\dagger h)^{-1}. \tag{10}$$

We defer the (easy) proofs to Appendix 12.2.

Remark 3.7. Observe that

$$\log\det(P, P) = \operatorname{Tr} P \log P.$$

Equivalently, $-\log\det(P, P)$ is the *von Neumann entropy* of P, denoted $S(P)$, which is equal to the Shannon entropy $H(p)$ of the spectrum of P. One can also draw some parallels with the quantum relative entropy. By the log-concavity of the determinant, $\log\det(P, X)$ is concave in X. Further, we will see that for fixed nonsingular P, $\det(P, X)$ is maximized at $X = I$ subject to $X \succeq 0$ and $\operatorname{Tr} XP = 1$. Thus, it can be intuitively helpful to think of $-\log\det(P, X)$ as a cross-entropy of P and X, and

$$-\log\det(P, XP^{-1})$$

as a relative entropy of P with respect to X, though it is *not* equal to the Von-Neumann relative entropy.

Finally we come to an extension of Gurvits' capacity.

Definition 3.13 (Capacity for specified marginals)**.** Here we take $0^0 = 1$. Recall from Definition 3.9 that $\eta_i : V \to F_i$, $\nu_j : W \to E_j$ are partial isometries. Define

$$\operatorname{cap}(T, P, Q) = \inf_{h \in \mathrm{GL}(V)_{F_\circ}} \frac{\det(Q, T(hPh^\dagger))}{\det(P, h^\dagger h)}, \tag{11}$$

If p-partial flags and q-partial flags $E_\circ$ and $F_\circ$ are given, then $\operatorname{cap}(T, p, q)$ refers to the quantity $\operatorname{cap}(T, P, Q)$ where P and Q are the unique operators with $\lambda(P) = p$, $\lambda(Q) = q$ and $F_\circ(P) \subset F_\circ$, $F_\circ(Q) \subset E_\circ$.

Note that $\det(X, I_V) = \det X$ and $\det h^\dagger h = \det hh^\dagger$. By the existence of Cholesky decompositions, $\{hh^\dagger : h \in \mathrm{GL}(V)_{F_\circ}\} = \{X : X \succ 0\}$. This implies $\operatorname{cap}(T, I_V, I_V) = \operatorname{cap} T$ for $V = W$, so $\operatorname{cap}(T, P, Q)$ is an extension of the usual capacity.

3.2 Main theorems

We are ready to state our analogue of Gurvits' characterization for block-upper-triangular scalings. Gurvits' characterization is the special case $V = W$ and $P = Q = I_V$ of the following theorem. Recall that if $F_\circ$ is a flag on V and G is a subgroup of $\mathrm{GL}(V)$, then $G_{F_\circ}$ is the subgroup of G fixing each subspace in $F_\circ$.

Theorem 3.8. *Suppose $T : L(V) \to L(W)$ is a completely positive map and $P \in L(V), Q \in L(W)$ are positive-semidefinite. The following are equivalent:*

1. T is (P,Q)-rank-nondecreasing.

2. $\operatorname{cap}(T,P,Q) > 0$.

3. T is approximately $(G_{F_\circ(Q)}, H_{F_\circ(P)})$-scalable to $(I_V \to Q, I_W \to P)$ for all G and H such that (G,H,P,Q,T) is block-diagonal.

For T,P,Q fixed, the completely positive map $T_{g,h}$ is either (P,Q)-rank-nondecreasing for *no* $(g,h) \in G \times H$, or $T_{g,h}$ is (P,Q)-rank-nondecreasing for *generic* $(g,h) \in G \times H$. Here "generic (g,h)", a strengthening of "almost all (g,h)", means "all (g,h) not in the zero set of some fixed finite set of polynomials, none of which vanishes on $G \times H$". This allows us to extend our characterization from $(G_{F_\circ(Q)}, H_{F_\circ(P)})$-scalability to a characterization of (G,H)-scalability.

Theorem 3.9. *Suppose (G,H,P,Q,T) is block-diagonal. The following are equivalent:*

1. $T_{g,h}$ is (P,Q)-rank-nondecreasing for generic $(g^\dagger, h) \in G \times H$.

2. $\operatorname{cap}(T_{g,h}, P, Q) > 0$ for generic $(g^\dagger, h) \in G \times H$.

3. T is approximately (G,H)-scalable to $(I_V \to Q, I_W \to P)$.

Recall that for (G,H,P,Q,T) block-diagonal, the (G,H)-scalability of T to $(I_V \to Q, I_W \to P)$ depends only on the spectra of P and Q. In fact, the spectra for which approximate scaling can be done form a convex polytope.

Theorem 3.10. *The spectra (p,q) of pairs (P,Q) of positive-semidefinite operators such that*

1. (G,H,P,Q,T) is block-diagonal and

2. T is approximately (G,H)-scalable to $(I_V \to Q, I_W \to P)$

forms a convex polytope, which we denote $\mathcal{K}(T,G,H)$.

We also obtain algorithmic counterparts of Theorem 3.8 and Theorem 3.9. Let p_{min}, q_{min} denote the least nonzero eigenvalues of P and Q, and let b be the total bit-complexity of the input T,p,q where T is given by Kraus operators written down in bases F and E in which Q and P, respectively, are diagonal.

Theorem 3.11. *Suppose (G,H,P,Q,T) is block-diagonal. There is a deterministic algorithm of time-complexity $\operatorname{poly}(\epsilon^{-1}, p_{min}^{-1}, q_{min}^{-1}, n, m, b)$ that takes as input T,P,Q,ϵ and outputs $g \in G_{E_\circ}, h \in H_{F_\circ}$ such that $T_{g,h}$ is an ϵ-$(I_V \to Q, I_W \to P)$-scaling whenever $\operatorname{cap}(T,P,Q) > 0$ and ERROR otherwise.*

Theorem 3.12. *Suppose (G,H,P,Q,T) is block-diagonal. There is a randomized algorithm of time-complexity $\operatorname{poly}(\epsilon^{-1}, p_{min}^{-1}, q_{min}^{-1}, n, m, b)$ that takes as input T,P,Q,ϵ and outputs $g \in G, h \in H$ such that $T_{g,h}$ is an ϵ-$(I_V \to Q, I_W \to P)$-scaling with probability at least 2/3 whenever T is approximately (G,H)-scalable to $(I_V \to Q, I_W \to P)$ and ERROR otherwise.*

We can give only a randomized exponential time algorithm for the decision version of our problem. It would be interesting to find a polynomial time algorithm for this, as it would be a step towards finding a truly polynomial time algorithm for membership in the Kronecker polytope. Note that the only exponential dependence is on the bit complexity of the spectra p and q.

Theorem 3.13. *Suppose (G,H,P,Q,T) is block-diagonal. There is a randomized algorithm of time-complexity $\operatorname{poly}(\epsilon^{-1}, n, m, 2^b)$ to decide if $(p,q) \in \mathcal{K}(T,G,H)$. Equivalently, the algorithm decides if T is approximately (G,H)-scalable to $(I_V \to \operatorname{diag}(q), I_W \to \operatorname{diag}(p))$.*

3.3 Proof overviews and discussion

Theorem 3.8:

(1 $\implies$ 2): We prove (P, Q)-rank-nondecreasingness implies $\operatorname{cap}(T, P, Q) > 0$ in Section 6 using the reduction to the doubly stochastic case from Section 4 and some concavity properties of capacity.

(2 $\implies$ 3): We prove $\operatorname{cap}(T, P, Q) > 0$ implies T is $(G_{E_\circ}, H_{F_\circ})$-scalable to $(I_V \to Q, I_W \to P)$ in Section 5 by analyzing Sinkhorn scaling as in [**Gu04**], but with $\operatorname{cap}(T)$ replaced by $\operatorname{cap}(T, P, Q)$.

(3 $\implies$ 1): That $(G_{E_\circ}, H_{F_\circ})$-scalability of T to $(I_V \to Q, I_W \to P)$ implies T is (P, Q)-rank-nondecreasing is a direct linear algebra argument presented in Section 7.

Theorem 3.9:

Theorem 3.9 is proved in Section 8. The implications 1 $\implies$ 2 and 2 $\implies$ 3 follow immediately from Theorem 3.8. The only hard work left is the implication 3 $\implies$ 1. It is not hard to see (Corollary 7.2) that approximate (G, H)-scalability implies there exists $(g, h) \in G \times H$ such that $T_{g,h}$ is (P, Q)-rank-nondecreasing. Next, via an algebraic geometry argument (Lemma 8.1), we show the existence of *any* such (g, h) implies a *generic* (g, h) has $T_{g,h}$ is (P, Q)-rank-nondecreasing.

Theorem 3.10:

Theorem 3.10 appears as Corollary 8.9 in Section 8.2, but we give an overview of the proof here. For $F_\circ$ and $E_\circ$ fixed, it is clear that the set $\{(p, q) : T \text{ is } (P, Q)\text{-rank-nondecreasing}\}$, which we denote $\mathcal{K}(T, E_\circ, F_\circ)$, is a convex polytope since it is defined by a finite number of linear constraints.

It is not hard to see (Corollary 7.2 and Theorem 3.8) that T is approximately (G, H)-scalable to $(I_V \to Q, I_W \to P)$ if and only if $(p, q) \in \mathcal{K}(T_{g,h}, E_\circ, F_\circ)$ for some $(g, h) \in G \times H$. In other words, the obtainable pairs of spectra are

$$\bigcup_{g \in G, h \in H} \mathcal{K}(T_{g,h}, E_\circ, F_\circ).$$

This set is not obviously convex, but due to the results of Section 8, we find that for generic $(g, h) \in G \times H$,

$$\mathcal{K}(T_{g,h}, E_\circ, F_\circ) = \bigcup_{g' \in G, h' \in H} \mathcal{K}(T_{g',h'}, E_\circ, F_\circ).$$

This tells us that for some $(g, h) \in G \times H$, the obtainable spectra comprise the convex polytope $\mathcal{K}(T_{g,h}, E_\circ, F_\circ)$, proving Theorem 3.10.

We remark that Theorem 3.10 could likely be obtained by other methods involving the representation theory of Lie algebras (see [**CDW13**]), using which one might be able to show $K(T, G, H)$ is what is known as a *moment polytope*. We could not see how to obtain Theorems 3.8 and 3.9 from those methods, however.

Theorems 3.11 and 3.12:

Our proofs of Theorem 3.8 and Theorem 3.9 are just shy of effective. While the approximate scalings in Theorem 3.8 are produced by iterated scaling (see Algorithm TOSI of Section 5.1 and Algorithm GOSI of Section 8.3), a priori the bit-complexity of the scaled operators could grow exponentially. In Appendix 12.4 we obtain the efficient algorithms Algorithm 12.8 and Algorithm 12.10 by modifying the iterative scaling algorithms and rounding. The running times of the modified algorithms are described by Theorems 12.9 and Theorem 12.10.

The reader might wonder if analyzing the performance of Sinkhorn scaling for operators, which is called "Algorithm G" in [**GGOW16**] and "OSI" in [**Gu04**], on the reduction in Section 4 would be sufficient to obtain our algorithmic results. However, the reduction only works if P and Q have integral spectra and results in a completely positive map from $\operatorname{Mat}_{\operatorname{Tr} P}(\mathbb{C}) \to \operatorname{Mat}_{\operatorname{Tr} Q}(\mathbb{C})$. Thus, the dimension of the reduction depends on a common denominator for all the entries in the spectra of P and Q, rendering the algorithms inefficient. In fact, operator Sinkhorn scaling on the reduction amounts to Algorithm TOSI anyway, which is simpler to state without using the reduction.

Theorem 3.13

We will prove Theorem 3.13 as Corollary 12.13 and present the algorithm (Algorithm 12.12) in Section 12.4, but the proof is straightforward so we summarize it here.

Corollary 7.2 states that if if $T_{g,h}$ is an ϵ-$(I_V \to Q, I_W \to P)$-scaling for ϵ-smaller than $1/\operatorname{poly}(n,m)$ times the inverse of the least common denominator of the entries of p and q, then $T_{g,h}$ must be (P,Q)-rank-nondecreasing. In other words, $(p,q) \in \mathcal{K}(T_{g,h}, E_\circ, F_\circ)$. However, $\mathcal{K}(T_{g,h}, E_\circ, F_\circ) \subset K(T,G,H)$. From Theorem 3.12, we have a $\operatorname{poly}(\epsilon^{-1}, p_{min}^{-1}, q_{min}^{-1}, n, m, b)$-time algorithm (Algorithm 12.10) which outputs an ϵ-$(I_V \to Q, I_W \to P)$-scaling with probability at least $2/3$ whenever $(p,q) \in K(T,G,H)$ and ERROR otherwise. In particular, if Algorithm 12.10 never outputs other than ERROR and ϵ-scalings (even if it did, we could easily check if they were).

Thus, running Algorithm 12.10 with ϵ small enough (in particular we can take $\epsilon = 1/\operatorname{poly}(n,m)2^b$) and outputting NO if and only if the Algorithm 12.10 outputs ERROR is a $\operatorname{poly}(n,m,2^b)$-time decision problem for membership in $K(T,G,H)$.

3.4 Additional background

Our algorithms will rely on the Cholesky decomposition.

Fact 3.14 (Existence and uniqueness of Cholesky decompositions)**.** Suppose $F_\circ$ is a flag in an n-dimensional vector space V with $\sigma(F_\circ) = \{i_1, \dots, i_k\}$, and that F is an adapted basis for $F_\circ$. If A is positive semi-definite operator on V, then there exists an operator $B \in L(V)_{F_\circ}$ such that $B^\dagger B = A$ (that is, B is a block-upper-triangular matrix with block sizes $i_1, i_2 - i_1, \dots n - i_k$ in the basis F). If A is nonsingular, then the choice of B is unique up to left-multiplication by a unitary U that fixes the orthogonal complement of F_{i_l} in $F_{i_{l+1}}$ for $l \in \{0, \dots, k\}$ (here we define $F_{i_0} = \{0\}$ and $F_{i_{k+1}} = V$). In the basis F, U is block-diagonal with block sizes $i_1, i_2 - i_1, \dots n - i_k$. Further, if the entries of A in the orthonormal basis F are b-bit binary numbers, then $B \in \operatorname{Mat}_n(\mathbb{C})_{F_\circ}$ such that $\|B^\dagger B - A\|_2 \le \epsilon$ can be computed in time $\operatorname{poly}(-\log(\epsilon), b)$ [**Lo03**]. The uniqueness claim follows because a unitary fixing a subspace also fixes the orthogonal complement of the subspace.

We will need a few definitions from algebraic geometry.

Definition 3.14 (Generic)**.** An affine variety $\mathcal{V}$ in $\mathbb{C}^n$ is the set of common zeros of an ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$. The Hilbert basis theorem says every such ideal is finitely generated. A set is *Zariski-closed* if it is an affine variety. A set is *Zariski-open* if it is the complement of an affine variety. A set $S \subset \mathbb{C}^n$ is *generic* if S is nonempty and the complement $\overline{S}$ is contained in an affine variety. If $S \subset \mathbb{C}^n$ is nonempty and Zariski-open, we say a *generic element of S has property P* if the elements with property P form a generic set. See [**MilneAG**].

Definition 3.15 (More flag conventions)**.** The symbols $p, P, F_\circ, F_\bullet$ (resp. $q, Q, E_\circ, E_\bullet$) will be defined according to the following conventions. If P (resp. Q) is given, we assume P (resp. Q) is positive-semidefinite and define $F_\circ = F_\circ(P)$ and $p = \lambda(P)$ (resp, $E_\circ = F_\circ(Q)$ and $q = \lambda(Q)$). If, rather, a nonincreasing sequence $p = (p_i : i \in [n])$ (resp. q) and partial flag $F_\circ$ of V (resp. $E_\circ$ of W) are given, we assume $F_\circ$ is p-partial (resp. $E_\circ$ is q-partial), and P (resp. Q) always refers to the unique operator satisfying $F_\circ(P) \subset F_\circ$ and $\lambda(P) = p$ (resp. $F_\circ(Q) \subset F_\circ$ and $\lambda(Q) = q$.

$F_\bullet$ (resp. $E_\bullet$) will always be a complete flag extending $F_\circ$ or $F_\circ(P)$, depending on whether $F_\circ$ or P is given. (resp. $E_\circ$ or $E_\circ(Q)$). This is consistent with the rest of our conventions because $F_\circ(A)$ is a subflag of the standard flag $F_\bullet$ if and only if A is diagonal with nonincreasing diagonal in the orthonormal basis F.

3.5 Reducing to nonsingular marginals

Unfortunately, Algorithm TOSI only works if P and Q are nonsingular. We address these two issues in the following subsection; Corollary 3.20 summarises the results. In case P or Q is singular, we can form a new operator $\underline{T}$ such that the map $T \mapsto \underline{T}$ preserves capacity and rank-nondecreasingness.

Definition 3.16 (Reductions to nonsingular marginals)**.**

1. If A is a Hermitian operator, $\operatorname{supp} A$ is defined to be $\ker A^\perp$, i.e. the span of the positive eigenspaces of A.
2. Let $\eta : V \to \operatorname{supp} P, \nu : W \to \operatorname{supp} Q$ be partial isometries.
3. Define $\underline{T} : L(\operatorname{supp} P) \to L(\operatorname{supp} Q)$ by
$$\underline{T} : X \mapsto \nu T(\eta^\dagger X \eta)\nu^\dagger.$$
4. Define $\underline{P} := \eta P \eta^\dagger \in \mathcal{S}_{++}(\operatorname{supp} P)$ and $\underline{Q} = \nu Q \nu^\dagger \in \mathcal{S}_{++}(\operatorname{supp} Q)$.
5. If $g \in \mathrm{GL}(W)_{F_\circ(Q)}$, let $\underline{g}$ denote $\nu g \nu^\dagger$. Similarly, $\underline{h} := \eta h \eta^\dagger$. If $G \subset \mathrm{GL}(W)_{F_\circ(Q)}$ and $H \subset \mathrm{GL}(V)_{F_\circ(P)}$, define $\underline{G} = \{\underline{g} : g \in G\}$ and $\underline{H} = \{\underline{h} : h \in H\}$.

We'll need an easy, but useful, lemma.

Lemma 3.15. *Suppose $L \subset V$ is a subspace and $h \in \mathrm{GL}(V)$. Let $\beta : V \to L$ be a partial isometry (or orthogonal projection with range restricted to L), so that $\beta^\dagger\beta$ is the orthogonal projection to L. Suppose $hL = L$; then*

$$\beta^\dagger \beta h \beta^\dagger = h\beta^\dagger \tag{12}$$
$$\text{and } (\beta h \beta^\dagger)^{-1} = \beta h^{-1}\beta^\dagger. \tag{13}$$

Proof. If h fixes L, then an embedding to L followed by an application of h is the same as an embedding of L followed by an application of h followed by a projection to L. This proves 12. The identity 13 follows from 12. ∎

We defer the proof of the next proposition to Appendix 12.2.

Proposition 3.16.

1. $\operatorname{cap}(\underline{T}, \underline{P}, \underline{Q}) = \operatorname{cap}(T, P, Q)$.

2. $\underline{T}$ *is* $(\underline{P}, \underline{Q})$*-rank-nondecreasing if and only if* T *is* (P, Q)*-rank-nondecreasing.*

Proposition 3.17. *For* $(g, h) \in \mathrm{GL}(W)_{E_\circ} \times \mathrm{GL}(V)_{F_\circ}$,

$$\overline{T_{g,h}} = \overline{T}_{\overline{g},\overline{h}}.$$

Proof. Note that $h \operatorname{supp} P = \operatorname{supp} P$ and $g \operatorname{supp} Q = \operatorname{supp} Q$. By Lemma 3.15,

$$\begin{aligned} \overline{T_{g,h}}(X) &= \nu g^\dagger T(h\eta^\dagger X \eta h^\dagger) g \nu^\dagger \\ &= \nu g^\dagger \nu^\dagger \nu T(\eta^\dagger \eta h \eta^\dagger X \eta h^\dagger \eta^\dagger \eta) \nu^\dagger \nu g \nu^\dagger \\ &\overline{T}_{\overline{g},\overline{h}}(X). \end{aligned}$$

□

We omit the proof of the following easy proposition.

Proposition 3.18. *If* $(G, H, F_\circ(P), F_\circ(Q), T)$ *is block-diagonal, then* $(\underline{G}, \underline{H}, F_\circ(\underline{P}), F_\circ(\underline{Q}), \underline{T})$ *is block-diagonal.*

To find an ϵ-$(I_V \to Q, I_W \to P)$-scaling, it suffices to find an ϵ-$(I_{\operatorname{supp} P} \to \underline{Q}, I_{\operatorname{supp} Q} \to \underline{P})$-scaling.

Lemma 3.19. *If* $(G, H, F_\circ(P), F_\circ(Q), T)$ *is block-diagonal and* T *has an* ϵ*-*$(I_{\operatorname{supp} P} \to \underline{Q}, I_{\operatorname{supp} Q} \to \underline{P})$*-scaling, then* T *has a* 2ϵ*-*$(I_V \to Q, I_W \to P)$*-scaling.*

Proof. Sppose $\underline{T}_{\underline{g},\underline{h}}$ is an ϵ-$(I_{\operatorname{supp} P} \to \underline{Q}, I_{\operatorname{supp} Q} \to \underline{P})$. We use a limiting argument: let h approach $\overline{g} \oplus \eta^\dagger h \eta$ and g approach $\nu^\dagger \underline{g} \nu$. Then

$$\begin{aligned} g^\dagger T(hh^\dagger) g &\to \nu^\dagger \underline{T}_{\underline{g},\underline{h}} \nu = \nu^\dagger \underline{Q} + \nu^\dagger X \nu = Q + \underline{X} \\ \text{and } h^\dagger T^*(gg^\dagger) h &\to \eta^\dagger \underline{T}^*_{\underline{h},\underline{h}} \eta = \eta^\dagger \underline{P} \eta + \eta^\dagger Y \eta = P + \underline{Y}. \end{aligned}$$

Where X and Y both have trace-norm at most ϵ, and so the same is true for $\underline{X}$ and $\underline{Y}$. □

The next corollary follows from Lemma 3.5 and Lemma 3.1. The claims about effectiveness follow because the proofs of the two lemmas can be made effective with exponentially small error. This is true because approximate roots and Cholesky decompositions can be computed efficiently.

Corollary 3.20. *Suppose* $(G, H, F_\circ(P), F_\circ(Q), T)$ *is block-diagonal. If* $\underline{T}$ *is approximately* $(\underline{G}_{E_\circ}, \underline{H}_{F_\circ})$*-scalable to* $(\underline{P} \to I_{\operatorname{supp} Q}, \underline{Q} \to I_{\operatorname{supp} P})$*, then* T *is approximately* $(G_{E_\circ}, H_{F_\circ})$*-scalable to* $(I_V \to Q, I_W \to P)$*.*

Further, 3ϵ*-*$(I_V \to Q, I_W \to P)$*-scalings can be computed from* ϵ*-*$(\underline{P} \to I_{\operatorname{supp} Q}, \underline{Q} \to I_{\operatorname{supp} P})$*-scalings in time* $\operatorname{poly}(-\log \epsilon, n, m, b)$ *provided* $r \leq mn$ *where* b *is the input size of* $P, \underline{Q}, \underline{T}_{\underline{g},\underline{h}}$ *written in the bases* E *and* F*.*

3.6 Notation

When possible, these symbols will hold their meaning throughout the paper.

1. If Y is an inner product space over $\mathbb{C}$, $\mathcal{S}_{++}(Y) \subset \mathcal{S}_{+}(Y) \subset L(Y)$ denote the positive-definite, positive-semidefinite, and Hermitian operators, respectively, on Y. If $L \subset Y$ is a subspace, then $\pi_L : Y \to Y$ denotes the orthogonal projection to L.

2. If B is a linear map between two inner product spaces, $B^\dagger$ denotes the adjoint of B. In bases such that the inner products are the standard inner products, $B^\dagger$ is just the conjugate transpose. If $g \in \mathrm{GL}(Y)$, $g^{-\dagger}$ denotes $(g^{-1})^\dagger = (g^\dagger)^{-1}$.

3. $\mathrm{GL}(Y)$ denotes the general linear group.

4. The Loewner ordering $\succeq$ on $L(Y)$ is defined by $X \succeq Z$ if $X - Z \in \mathcal{S}_+(Z)$. Similarly $X \succ Z$ if $X - Z \in \mathcal{S}_{++}(Y)$.

5. If $A \in L(Y)$ is a Hermitian operator on an n-dimensional vector space Y,
$$\lambda(A) = (\lambda_1(A), \dots \lambda_n(A))$$
denotes the spectrum of A (in non-increasing order), but we also define $\lambda_0(A) = \infty$ and $\lambda_{n+1} = 0$.

6. If A is positive-semidefinite, $\operatorname{supp} A$ denotes the orthogonal complement of the kernel of A; alternatively, $\operatorname{supp} A$ is the largest subspace in $F_\circ(A)$ or the span of the positive eigenspaces of A.

7. W (resp. V) is an inner product space over $\mathbb{C}$ of dimension n (resp. m).

8. $T : L(V) \to L(W)$ is a completely positive map and $A_i : i \in [r]$ are the Kraus operators of T.

9. E (resp. F) is an orthonormal basis of W (resp. V).

10. $E_\bullet$ (resp. $F_\bullet$) is the standard flag on W (resp. V).

11. $P \in \mathcal{S}_+(V)$, $Q \in \mathcal{S}_+(W)$ are positive-semidefinite operators that are diagonal with non-increasing diagonal the bases F and E, respectively with $\operatorname{Tr} P = \operatorname{Tr} Q$. Equivalently, E and F are adapted bases for $F_\circ(Q)$ and $F_\circ(P)$, respectively.

12. p denotes $\lambda(P)$ and q denotes $\lambda(Q)$.

13. If α is a non-increasing sequence of length n, $\Delta\alpha$ denotes the sequence denotes the sequence $(\alpha_i - \alpha_{i+1} : i \in [n])$ where $\alpha_{n+1} := 0$.

14. Recall the conventions of Definition 3.15. If P and Q are given, $E_\circ = F_\circ(Q)$ and $F_\circ = F_\circ(P)$. If $E_\circ$ and $F_\circ$ and p and q are given without P and Q, we always assume $E_\circ$ and $F_\circ$ are q- and p-partial, respectively, and P and Q always refer to the unique operators satisfying $F_\circ(P) \subset F_\circ$, $F_\circ(Q) \subset E_\circ$ and $\lambda(Q) = q$, $\lambda(P) = p$.

15. As in Definition 3.13, for $j \in \sigma(F_\circ)$, let $\eta_j : V \to F_j$ be a partial isometry. That is, $\eta_j^\dagger \eta_j$ is π_{F_j}, the orthogonal projection to F_j. Define $\nu_i : W \to E_i$ for $i \in \sigma(E_\circ)$ to be the analogous partial isometries.

16. Sometimes we'll want to obtain "invertible versions" of P and Q. As per Definition 3.16, let $\eta : V \to \operatorname{supp} P$ and $\nu : W \to \operatorname{supp} Q$ be partial isometries, and define $\underline{P} = \eta P \eta^\dagger : \operatorname{supp} P \to \operatorname{supp} P$ and $\underline{Q} = \nu Q \nu^\dagger : \operatorname{supp} Q \to \operatorname{supp} Q$; note that $\underline{P}$ and $\underline{Q}$ are actually positive-*definite*.

17. If $A : Y \to Z$ is a linear operator, $\|A\|$ is the trace-norm (or Frobenius norm) $\sqrt{\operatorname{Tr} A^\dagger A}$ of A and $\|A\|_2$ is the spectral norm $\sqrt{\lambda_1(A^\dagger A)}$ of A.

4 A reduction to the doubly stochastic case

In this section we describe a reduction from $(P \to I_W, Q \to I_V)$-scalability to $(I \to I, I \to I)$-scalability by block-upper-triangular scalings. The reduction makes sense only when P and Q have integral spectra, and sends the completely positive map T to the larger completely positive map $\operatorname{Trun}_{P,Q} T$ which maps $\operatorname{Tr} P \times \operatorname{Tr} P$ positive semidefinite operators to $\operatorname{Tr} Q \times \operatorname{Tr} Q$ positive semidefinite operators. The only properties we need are that $\operatorname{cap}(T, P, Q) = \operatorname{cap} \operatorname{Trun}_{P,Q} T$ and that T is (P, Q)-rank-nondecreasing if and only if $\operatorname{Trun}_{P,Q} T$ is rank-nondecreasing. Though we do not need it in this paper, one can prove directly that for P and Q nonsingular, $\operatorname{Trun}_{P,Q} T$ is approximately scalable to doubly stochastic if and only if T is approximately scalable to $(P \to I_W, Q \to I_V)$ by block-upper-triangular scalings.

The key to designing the reduction is Fact 3.2, which allows us to identify a positive-semidefinite operator with its spectrum and its flag. We motivate the formal definition with a small example.

Example 3. Suppose $P = \operatorname{diag}(2, 2, 1)$ and $Q = \operatorname{diag}(2, 1, 1)$. Then $P = \operatorname{diag}(1, 1, 0) + \operatorname{diag}(1, 1, 1)$ and $Q = \operatorname{diag}(1, 0, 0, 0) + \operatorname{diag}(1, 1, 1, 1)$. Suppose T has one Kraus operator A (for simplicity, suppose it has real entries). We want $APA^t = I$ and $A^tQA = I$. Our starting point is to build a function sending operators mapping $(P \to I_W, Q \to I_V)$ to doubly stochastic operators.

First note that

$$APA^t = AA^t + A \operatorname{diag}(1, 1, 0)A^t =$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}^t + \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \\ a_{41} & a_{42} & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \\ a_{41} & a_{42} & 0 \end{bmatrix}^t \tag{14}$$

and

$$A^tQA = A^tA + A^t \operatorname{diag}(1, 0, 0, 0)A =$$
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}^t \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^t \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By looking at the expressions above, on can see that the i, j entry of $APA^\dagger$ can be thought of as the dot product of the rows i and j of A, *plus* the pairwise dot products of the rows i and j of A each *truncated* to two entries. That is, the i and j rows of $A\eta_2^\dagger$ where

$$\eta_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We need to simulate this with one set of Kraus operators. We hope that one product $A_k A_k^\dagger$ of Kraus operators of $\operatorname{Trun}_{P,Q} T$ will contribute the inner product of the i and j rows of A to $\operatorname{Trun}_{P,Q} T(I)_{ij}$

and another will contribute the inner product of the i and j rows of $A\eta_2^\dagger$.

Likewise, one product $A_k^\dagger A_k$ of Kraus operators of $(\operatorname{Trun}_{P,Q} T)^*$ will contribute the inner product of the i and j *columns* of A to $(\operatorname{Trun}_{P,Q} T)^*(I)_{ij}$ and another will contribute the inner product of the i and j columns of $\nu_1 A$.

For $i \in [3]$ and $j \in [4]$, the below Kraus operators achieve this. Note that the rows are blocked into sizes $(4,1)$ and the columns into $(3,2)$, the respective conjugate partitions of $(2,1,1,1)$ and $(2,2,1)$.

$$A_{1,1} = \left[\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 \\ a_{41} & a_{42} & a_{43} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right], \qquad A_{1,2} = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & a_{21} & a_{22} \\ 0 & 0 & 0 & a_{31} & a_{32} \\ 0 & 0 & 0 & a_{41} & a_{42} \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$A_{2,1} = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline a_{11} & a_{12} & a_{13} & 0 & 0 \end{array}\right], \qquad A_{2,2} = \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & a_{11} & a_{12} \end{array}\right].$$

With these Kraus operators, $\operatorname{Trun}_{P,Q} T(I)$ will be block-diagonal with one 4×4 block and one 1×1 block. The 4×4 block is exactly $T(P)$. The bottom right 1×1 block of $\operatorname{Trun}_{P,Q} T$ is simply the top-left 1×1 principal submatrix of $T(P)$.

Similarly, $(\operatorname{Trun}_{P,Q} T)^*(I)$ will be block-diagonal with a 3×3 block in the top-left that is $T^*(Q)$ and a 2×2 block that is the top-left 2×2 principal submatrix of $T^*(Q)$. Together, this shows T maps $(P \to I_W, Q \to I_V)$ if and only if $\operatorname{Trun}_{P,Q} T$ is doubly stochastic.

In general, we will try to make $\operatorname{Trun}_{P,Q} T(I)$ block-diagonal with the i^{th} diagonal block equal to

$$\nu_i T\left(\sum_{j=1}^{p_1} \eta_{v(j)}^\dagger \eta_{v(j)}\right)\nu_i^\dagger,$$

where $v(j)$ is the conjugate partition to p. Critically, $P = \sum_{j=1}^{p_1} \eta_{v(j)}^\dagger \eta_{v(j)}$. We make this plan precise below.

Definition 4.1 (Reduction to doubly stochastic case)**.** Suppose $T : L(V) \to L(W)$ is a completely positive map and that $P \in \mathcal{S}_+(V)$ and $Q \in \mathcal{S}_+(W)$ are positive-semidefinite operators with integral spectra.

1. Let $(v(j) : j \geq 1)$ be the conjugate partition to p. Let $V^\bullet = (V^1 \supset V^2 \supset \cdots \supset V^{p_1})$ be the decreasing sequence of subspaces defined by $V^j = F_{v(j)}$ for $j \geq 1$. Let w be the analogous partition and $W^\bullet = (W^1 \supset W^2 \supset \dots W^{q_1})$ be the analogous sequence of subspaces for Q. Note that the number of times $v(j) = k$ occurs for $k \in \sigma(F_\circ)$ is exactly $p_k - p_{k+1}$.

2. For each $i \in [p_1]$, define $\tau_i : V \to \bigoplus_{j=1}^{p_1} V^j$ by
$$\tau_i : x \mapsto (0_{V^1}, \dots, 0_{V^{i-1}}, \eta_{v(i)}x, 0_{V^{i+1}}, \dots, 0_{V^{p_1}})$$
Here 0_{V^j} denotes the zero vector in V^j, which is itself a copy of $F_{v(j)}$. The vector $\eta_{v(i)}x$ appears in the i^{th} spot. Note that $\tau_i^\dagger : \bigoplus_{i=1}^{p_1} V^i \to V$ is given by $\tau_i^\dagger : (x_1, \dots, x_{p_1}) \mapsto \eta_{v(i)}^\dagger x_i$. Let $\kappa_i : W \to \bigoplus_{i=1}^{q_1} W^i$ be the analogous map for $W^\bullet$, using ν_i's instead of η_i's.

3. Define the completely positive rectangular operator $\mathrm{Trun}_{P,Q} T : L\left(\bigoplus_{i=1}^{p_1} V^i\right) \to L\left(\bigoplus_{i=1}^{q_1} W^i\right)$ by

$$\mathrm{Trun}_{P,Q} T(X) = \sum_{i\in[q_1], j\in[p_1]} \kappa_i T(\tau_j^\dagger X \tau_j)\kappa_i^\dagger \tag{15}$$

Remark 4.1 (Kraus operators of the reduction)**.** From 15 one can see that if $A_i, i \in [r]$ are Kraus operators for T, then $\mathrm{Trun}_{P,Q} T$ has Kraus operators

$$A_l^{ij} := \kappa_i A_l \tau_j^\dagger$$

for $l \in [r], i \in [q_1], j \in [p_1]$. If F is an orthonormal basis for V such that the standard flag $F_\bullet$ contains $F_\circ$, then $\{\tau_i f_j : i \in [p_1], j \in [v(i)]\}$ is a natural choice basis for $\bigoplus_{i=1}^{p_1} V^i$. Let $0_{s,t}$ denote the $s \times t$ all-zero matrix. In this basis,

$$A_l^{ij} = \begin{bmatrix} 0_{w(1),v(1)} & \cdots & 0_{w(1),v(j)} & \cdots & 0_{w(1),v(p_1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0_{w(k),v(1)} & \cdots & \nu_{w(k)} A_l \eta_{v(j)}^\dagger & \cdots & 0_{w(k),v(p_1)} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0_{w(q_1),v(1)} & \cdots & 0_{w(q_1),v(j)} & \cdots & 0_{w(q_1),v(p_1)} \end{bmatrix}, \tag{16}$$

and the nonzero matrix $\nu_{w(k)} A_l \eta_{v(j)}^\dagger$ is just the submatrix of A_l indexed by the first $w(k)$ rows and first $v(j)$ columns of A_l.

As promised, we accomplished at least part of what we set out to do before Definition 4.1. The below observation demonstrates this when applied with $X = I$; note that $\sum_{j\in[p_1]} \tau_j^\dagger \tau_j = P$.

Observation 4.2. *Let $\iota_i : V^i \to \bigoplus_{j=1}^{p_1} V^j$ be an inclusion from V^i into $\bigoplus_{j=1}^{p_1} V^j$ so that $\tau_i = \iota_i \circ \eta_{v(i)}$. We can write*

$$\mathrm{Trun}_{P,Q} T(X) = \bigoplus_{i\in[q_1]} \nu_{w(i)} T\left(\sum_{j\in[p_1]} \tau_j^\dagger X \tau_j\right) \nu_{w(i)}^\dagger \tag{17}$$

All we need from the reduction is its preservation of capacity and rank-nondecreasingness. We prove this in the next two subsections.

Theorem 4.3. *Suppose P and Q are positive definite operators with integral spectra. Then*

1. $\operatorname{cap} \mathrm{Trun}_{P,Q} T = \operatorname{cap}(T, P, Q)$*, and*

2. $\mathrm{Trun}_{P,Q} T$ *is rank-nondecreasing if and only if* T *is* (P,Q)*-rank-nondecreasing.*

4.1 Capacity under the reduction

The first item of Theorem 4.3 follows immediately from the next two lemmas, simply because the right-hand side of the first is the left-hand side of the second. We present them separately because Lemma 4.5 will be useful for proving the existence of *exact* scalings. Throughout, this subsection, $A = \sigma(F_\circ(P))$ and $B = \sigma(F_\circ(Q))$.

Lemma 4.4.

$$\operatorname{cap}\operatorname{Trun}_{P,Q} T = \inf_{0 \prec Y_j : F_j \to F_j, j \in A} \frac{\det\left(Q, T\left(\sum_{j\in A} \Delta p_j \eta_j^\dagger Y_j \eta_j\right)\right)}{\prod_{j\in A} (\det Y_j)^{\Delta p_j}}. \tag{18}$$

Further, if

$$\inf_{X \succeq 0, \det X = 1} \frac{\det \operatorname{Trun}_{P,Q} T(X)}{X}$$

is attained (resp. uniquely attained) then the infimum in 18 is attained (resp. uniquely attained subject to $\prod_{j\in A}(\det Y_j)^{\Delta p_j} = 1$*).*

Lemma 4.5.

$$\inf_{0 \prec Y_j : F_j \to F_j, j \in A} \frac{\det\left(Q, T\left(\sum_{j\in A} \Delta p_j \eta_j^\dagger Y_j \eta_j\right)\right)}{\prod_{j\in A} (\det Y_j)^{\Delta p_j}} = \operatorname{cap}(T, P, Q) \tag{19}$$

Proof of Lemma 4.4. We prove Lemma 4.4 by showing we may assume the infimum in $\operatorname{cap}(\operatorname{Trun}_{P,Q} T)$ takes a very special form.

$$\begin{aligned}\operatorname{cap}\operatorname{Trun}_{P,Q} T &= \inf_{X \succ 0} \frac{\det \operatorname{Trun}_{P,Q} T(X)}{\det X} \\ &= \inf_{X \succ 0} \frac{\det \bigoplus_{i\in[q_1]} \nu_{w(i)} T\left(\sum_{j\in[p_1]} \tau_j^\dagger X \tau_j\right) \nu_{w(i)}^\dagger}{\det X}.\end{aligned} \tag{20}$$

Since τ_j is $\iota_j \circ \eta_{v(j)}$, $\sum_{j\in[p_1]} \tau_j^\dagger X \tau_j = \sum_{j\in[p_1]} \eta_{v(j)}^\dagger \iota_j^\dagger X \iota_j \eta_{v(j)}$. Then for each $j \in [p_1]$, $\iota_j^\dagger X \iota_j : V_j \to V_j$. Since $\iota_j^\dagger X \iota_j := X_j$ is all that appears in the numerator, we may assume the denominator of the expression 20 is maximized subject to *fixed* X_j, $j \in [p_1]$. In a certain basis for $\bigoplus_{i\in[p_1]} V_i$ (the basis using the image under ι_j of the first $v(j)$ vectors of the same ordered eigenbasis for P to span the j^{th} summand), the map X_j is just the j^{th} diagonal block of X. X_j can be any positive-semidefinite operator $V_j \to V_j$. Note that

$$\det X \le \det \bigoplus_{j=1}^{p_1} X_j,$$

because the determinant of a positive semidefinite matrix is at most the product of the determinants of its diagonal blocks [**GGOW16**]. Hence, we can replace X by $\bigoplus_{j=1}^{p_1} X_j$, so the infimum becomes

$$\begin{aligned} &\operatorname{cap}\operatorname{Trun}_{P,Q} T = \\ &\inf_{0 \prec X_j : V_j \to V_j,\, j\in[p_1]} \frac{\det \bigoplus_{i\in[q_1]} \nu_{w(i)} T\left(\sum_{j\in[p_1]} \eta_{v(j)}^\dagger X_j \eta_{v(j)}\right) \nu_{w(i)}^\dagger}{\prod_{j=1}^{p_1} \det X_j} \\ &= \inf_{0 \prec X_j : V_j \to V_j,\, j\in[p_1]} \frac{\prod_{i=1}^{q_1} \det \nu_{w(i)} T\left(\sum_{j\in[p_1]} \eta_{v(j)}^\dagger X_j \eta_{v(j)}\right) \nu_{w(i)}^\dagger}{\prod_{j=1}^{p_1} \det X_j}. \end{aligned} \tag{21, 22}$$

Since the replacement $X \leftarrow \bigoplus_{j=1}^{p_1} X_j$ only decreased the left-hand of 20, if the infimum was attained then it is still attained after the replacement. We can scale at will without changing the value of If it was uniquely attained, then it is still uniquely attained as restricting X to be block diagonal only restricts the set over which the infimum is taken to another set containing the minimizer.

We use the assumption that the function is minimal to make further deductions about the structure of the X_i's. Note that there are many duplicates among the V_i. The number of $i \in [p_1]$ such that $V_i = F_j(P)$ is exactly Δp_j. Fix j and suppose

$$V_l = \cdots = V_{l-1+\Delta p_j} = F_j(P).$$

Assume Δp_j is non-zero. Then the X_i such that $V_i = F_j(P)$ appears in the numerator in 22 as $\sum_{i=l}^{l-1+\Delta p_j} \eta_{v(i)}^\dagger X_j \eta_{v(i)} = \eta_j^\dagger \sum_{i=l}^{l-1+\Delta p_j} X_i \eta_j$. However, since the logarithm of the determinant is concave,

$$\prod_{i=l}^{l-1+\Delta p_j} \det(X_i) \leq \det\left(\frac{1}{\Delta p_j} \sum_{i=l}^{l-1+\Delta p_j} X_i\right)^{\Delta p_j}.$$

Thus, the denominator in 22 would increase if we replaced all X_i by $\frac{1}{\Delta p_j} \sum_{i=l}^{l-1+\Delta p_j} X_i$ for $i \in \{l, \ldots l-1+\Delta p_l\}$, and the numerator wouldn't change. Thus we can assume if $V_i = V_{i'} = F_j(P)$, then $X_i = X_{i'}$, which we can rename Y_j. Further, if the infimum was attained, the replacement of X_i by Y_j decreases the argument so it still attained. If it was uniquely attained, then we have restricted the set over which the infimum is being taken to another set containing the minimizer, so it is still uniquely attained. Now

$$\operatorname{cap} \operatorname{Trun}_{P,Q} T = \inf_{0 \prec Y_j : F_j(P) \to F_j(P), j \in A} \frac{\prod_{i=1}^{q_1} \det \nu_{w(i)} T\left(\sum_{j \in A} \eta_j^\dagger Y_j \eta_j\right) \nu_{w(i)}^\dagger}{\prod_{j \in A} (\det Y_j)^{\Delta p_j}}.$$

Now observe that for $j \in B$, the $\nu_{w(i)} = \nu_j$ for exactly Δq_j different $i's$, which shows

$$\operatorname{cap} \operatorname{Trun}_{P,Q} T = \inf_{0 \prec Y_j : F_j(P) \to F_j(P), j \in A} \frac{\prod_{i \in B} \left(\det \nu_i T\left(\sum_{j \in A} \Delta p_j \eta_j^\dagger Y_j \eta_j\right) \nu_i^\dagger\right)^{\Delta q_i}}{\prod_{j \in A} (\det Y_j)^{\Delta p_j}}.$$

On the other hand,

$$\prod_{i \in B} \left(\det \nu_i T\left(\sum_{j \in A} \Delta p_j \eta_j^\dagger Y_j \eta_j\right) \nu_i^\dagger\right)^{\Delta q_i} = \det\left(Q, T\left(\sum_{j \in A} \Delta p_j \eta_j^\dagger Y_j \eta_j\right)\right).$$

□

To prove Lemma 4.5, we will use minimality to reduce the number of degrees of freedom in the denominator. We'll use the following claim.

Claim 4.6. *Suppose $h \in \mathrm{GL}(V)_{F_\circ(P)}$. Then*

$$\max \prod_{j \in A} (\det Y_j)^{\Delta p_j}$$

subject to

$$\sum_{j \in A} \Delta p_j \eta_j^\dagger Y_j \eta_j = hPh^\dagger,$$

and $0 \prec Y_j : F_j \to F_j$ for $j \in A$ is

$$\det(P, h^\dagger h)$$

and is uniquely attained at the tuple

$$Y^* = (\eta_j h \eta_j^\dagger \eta_j h^\dagger \eta_j^\dagger : j \in A)$$

of positive-definite operators.

Before proving Claim 4.6, let's see why it implies Lemma 4.5.

Proof of Lemma 4.5. Consider the argument of T in the numerator of 19, $\sum_{j=1}^n \Delta p_j \eta_j^\dagger Y_j \eta_j$. We can write

$$\sum_{j=1}^n \Delta p_j \eta_j^\dagger Y_j \eta_j = hPh^\dagger$$

for some $h \in \mathrm{GL}(V)_{F_\circ(P)}$; by the existence of Cholesky decompositions there exists $h' \in \mathrm{GL}(V)_{F_\circ(\underline{P})}$ such that $h'h'^\dagger = \eta \sum_{j=1}^n \Delta p_j \eta_j^\dagger Y_j \eta_j \eta^\dagger$, and so setting $h = h' \oplus I_{\ker P}$ will suffice. The numerator of the left-hand-side of 19 is now $\det(Q, T(hPh^\dagger))$, and by Claim 4.6, the denominator can be replaced by $\det(h^\dagger h, P)$. Thus,

$$\operatorname{cap} \operatorname{Trun}_{P,Q} T = \inf_{h \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(Q, T(hPh^\dagger))}{\det(P, h^\dagger h)} = \operatorname{cap}(T, P, Q).$$

□

We continue with the proof of the claim.

Proof of Claim 4.6. By the strict log-concavity of $\det X$ in the positive-definite cone, the quantity

$$f(Y) = \log \prod_{j \in A} (\det Y_j)^{\Delta p_j} = \sum_{j \in A} \Delta p_j \log \det Y_j$$

is strictly concave as a function of $Y = (Y_j : j \in A)$. Further,

$$\sum_{j \in A} \Delta p_j \eta_j^\dagger Y_j \eta_j = hPh^\dagger$$

is a linear constraint. Thus, maximizing f over the domain D described in the claim is a (strictly) convex program, and it is not hard to see that $f : \overline{D} \to [-\infty, \infty)$ is upper-semicontinuous. Further, D is nonempty, relatively open, and $\overline{D}$ is compact. To see that D is nonempty we may set $Y^* = (\eta_j h \eta_j^\dagger \eta_j h^\dagger \eta_j^\dagger : j \in A)$ to obtain

$$\begin{aligned}
&\sum_{j=1}^n \Delta p_j \eta_j^\dagger Y_j^* \eta_j \\
&= \sum_{j=1}^n \Delta p_j \eta_j^\dagger \eta_j h \eta_j^\dagger \eta_j h^\dagger \eta_j^\dagger \eta_j \\
&= \sum_{j=1}^n \Delta p_j h \eta_j^\dagger \eta_j h^\dagger \\
&= h \left(\sum_{j=1}^n \Delta p_j \eta_j^\dagger \eta_j \right) h^\dagger = hPh^\dagger.
\end{aligned}$$

The second equality follows from $h \in F_\circ(P)$ and Lemma 3.15.

Since f is upper-semicontinous on the compact set $\overline{D}$, f achieves a maximum on $\overline{D}$. However, the maximum cannot be $\overline{D} \setminus D$, because this implies some Y_j, $j \in A$ is singular and hence

$f(Y) = -\infty$. Thus, f takes a maximum Y on D, which is unique by strict convexity.

We must have that the directional derivative vanishes at Y, i.e. $\Delta_X f|_Y = 0$ for every $X = (X_j \in L(F_j(P)) : j \in A)$ such that

$$\sum_{j \in A} \Delta p_j \eta_j^\dagger X_j \eta_j = 0,$$

and the maximum M must be uniquely attained here. To show that M is attained at Y^*, it is enough to show that $\Delta_X f|_{Y^*} = 0$.

Let us expand the equation $\Delta_X f|_Y$.

$$\Delta_X f|_Y = \sum_{j \in A} \Delta p_j \nabla_X \log\det Y_j$$

By the formula $\nabla_B \log\det C = \operatorname{Tr} C^{-1} B$,

$$\Delta_X f|_Y = \sum_{j \in A} \Delta p_j \nabla_X \log\det Y_j = \sum_{j=1}^{n} \Delta p_j \operatorname{Tr} Y_j^{-1} X_j.$$

If we insert $Y_j = Y_j^*$, we obtain

$$\begin{aligned}
\Delta_X f|_{Y^*} &= \sum_{j \in A} \Delta p_j \operatorname{Tr} (Y_j^*)^{-1} X_j \\
&= \sum_{j \in A} \Delta p_j \operatorname{Tr} \eta_j h^{-\dagger} h^{-1} \eta_j^\dagger X_j \\
&= \sum_{j \in A} \Delta p_j \operatorname{Tr} h^{-\dagger} h^{-1} \eta_j^\dagger X_j \eta_j^\dagger \\
&= \operatorname{Tr} h^{-\dagger} h^{-1} \left(\sum_{j \in A} \Delta p_j \eta_j^\dagger X_j \eta_j^\dagger \right) \\
&= 0.
\end{aligned}$$

The second equality is by $h \in F_\circ(P)$ and Lemma 3.15. $\square$

4.2 Rank-nondecreasingness under the reduction

Proof of item 2 of Theorem 4.3. The second item of Theorem 4.3 follows immediately from Lemma 4.7: recall that the number of j such that $W^j = E_i$ for $i \in \sigma(F_\circ(Q))$ is exactly $q_i - q_{i+1}$ and the number of j such that $V^j = F_i$ for $i \in \sigma(E_\circ(Q))$ is $p_i - p_{i+1}$, so

$$\sum_{i=1}^{q_1} \dim(L \cap W^i) + \sum_{i=1}^{p_1} \dim(R \cap V^i) = \sum_{i \in \sigma(E_\circ)} \Delta q_i \dim E_i \cap L + \sum_{j \in \sigma(F_\circ)} \Delta p_i \dim F_j \cap R \tag{23}$$

$\square$

Lemma 4.7. $\mathrm{Trun}_{P,Q}\, T$ *is rank-nondecreasing if and only if*

$$\sum_{i=1}^{q_1} \dim(L \cap W^i) + \sum_{i=1}^{p_1} \dim(R \cap V^i) \leq \operatorname{Tr} P \tag{24}$$

for all T-independent pairs (L, R).

Proof. $\mathrm{Trun}_{P,Q}\, T$ is rank non-decreasing if and only if $\dim \mathcal{L} + \dim \mathcal{R} \leq N$ for any $\mathrm{Trun}_{P,Q}\, T$-independent pair $(\mathcal{L}, \mathcal{R})$. However, $\mathrm{Trun}_{P,Q}\, T$ is in fact rank non-decreasing if and only if $\dim L + \dim R \leq n$ for all $\mathrm{Trun}_{P,Q}\, T$-independent pairs such that L and R are each maximal (holding the other fixed) subject to $(\mathcal{L}, \mathcal{R})$ being $\mathrm{Trun}_{P,Q}\, T$-independent.

Claim 4.8. *Suppose $(\mathcal{L}, \mathcal{R})$ is a $\mathrm{Trun}_{P,Q}\, T$-independent pair.*

$$\left(\sum_{i=1}^{q_1} \kappa_i \kappa_i^\dagger \mathcal{L}, \sum_{j=1}^{p_1} \tau_j \tau_j^\dagger \mathcal{R} \right) \tag{25}$$

is also a $\mathrm{Trun}_{P,Q}\, T$-independent pair. Further, $\mathcal{L} \subset \sum_{i=1}^{q_1} \kappa_i \kappa_i^\dagger \mathcal{L}$ and $\mathcal{R} \subset \sum_{j=1}^{p_1} \tau_j \tau_j^\dagger \mathcal{R}$.

Proof of Claim. First we show $\mathcal{L} \subset \sum_{i=1}^{q_1} \kappa_i \kappa_i^\dagger \mathcal{L}$. This follows from $\sum_{i=1}^{q_1} \kappa_i \kappa_i^\dagger = I$, because

$$\mathcal{L} = \left(\sum_{i=1}^{q_1} \kappa_i \kappa_i^\dagger \right) \mathcal{L} \subset \sum_{i=1}^{q_1} \kappa_i \kappa_i^\dagger \mathcal{L}.$$

The proof of the analogous statement for $\mathcal{R}$ is similar.

To show that 25 is $\mathrm{Trun}_{P,Q}\, T$-independent, consider a pair of vectors

$$u := \sum_{i=1}^{q_1} \kappa_i \kappa_i^\dagger u_i \in \sum_{i=1}^{q_1} \kappa_i \kappa_i^\dagger \mathcal{L}$$

where $u_i \in \mathcal{L}$ and

$$v := \sum_{j=1}^{p_1} \tau_j \tau_j^\dagger v_i \in \sum_{j=1}^{p_1} \tau_j \tau_j^\dagger \mathcal{R}.$$

where $v_i \in \mathcal{L}$. Recall that $\{A_l^{\alpha\beta} := \kappa_\alpha A_l \tau_\beta^\dagger : l \in [r], \alpha \in [q_1], \beta \in [p_1]\}$ are Kraus operators $\mathrm{Trun}_{P,Q}\, T$. It is enough to show $v^\dagger A_l^{\alpha\beta} u = 0$ for all $l \in [r], \alpha \in [q_1], \beta \in [p_1]$. Indeed,

$$v^\dagger A^{l\alpha\beta} u$$

$$= \left(\sum_{i=1}^{q_1} \kappa_i \kappa_i^\dagger u_i \right) A_l^{\alpha\beta} \left(\sum_{j=1}^{p_1} \tau_j \tau_j^\dagger v_i \right)$$

$$= \left(\sum_{i=1}^{q_1} \kappa_i \kappa_i^\dagger u_i \right) \kappa_\alpha A_l \tau_\beta^\dagger \left(\sum_{j=1}^{p_1} \tau_j \tau_j^\dagger v_i \right)$$

$\tau_j\tau_j^\dagger$ is an orthogonal projection to the row space of $\tau_j^\dagger$, which is orthogonal to the row space of $\tau_\beta^\dagger$ for any $\beta \neq i$, so $\tau_\beta^\dagger\tau_j\tau_j^\dagger = \delta_{j\beta}\tau_\beta^\dagger$. Expanding the sum and applying this identity gives

$$v^\dagger A_l^{\alpha\beta} u = v_\alpha^\dagger \kappa_\alpha A_l \tau_\beta^\dagger u_\beta.$$

However, $v_\alpha^\dagger \kappa_\alpha A_l \tau_\beta^\dagger u_\beta = 0$ by the $\mathrm{Trun}_{P,Q}\, T$-independence of $(\mathcal{L}, \mathcal{R})$. This proves the claim. □

By the discussion before Claim 4.8, we may assume $(\mathcal{L}, \mathcal{R})$ is a $\mathrm{Trun}_{P,Q}\, T$-independent pair such that $\mathcal{L}$ and R are each maximal (holding the other fixed) subject to $(\mathcal{L}, \mathcal{R})$ being $\mathrm{Trun}_{P,Q}\, T$-independent. By the claim, $(\mathcal{L}, \mathcal{R})$ is of the form 25. Equivalently, $(\mathcal{L}, \mathcal{R})$ is equal to

$$\left(\sum_{i=1}^{q_1} \kappa_i L_i, \sum_{j=1}^{p_1} \tau_j R_j\right) \tag{26}$$

for some $L_i \subset W^i$ and $R_j \subset V^j$. Observe that

$$\dim \mathcal{L} = \sum_{i=1}^{q_1} \dim L_i.$$

and

$$\dim \mathcal{R} = \sum_{j=1}^{p_1} \dim R_i.$$

Further, $(\mathcal{L}, \mathcal{R})$ are $\mathrm{Trun}_{P,Q}\, T$-independent if and only if (L_i, R_j) are T-independent for all $i \in [q_1], j \in [p_1]$.

We use the maximality of $\mathcal{L}$ to make further assumptions. Since (L_i, R_j) and (L_{i+1}, R_j) are T-independent for all $i \in [q_1 - 1], j \in [p_1]$, $(L_i + L_{i+1}, R_j)$ are T-independent for all $i \in [q_1], j \in [p_1]$. Since $L_{i+1} \subset W^{i+1} \subset W^i$, we have $L_i \supset L_{i+1}$. Further, since (L_i, R_j) is T-independent, $(L_i \cap W^{i+1}, R_j)$ is T-independent and $(L_{i+1} + L_i \cap W^{i+1}, R_j)$ is T-independent. Thus, $L_{i+1} \supset L_i \cap W^{i+1}$. $L_{i+1} \supset L_i$ and $W^{i+1} \supset L_{i+1} \supset L_i \cap W^{i+1}$ imply $L_{i+1} = W^{i+1} \cap L_i$ for $i \in [q_1 - 1]$. By induction, this gives us

$$L_i = L_1 \cap W^i$$

for $i \in [q_1]$. The same argument for $\mathcal{R}$ tells us $R_j = R_1 \cap V^j$ for $j \in [p_1]$. A priori, L_1 and R_1 can be any subspaces of V^1 and W^1, respectively. $\mathrm{Trun}_{P,Q}\, T$ is rank-nondecreasing if and only if

$$\sum_{i=1}^{q_1} \dim(L \cap W^i) + \sum_{i=1}^{p_1} \dim(R \cap V^i) \leq N$$

for all T-independent pairs (L, R) where $L \subset W^1$ and $R \subset V^1$. The inequality above holds for all $(L \subset W, R \subset V)$ if and only if it holds for all $(L \subset W^1, R \subset V^1)$, proving the lemma. □

5 Positive capacity implies scalability by upper triangulars

Here we show the implication 2 $\implies$ 3 of Theorem 3.8. Proceeds by "fixing" first $T(P)$ by a left scaling g, fixing $T^*_{g,I}(Q)$ by a right scaling h, and so on. This idea goes back to Sinkhorn and has many variants. One of which, due to Gurvits, was called operator sinkhorn iteration (or OSI).

Our version has the additional arguments P and Q and only uses g and h that respect $F_\circ(Q)$ and $F_\circ(P)$, hence our name "triangular operator Sinkhorn iterations," or Algorithm TOSI. We use the standard methods to show capacity increases by a function of ϵ in each iteration unless T is already an ϵ-$(P \to I_W, Q \to I_V)$-scaling. Thus, if the capacity is nonzero to begin with, Sinkhorn scaling eventually results in an ϵ-$(P \to I_W, Q \to I_V)$-scaling.

5.1 Algorithm TOSI

We will need another notion of how far T is from being a $(P \to I_W, Q \to I_V)$-scaling.

Definition 5.1. Let

$$\operatorname{ds}_{P,Q} T = \sum_{i=1}^{n} \Delta p_i \left\| \eta_i (T^*(Q) - I_V)\eta_i^\dagger \right\|^2 + \sum_{i=1}^{m} \Delta q_i \left\| \nu_i (T(P) - I_W)\nu_i^\dagger \right\|^2 .$$

In particular, $\operatorname{ds}_{P,Q} T \geq p_n \|T^*(Q) - I_V\|^2 + q_m \|T(P) - I_W\|^2$, so if P and Q are invertible and $\operatorname{ds}_{P,Q} T_{g,h} < \epsilon$ then $T_{g,h}$ is an $\frac{\epsilon}{\max\{p_n, q_m\}}$-$(P \to I_W, Q \to I_V)$-scaling of T.

Algorithm TOSI (Triangular operator Sinkhorn iterations)**.**

Input: $P \in \mathcal{S}_{++}(V)$, $Q \in \mathcal{S}_{++}(W)$, a completely positive map T, and a real number $\epsilon > 0$.

Output: $g \in G_{E_\circ}$ and $h \in H_{F_\circ}$ such that $\operatorname{ds}_{P,Q} T_{g,h} \leq \epsilon$.

Set $g_0 = I$, $h_0 = I$, and $j = 0$.

1. Increment j.
2. **If j is odd:** Find $g \in G_{E_\circ}$ such that $g^\dagger T(h_{j-1} P h_{j-1}^\dagger) g = I$. Set $g_j = g$ and $h_j = h_{j-1}$.

   If j is even: Find $h \in H_{F_\circ}$ such that $h^\dagger T^*(g_{j-1} Q g_{j-1}^\dagger) h = I$. Set $g_j = g_{j-1}$ and $h_j = h$.
3. If $\operatorname{ds}_{P,Q} T_{g_j,h_j} > \epsilon$, go to Step 1. Else,

   Return: g_j and h_j.

Definition 5.2. It will be convenient to define $T_j := T_{g_j,h_j}$ where g_j, h_j are as in Algorithm TOSI.

Theorem 5.1. *If $\operatorname{cap}(T, P, Q) > 0$, then Algorithm TOSI terminates in at most*

$$t = \frac{-7 \log \operatorname{cap}(T_1, P, Q)}{\min\{\epsilon, p_n\} + \min\{\epsilon, q_m\}}$$

iterations.

We show algorithm terminates, but delay the proof of the required claims and lemmas to 5.1.1.

Proof of Theorem 5.1. Lemma 5.4 implies that $T(h_j P h_j^\dagger)$ (resp. $T^*(g_j Q g_j^\dagger)$) will always be invertible. Next we need to be sure that we can always take $g \in G_{E_\circ}$ (resp. $h \in H_{F_\circ}$) in odd (resp. even) steps of the algorithm, g (resp. h) just needs to be a block-Cholesky decomposition of $T(h_{j-1} h_{j-1}^\dagger)^{-1}$ (resp. $T^*(g_{j-1} g_{j-1}^\dagger)^{-1}$). That is, $g \in \mathrm{GL}(W)_{E_\circ}$ and $gg^\dagger = T(h_{j-1} h_{j-1}^\dagger)^{-1}$. However, by assumption, $T(h_{j-1} h_{j-1}^\dagger)^{-1}$ is block-diagonal so we can take the block-Cholesky decomposition g to be block-diagonal with the same blocks, or $g \in G$. The reasoning for even steps is analogous.

Since $\operatorname{cap}(T,P,Q) > 0$, Lemma 5.3 implies $\operatorname{cap}(T_1,P,Q) > 0$. Suppose $j \geq 2$ and j even. Provided $\operatorname{ds}_{P,Q} T_j \geq \epsilon$, Lemmas 5.3 and 5.5 implies $\operatorname{cap}(T_j,P,Q) \geq e^{.3\min\{p_n,\epsilon\}} \operatorname{cap}(T_j,P,Q)$. If j is odd and $j \geq 3$, then Lemma 5.3 and Claim 5.5 applied to T^* with the roles of P and Q reversed and the roles of G and H reversed implies $\operatorname{cap}(T_j,P,Q) \geq e^{.3\min\{q_m,\epsilon\}} \operatorname{cap}(T_j,P,Q)$. By Claim 5.6, $\operatorname{cap}(T_1,P,Q) \leq 1$. Thus, Algorithm TOSI terminates in t iterations. $\square$

Corollary 5.2. *If $\operatorname{cap}(T,P,Q) > 0$, then T is approximately $(G_{E_\circ}, H_{F_\circ})$-scalable to $(I_V \to Q, I_W \to P)$.*

Proof. By Lemma 3.20, $\operatorname{cap}(T,P,Q) > 0$ implies $\operatorname{cap}(\underline{T},\underline{P},\underline{Q}) > 0$. By Theorem 5.1, $\underline{T}$ is approximately $(\underline{G}_{F_\circ(Q)}, \underline{H}_{F_\circ(\underline{P})})$-scalable to $(\underline{P} \to I_{\operatorname{supp} Q}, \underline{Q} \to I_{\operatorname{supp} P})$. Now Lemma 3.20 implies T is approximately $(G_{E_\circ}, H_{F_\circ})$-scalable to $(I_V \to Q, I_W \to P)$. $\square$

5.1.1 Capacity under scaling

Lemma 5.3. *If $h \in \mathrm{GL}(V)_{F_\circ}$ and $g \in \mathrm{GL}(W)_{E_\circ}$, then*

$$\operatorname{cap}(T_{g,h},P,Q) = \det(Q, g^\dagger g)\det(P, h^\dagger h)\operatorname{cap}(T,P,Q).$$

Proof.

$$\begin{aligned}
\operatorname{cap}(T_{g,h},P,Q) &= \inf_{x \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(Q, T_{g,h}(xPx^\dagger))}{\det(P, x^\dagger x)} \\
&= \inf_{x \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(Q, g^\dagger T(hxPx^\dagger h^\dagger))g}{\det(P, x^\dagger x)} \\
&= \inf_{y \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(Q, g^\dagger T(yPy^\dagger)g)}{\det(P, y^\dagger h^{-\dagger} h^{-1} y)} \\
&= \inf_{y \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(Q, g^\dagger g)\det(Q, T(yPy^\dagger))}{\det(P, y^\dagger y)\det(P, h^{-\dagger} h^{-1})} \\
&= \det(Q, g^\dagger g)\det(P, h^\dagger h) \inf_{y \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(Q, T(yPy^\dagger))}{\det(P, y^\dagger y)}.
\end{aligned}$$

The second two inequalities follow from 9 of Lemma 3.6, and the last from 10 of Lemma 3.6. $\square$

Lemma 5.4. *Suppose P and Q are invertible and $\operatorname{cap}(T,P,Q) > 0$. Then T and T^* both map positive-definite operators to positive-definite operators.*

Proof. First we prove the claim for T. We can rewrite

$$\begin{aligned}
\operatorname{cap}(T,P,Q) &= \inf_{h \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(Q, T(hPh^\dagger))}{\det(P, h^\dagger h)} \\
& \inf_{\tilde{h} \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(Q, T(\tilde{h}\tilde{h}^\dagger))}{\det(P, P^{-1/2}\tilde{h}^\dagger \tilde{h} P^{-1/2})} \\
&= \det(P,P) \inf_{h \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(Q, T(hh^\dagger))}{\det(P, h^\dagger h)}
\end{aligned} \tag{27}$$

$\det(P,P) \neq 0$, so $\operatorname{cap}(T,P,Q) > 0$ if and only if $\inf_{h \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(Q, T(hh^\dagger))}{\det(P, h^\dagger h)} > 0$.

For any $X \in \mathcal{S}_{++}(V)$, we can write $X = hh^\dagger$ by the existence of Cholesky decompositions. Since $\operatorname{cap}(T, P, Q) \neq 0$, $\det(Q, T(hh^\dagger)) > 0$. This implies $hh^\dagger$ must be nonsingular because $q_m > 0$.

We now prove the claim for T^*. Suppose $T^*(Y)$ is singular for $Y \succ 0$. Since $T^*(Y) = \sum_{i=1}^r A_i^\dagger Y A_i$,

$$\ker T^*(Y) \subset \bigcap_i \ker A_i := R.$$

Notice that (W, R) is a T-independent pair. Let $d = \dim R > 0$. For $c > 1$, let $h_c h_c^\dagger = c\pi_R + \pi_{R^\perp}$. Because $R \subset \ker A_i$, we have $A_i h_c h_c^\dagger A_i^\dagger = A_i \pi_{R^\perp} A_i^\dagger$ for all i, or $T(h_c h_c^\dagger) = T(\pi_{R^\perp})$. Then $\det(Q, T(h_c h_c^\dagger)) = \det(Q, T(\pi_{R^\perp}))$.

On the other hand, $h_c^\dagger h_c$ has the same spectrum as $h_c h_c^\dagger$, so it has all eigenvalues at least 1 and an eigenspace of eigenvalue c of dimension at least d. Since P is invertible, $p_n > 0$ and

$$\det(P, h_c^\dagger h_c) > c^{dp_n}.$$

Plugging h_c into 27 and letting $c \to \infty$ shows $\operatorname{cap}(T, P, Q) = 0$, a contradiction. □

Claim 5.5. *Suppose $T(P) = I_W$, $\operatorname{Tr} P = \operatorname{Tr} Q = 1$, $\operatorname{ds}_{P,Q} T \geq \epsilon$, and $h \in H_{F_\circ(P)}$ such that $h^\dagger T^*(Q) h = I$. Then*

$$\det(P, h^\dagger h) \geq e^{.3\min\{\epsilon, p_n\}}.$$

Proof. By 9 of Lemma 3.6, if $h^\dagger T^*(Q) h = I$, then

$$\det(P, h^\dagger h) = \frac{1}{\det(P, T^*(Q))}.$$

Thus, it is enough to show

$$\log\det(P, T^*(Q)) = \log\left(\prod_{i=1}^n (\det \eta_i T^*(Q) {\eta_i}^\dagger)^{\Delta p_i}\right) \leq -.3\min\{\epsilon, p_n\}.$$

For $i \in [n]$ and $j \in [i]$, let λ_{ij} be the j^{th} eigenvalue of $\eta_i T^*(Q) {\eta_i}^\dagger$. Then

$$\log\left(\prod_{i=1}^n (\det \eta_i T^*(Q) {\eta_i}^\dagger)^{\Delta p_i}\right) = \sum_{i=1}^n \Delta p_j \sum_{j=1}^i \log \lambda_{ij}.$$

Since $\sum_{i=1}^n i\Delta p_i = \operatorname{Tr} P = 1$, we may define a discrete random variable X by assigning probability a_i to λ_{ij}. Then

$$\mathbb{E}[X] = \sum_{i=1}^n a_i \operatorname{Tr} \eta_i T^*(Q) {\eta_i}^\dagger = \sum_{i=1}^n \operatorname{Tr} \Delta p_i {\eta_i}^\dagger \eta_i T^*(Q) = \operatorname{Tr} PT^*(Q) = \operatorname{Tr} T(P) Q = \operatorname{Tr} Q = 1,$$

and it is enough to show $\mathbb{E}[\log X] \leq -\min\{.3\epsilon, .3p_n\}$. By definition,

$$\operatorname{ds}_{P,Q} T = \sum_{i=1}^n \Delta p_i \operatorname{Tr}(\eta_i T^*(Q) {\eta_i}^\dagger - I_{F_i})^2 = \sum_{i=1}^n \Delta p_i \sum_{j=1}^i (\lambda_{ij} - 1)^2 = \mathbb{V}[X] \geq \epsilon.$$

If a concave function has high variance, then the expectation of the function should be strictly less than the function of the expectation. However, X may have outliers which limits our ability to use this to our advantage. We split into the case where all $\lambda_{ij} \leq 2$ and the case where there is some $\lambda_{ij} > 2$.

Define $\epsilon_1 = \mathbb{V}[X|X \leq 2]\Pr[X \leq 2]$, and $\epsilon_2 = \mathbb{V}[X|X > 2]\Pr[X > 2]$ so that $\mathbb{V}[X] = \epsilon_1 + \epsilon_2 \geq \epsilon$. If $0 < x \leq 2$, then $\log x \leq (x-1) - .3(x-1)^2$, and by convexity for $x > 2$, $\log x \leq (\log 2 - 1)(x-1) \leq .7(x-1)$. Hence,

$$\begin{aligned}
\mathbb{E}[\log X] &= \mathbb{E}[\log X|X \leq 2]\Pr[X \leq 2] + \mathbb{E}[\log X|X > 2]\Pr[X > 2] \\
&\leq \mathbb{E}[(X-1) - .3(X-1)^2|X \leq 2]\Pr[X \leq 2] + \mathbb{E}[.7(X-1)|X > 2]\Pr[X > 2] \\
&= -.3\epsilon_1 + \mathbb{E}[(X-1)|X \leq 2]\Pr[X \leq 2] + .7\mathbb{E}[(X-1)|X > 2]\Pr[X > 2] \\
&= -.3\epsilon_1 + \mathbb{E}[(X-1)] - .3\mathbb{E}[(X-1)|X > 2]\Pr[X > 2] \\
&= -.3\epsilon_1 - .3\mathbb{E}[(X-1)|X > 2]\Pr[X > 2] \\
&\leq -.3\epsilon_1 - .3\Pr[X > 2].
\end{aligned}$$

If $\Pr[X > 2] = 0$, then $\epsilon_1 = \mathbb{V}[X] \geq \epsilon$. Else, there is at least one $\lambda_{ii} \geq \lambda_{ij} > 2$. However, by Cauchy interlacing, $\lambda_{n1} \geq 2$, which occurs with probability $\Delta p_n = p_n$. Thus, if $\Pr[X > 2] > 0$, then $\Pr[X > 2] \geq p_n$. □

Claim 5.6 (Capacity upper bound)**.** *Suppose $T(P) = I$ or $T^*(Q) = I$. Then* $\operatorname{cap}(T, P, Q) \leq 1$.

Proof. Plug in $h = I$. By definition,

$$\operatorname{cap}_H(T, P, Q) \leq \frac{\det(Q, T(I_V P I_V))}{\det(P, I_V)} = \prod_{i:\Delta q_i \neq 0}^{m} \left(\det \nu_i T(P) \nu_i^\dagger\right)^{\Delta q_i}$$

If λ_{ij} is the i^{th} eigenvalue of $\nu_i T(P) \nu_i^\dagger$, then if $T(P) = I_W$ we have $\sum_{i=1}^m \Delta q_i \sum_{j=1}^i \lambda_{ij} = \sum_{i=1} i\Delta q_i = \operatorname{Tr} Q = 1$. If $T^*(Q) = I_V$, then

$$\sum_{i=1}^m \Delta q_i \sum_{j=1}^i \lambda_{ij} = \sum_{i=1}^m \Delta q_i \operatorname{Tr} \nu_i T(P) {\nu_i}^\dagger = \sum_{i=1}^m \operatorname{Tr} \Delta q_i {\nu_i}^\dagger \nu_i T(P) = \operatorname{Tr} QT(P) = \operatorname{Tr} T^*(Q)P = \operatorname{Tr} P = 1,$$

In either case, the AM-GM inequality implies

$$\prod_{i:\Delta q_i \neq 0}^{m} \left(\det \nu_i T(P) \nu_i^\dagger\right)^{\Delta q_i} = \prod_{i:\Delta q_i \neq 0}^{m} \left(\prod_{j=1}^{i} \lambda_{ij}\right)^{\Delta q_i} \leq 1.$$

□

5.1.2 Number of iterations of Algorithm TOSI

For the proofs of Theorems 3.8 and 3.9, it was enough to show that Algorithm TOSI terminates. In order to obtain an algorithm, however, we must bound the number of iterations. Even if we do bound the number of iterations, numeircal issues remain, which will be handled in Appendix 12.4.

Assumption 1. T has at most nm Kraus operators with complex entries of the form $a + ib$ where a and b are signed $\leq \log M$ digit binary numbers. In Appendix 12.3 we show the assumption that there are at most nm Kraus-operators is without loss of generality.

In order to use the guarantees from the previous subsection to bound the running time of Algorithm TOSI, we must bound the capacity above and below. The upper bound is fairly easy, but our lower bound requires both log-concavity of our capacity and the reduction to the doubly stochastic case. For this we will need to use a nontrivial lower bound on $\operatorname{cap} T$ which was more or less implicit in the proof of Theorem 2.21 in [**GGOW16**].

Theorem 5.7 ([**GGOW16**])**.** *If $T : \operatorname{Mat}_N \mathbb{C} \to \operatorname{Mat}_N \mathbb{C}$ is a completely positive operator with Kraus operators $A_1 \dots A_R$ with Gaussian integer entries, then*

$$\operatorname{cap} T \geq e^{-N \log(RN^4)}.$$

Proof. This is implicit in the proof of Theorem 2.21 in [**GGOW16**], but one of the bounds used there has since been improved. Here we discuss the improved bound.

Definition 5.3. Let $T : \operatorname{Mat}_N \mathbb{C} \to \operatorname{Mat}_N \mathbb{C}$ be a completely positive map with Kraus operators $A_1 \dots A_R$. $\sigma(N, R)$ denotes the minimal d such that T is rank-nondecreasing if and only if the polynomial $p : \operatorname{Mat}_{N-1}(\mathbb{C})^R \to \mathbb{C}$ given by

$$p(R_1, \dots, R_r) = \det\left(\sum_{i=1}^{R} A_i \otimes R_i\right)$$

is not identically zero.

It is surprising that $\sigma(N, R)$ even exists; the bound $\sigma(N, R) \leq (N+1)!$ was used in [**GGOW16**], but a better bound appeared afterwards.

Theorem 5.8 ([**DM17**])**.** $\sigma(N, R) \leq N - 1$.

Let $d = \sigma(N, R)$. In the proof of Theorem 2.21 of [**GGOW16**], the authors show that if T satisfies the hypotheses of Thorem 5.7 then

$$\operatorname{cap} T \geq (\operatorname{cap} T')^{1/d} \tag{28}$$

where $T' : \operatorname{Mat}_{Nd} \mathbb{C} \to \operatorname{Mat}_{Nd} \mathbb{C}$ is a completely positive map with Gaussian integer entries and whose Kraus operators contain in their span an invertible matrix. They then apply their Lemma 2.17. Implicit in the proof of Lemma 2.17 is a proof of the following:

Lemma 5.9. *Let $T : \operatorname{Mat}_N(\mathbb{C}) \to \operatorname{Mat}_N(\mathbb{C})$ be a completely positive map with Kraus operators $A_1, \dots, A_R$ which are $N \times N$ matrices with* Gaussian *integer entries. Further suppose the subspace of matrices spanned by $A_1 \dots A_R$ contains a non-singular matrix. Then*

$$\operatorname{cap}(T) \geq \frac{1}{(RN^2)^N}.$$

In fact, they prove this for *integer* Kraus operators rather than *Gaussian integer* Kraus operators, but the only place they use the hypothesis that A_i are integer matrices is in showing $|\det(\sum_{i=1}^R z_i A_i)| \geq 1$ if z_i are integers and $\sum_{i=1}^R z_i A_i$ is nonsingular. However, this still holds if A_i are Gaussian integer matrices because the Gaussian integers are a ring in which every nonzero element has magnitude at least 1.

Applying Lemma 5.9 to T' in Equation 28 and using the bound $d \le N$ we have

$$\begin{aligned}\operatorname{cap}(T) \ge (\operatorname{cap} T')^{1/d} &\ge \left(\frac{1}{(R(Nd)^2)^{Nd}}\right)^{1/d} \\ &\ge e^{-N\log(RN^4)}\end{aligned}$$

□

Theorem 5.10 (Capacity lower bound)**.** *Suppose T satisfies Assumption 1 and P, Q are as in 3.6. If T is (P,Q)-rank-nondecreasing, then*

$$\operatorname{cap}(T,p,q) \ge e^{-5(n+m)\log(n+m)-2\log M}.$$

Proof. First note that if T' is the completely positive map obtained by scaling the Kraus operators of T by M, then the Kraus operators of T' have Gaussian integer entries and

$$\operatorname{cap}(T',p,q) = M^2 \operatorname{cap}(T,p,q).$$

We now bound $\operatorname{cap}(T',p,q)$ from below. Denote by $\mathcal{K}_1(T,E_\bullet,F_\bullet)$ the bounded convex polytope $\mathcal{K}(T,E_\bullet,F_\bullet) \cap \{(p,q) : \sum p_j = \sum q_i = 1\}$. By Proposition 6.1, $e^{H(p)}\operatorname{cap}(T',p,q)$ is log-concave in (p,q), so $e^{H(p)}\operatorname{cap}(T',p,q)$ takes a minimum at some vertex (p^*,q^*) of $\mathcal{K}_1(T,E_\bullet,F_\bullet)$, or

$$\operatorname{cap}(T',p,q) \ge e^{H(p^*)-H(p)}\operatorname{cap}(T',p^*,q^*) \ge \frac{1}{n}\operatorname{cap}(T',p^*,q^*). \tag{29}$$

By Lemma 7.4, each vertex of $\mathcal{K}_1(T,E_\bullet,F_\bullet)$ is the solution to the $n+m$ affine equations $\operatorname{Tr} P = \operatorname{Tr} Q = 1$ and

$$\sum_{i\in I} q_i + \sum_{j\in J} p_j = 1$$

for some choice of $(I_1,J_1),\dots(I_{n+m-2},J_{n+m-2}) \in S_T$. Thus, the vertices are of the form $(p,q) = x$ such that $Ax = b$ for an invertible matrix $A \in \operatorname{Mat}_{(n+m)\times(n+m)}(\{0,\pm1\})$ and $b = (1,1,0_{m+n-2})^\dagger$. $x = A^{-1}b$, which is the sum of two columns of A^{-1}. Each entry of this vector has the form $(\det M_1 + \det M_2)/\det A$, where M_1 and M_2 are $(n+m-1)\times(n+m-1)$ minors of A. Since A is an invertible matrix with entries in $\{0,\pm1\}$, its determinant is at most $(n+m)^{\frac{n+m}{2}}$ in absolute value by Hadamard's inequality. Thus, each entry of (p^*,q^*) can be written as a ratio c/γ of integers where γ is of absolute value at most $(n+m)^{\frac{n+m}{2}}$. By Claim 6.3,

$$\operatorname{cap}(T',p^*,q^*) = \frac{1}{\gamma}\left(\operatorname{cap}\operatorname{Trun}_{\gamma P^*,\gamma Q^*} T'\right)^{1/\gamma} \tag{30}$$

The Kraus operators of $\operatorname{Trun}_{\gamma P^*,\gamma Q^*} T'$ are $\gamma\times\gamma$ Gaussian integer matrices, and there are $R \le \gamma^2 mnp_1^*q_1^*$ of them by Assumption 1. By Theorem 5.7,

$$\frac{1}{\gamma}\left(\operatorname{cap}\operatorname{Trun}_{\gamma P^*,\gamma Q^*} T'\right)^{1/\gamma} \ge \frac{1}{\gamma}e^{-\log(mnp_1^*q_1^*\gamma^6)} = \frac{1}{mnp_1^*q_1^*\gamma^7} \ge \frac{1}{mn\gamma^7}.$$

Combining 29, 30 and the inequality directly above, we obtain $\operatorname{cap}(T',p,q) \ge \frac{1}{mn^2\gamma^7} \ge \frac{1}{\gamma^{10}}$. Now

$$\operatorname{cap}(T,p,q) \ge \frac{1}{M^2\gamma^{10}} \ge e^{-5(n+m)\log(n+m)-2\log M}.$$

□

We also need to ensure that the capacity does not decrease too much after the first step of Algorithm TOSI.

Lemma 5.11. *Suppose $(p, q) \in \mathcal{K}_1(T, E_\bullet, F_\bullet)$ and T satisfies Assumption 1 and T_1 is the operator obtained from the first step of Algorithm TOSI applied to T, P, Q. Then*

$$\operatorname{cap}(T_1, P, Q) \geq e^{-8(m+n)\log(m+n)-4\log M}.$$

Proof. By Lemma 5.3,

$$\operatorname{cap}(T_1, P, Q) = \operatorname{cap}(T_{g,h}, P, Q) = \det(Q, g^\dagger g)\operatorname{cap}(T, P, Q).$$

However, because $gT(P)g^\dagger = I$, 9 of Lemma 3.6 shows $\det(Q, g^\dagger g) = \det(Q, T(P))^{-1}$ and so

$$\operatorname{cap}(T_1, P, Q) = \det(Q, T(P))^{-1}\operatorname{cap}(T, P, Q).$$

Thus, it is enough to bound $\det(Q, T(P))^{-1}$. By Lemma 12.16, $T(P) \preceq m^2n^2M^2I$, so

$$\det(Q, T(P))^{-1} \geq m^{-2}n^{-2}M^{-2}.$$

Multiplying the above by the bound from Lemma 5.10 easily implies Lemma 5.11.

□

Now we can just plug the bound from Lemma 5.11 into Theorem 5.1.

Corollary 5.12. *Suppose T satisfies Assumption 1, $\operatorname{Tr} P = \operatorname{Tr} Q = 1$, and T is (P, Q)-rank-nondecreasing. Then Algorithm TOSI terminates in at most*

$$\frac{56(m+n)\log(m+n) + 28\log M}{\min\{\epsilon, p_n\} + \min\{\epsilon, q_m\}}$$

steps.

6 Rank-nondecreasingness implies positive capacity

As stated at the beginning of this section, we use the reduction for this part of the proof. We also need a nice concavity property of $\operatorname{cap}(T, P, Q)$. For this subsection we assume only that $\operatorname{Tr} P = \operatorname{Tr} Q$ without assuming $\operatorname{Tr} P = 1$. Define

$$\mathcal{K}'(T, E_\bullet, F_\bullet) := \{(p, q) : \operatorname{cap}(T, p, q) > 0\} = \{(p, q) : \log\operatorname{cap}(T, p, q) > -\infty\}.$$

The next proposition and claim show $\mathcal{K}'(T, E_\bullet, F_\bullet)$ is a convex cone.

Proposition 6.1. *The function*

$$(p, q) \mapsto e^{H(p)}\operatorname{cap}(T, p, q)$$

is log*-concave in (p, q). Here $H(p)$ is the Shannon entropy of p. In particular, $\mathcal{K}'(T, E_\bullet, F_\bullet)$ is a convex subset of $\mathbb{R}^{m+n}$.*

We use an alternate expression for the capacity to prove the proposition.

Claim 6.2.

$$\operatorname{cap}(T,p,q) = e^{-H(p)} \inf_{h \in \mathrm{GL}(V)_{F_\bullet}} \frac{\det(Q, T(hh^\dagger))}{\det(P, h^\dagger h)}$$

Proof. Choose $x \in \mathrm{GL}(V)_{F_\circ(P)}$ such that $\eta x^\dagger \eta^\dagger \eta x \eta^\dagger = \eta^\dagger P \eta^\dagger$. By the uniqueness of the block Cholesky decomposition, $\eta x \eta^\dagger$ is $\eta\sqrt{P}\eta^\dagger$ up to a unitary commuting with $\eta\sqrt{P}\eta^\dagger$ (fixing the eigenspaces of $\eta\sqrt{P}\eta^\dagger$). In particular, $\eta x \eta^\dagger \eta x^\dagger \eta^\dagger = \eta^\dagger P \eta^\dagger$ and so

$$x\eta^\dagger \eta x^\dagger = \eta^\dagger \eta x \eta^\dagger \eta x^\dagger \eta^\dagger \eta = \eta^\dagger \eta^\dagger P \eta^\dagger \eta = P.$$

Under the change of variables $\tilde{h} = hx$, we will have

$$\begin{aligned}
\operatorname{cap}(T,p,q) &= \inf_{h \in \mathrm{GL}(V)_{F_\circ(p)}} \frac{\det(Q, T(hPh^\dagger))}{\det(P, h^\dagger h)} \\
&= \inf_{\tilde{h} \in \mathrm{GL}(V)_{F_\circ(p)}} \frac{\det(Q, T(\tilde{h}\eta^\dagger \eta \tilde{h}^\dagger))}{\det(P, x^{-\dagger}\tilde{h}^\dagger \tilde{h} x^{-1})}, \\
&= \inf_{\tilde{h} \in \mathrm{GL}(V)_{F_\circ(p)}} \det(x^\dagger x, P) \frac{\det(Q, T(\tilde{h}\eta^\dagger \eta \tilde{h}^\dagger))}{\det(P, \tilde{h}^\dagger \tilde{h})}.
\end{aligned}$$

by 9 and 10 of Lemma 3.6. One can easily see that $\det(X,P) = \det(\eta^\dagger \eta X \eta^\dagger \eta, P)$, so

$$\begin{aligned}
\operatorname{cap}(T,p,q) &= \inf_{\tilde{h} \in \mathrm{GL}(V)_{F_\circ(p)}} \det(\eta^\dagger \eta x^\dagger x \eta^\dagger \eta, P) \frac{\det(Q, T(\tilde{h}\eta^\dagger \eta \tilde{h}^\dagger))}{\det(P, \tilde{h}^\dagger \tilde{h})} \\
&= \inf_{\tilde{h} \in \mathrm{GL}(V)_{F_\circ(p)}} \det(\eta^\dagger \eta P \eta^\dagger \eta, P) \frac{\det(Q, T(\tilde{h}\eta^\dagger \eta \tilde{h}^\dagger))}{\det(P, \tilde{h}^\dagger \tilde{h})} \\
&= e^{-H(p)} \inf_{\tilde{h} \in \mathrm{GL}(V)_{F_\circ(p)}} \frac{\det(Q, T(\tilde{h}\eta^\dagger \eta \tilde{h}^\dagger))}{\det(P, \tilde{h}^\dagger \tilde{h})}.
\end{aligned}$$

See Remark 3.7 for a discussion of the identity $\det(P,P) = e^{-H(p)}$, although it is a straightforward calculation. We can also get rid of the $\eta^\dagger \eta$ in the numerator:

$$\begin{aligned}
\inf_{\tilde{h} \in \mathrm{GL}(V)_{F_\circ(p)}} \frac{\det(Q, T(\tilde{h}\eta^\dagger \eta \tilde{h}^\dagger))}{\det(P, \tilde{h}^\dagger \tilde{h})} &\leq \inf_{\tilde{h} \in \mathrm{GL}(V)_{F_\circ(p)}} \frac{\det(Q, T(\tilde{h}\tilde{h}^\dagger))}{\det(P, \tilde{h}^\dagger \tilde{h})} \qquad (31) \\
&\leq \underline{\inf_{\tilde{h} \in \mathrm{GL}(V)_{F_\circ(p)}}} \liminf_{x^\dagger x \to \eta^\dagger \eta} \frac{\det(Q, T(\tilde{h} x x^\dagger \tilde{h}^\dagger))}{\det(P, x^\dagger \tilde{h}^\dagger \tilde{h} x)} \\
&= \underline{\inf_{\tilde{h} \in \mathrm{GL}(V)_{F_\circ(p)}}} \liminf_{x^\dagger x \to \eta^\dagger \eta} \frac{\det(Q, T(\tilde{h} x x^\dagger \tilde{h}^\dagger))}{\det(P, \tilde{h}^\dagger \tilde{h}) \det(P, x^\dagger x)} \\
&= \inf_{\tilde{h} \in \mathrm{GL}(V)_{F_\circ(p)}} \frac{\det(Q, T(\tilde{h}\eta^\dagger \eta \tilde{h}^\dagger))}{\det(P, \tilde{h}^\dagger \tilde{h})}.
\end{aligned}$$

In the above, $xx^\dagger \to \eta^\dagger \eta$ because $xx^\dagger$ and $x^\dagger x$ have the same spectrum and because $x^\dagger$ fixes $\ker P$. This implies

$$xx^\dagger (I - \eta^\dagger \eta) = x(I - \eta^\dagger \eta) x^\dagger \to 0,$$

which means $xx^\dagger$ tends to a projection to $\operatorname{supp} P$. The chain of inequalities shows 31 is actually an *equality*, and so

$$\operatorname{cap}(T,p,q) = e^{-H(p)} \inf_{\tilde{h}\in \mathrm{GL}(V)_{F_\circ(p)}} \frac{\det(Q, T(\tilde{h}\tilde{h}^\dagger))}{\det(P, \tilde{h}^\dagger\tilde{h})}.$$

Finally, observe that

$$\inf_{\tilde{h}\in \mathrm{GL}(V)_{F_\circ(p)}} \frac{\det(Q, T(\tilde{h}\tilde{h}^\dagger))}{\det(P, \tilde{h}^\dagger\tilde{h})} = \inf_{\mathrm{GL}(V)_{F_\bullet}} \frac{\det(Q, T(hh^\dagger))}{\det(P, h^\dagger h)}$$

because by the existence and uniqueness of Cholesky decompositions we can find a unitary U commuting with P such that $h = \tilde{h}U$ is upper triangular in the basis E; replacing $\tilde{h}$ by h preserves $\det(Q, T(\tilde{h}\tilde{h}^\dagger))$ and $\det(P, \tilde{h}^\dagger\tilde{h})$. □

Proof of Proposition 6.1. The function $e^{H(p)}\operatorname{cap}(T,p,q)$ is just $\log\inf_{\mathrm{GL}(V)_{F_\bullet}} \frac{\det(Q,T(hh^\dagger))}{\det(P,h^\dagger h)}$, which is almost manifestly log-convex in p and q. To see why, write

$$\begin{aligned}
&\log \inf_{h\in \mathrm{GL}(V)_{F_\bullet}} \frac{\det(Q, T(hh^\dagger))}{\det(P, h^\dagger h)} \\
&= \inf_{h\in \mathrm{GL}(V)_{F_\bullet}} \log \frac{\det(Q, T(hh^\dagger))}{\det(P, h^\dagger h)} \\
&= \inf_{h\in \mathrm{GL}(V)_{F_\bullet}} \sum_{i=1}^m \Delta q_i \log\det \nu_i T\left(hh^\dagger\right)\nu_i^\dagger - \sum_{j=1}^n \Delta p_j \log\det \eta_j h^\dagger h \eta_j^\dagger
\end{aligned}$$

with the convention $0\log 0 = 0$. Note that

$$\sum_{i=1}^m \Delta q_i \log\det \nu_i T\left(hh^\dagger\right)\nu_i^\dagger - \sum_{j=1}^n \Delta p_j \log\det \eta_j h^\dagger h \eta_j^\dagger$$

is of the form $f : \mathbb{R}_+^{m+n} \times U \to \mathbb{R}\cup\{-\infty\}$ defined by

$$f : (x,u) \mapsto x \cdot g(u)$$

where U is some set (in this case $U = \mathrm{GL}(V)_{F_\bullet}$). Here $g(u)$ may have some coordinates that are $-\infty$, but our convention is to treat $x_i g(u)_i = 0$ if $x_i = 0$. We claim that

$$f(\lambda x + (1-\lambda)x', u) = \lambda f(x,u) + (1-\lambda) f(x', u) \tag{32}$$

holds in the extended reals for $\lambda \in (0,1)$. Note that $(\lambda x + (1-\lambda)x')_i > 0$ if and only if $x_i > 0$ or $x'_i > 0$, so the coordinates where both $x_i, x'_i = 0$ whenever are effectively ignored so 32 holds whenever the coordinates where $g(u)$ is negative infinity are zero in x and x'. The only case remaining is where some coordinate i has $x_i > 0$ or $x'_i > 0$ and $g(u)_i = -\infty$. However, in this case, both sides of 32 are $-\infty$ in that case.

Using 32, we find that $x \mapsto \inf_{u\in U} f(x,u)$ is a concave function of x, because

$$\begin{aligned}
\inf_{u\in U} f(\lambda x_1 + (1-\lambda)x_2, u) &= \inf_{u\in U} (\lambda f(x_1,u) + (1-\lambda) g(x_2,u)) \\
&\geq \lambda \inf_{u\in U} f(x_1,u) + (1-\lambda) \inf_{u\in U} f(x_2,u)
\end{aligned}$$

for $\lambda \in (0,1)$. □

Claim 6.3.

1. *$\mathcal{K}'(T,E_\bullet,F_\bullet)$ is a cone. In particular,* $\operatorname{cap}(T,P,Q) = \gamma^{-\operatorname{Tr} Q}\operatorname{cap}(T,\gamma P,\gamma Q)^{\frac{1}{\gamma}}$ *for all* $\gamma > 0$.

2. *If γP and γQ have integral spectra, then* $\operatorname{cap}(T,P,Q) = \gamma^{-\operatorname{Tr} Q}\left(\operatorname{cap}\operatorname{Trun}_{\gamma P,\gamma Q} T\right)^{1/\gamma}$.

Proposition 6.4. *If T is (P,Q)-rank-nondecreasing, then $\operatorname{cap}(T,P,Q) > 0$. Equivalently,*

$$\mathcal{K}'(T,E_\bullet,F_\bullet) \supset \mathcal{K}(T,E_\bullet,F_\bullet) := \{(p,q) : T \text{ is } (P,Q)\text{-rank-nondecreasing}\}.$$

Proof of Claim 6.3. The proof of the first item is a straightforward calculation.

$$\begin{aligned}
&\gamma^{-\operatorname{Tr} Q}\operatorname{cap}(T,\gamma P,\gamma Q)^{\frac{1}{\gamma}}\\
&= \gamma^{-\operatorname{Tr} Q}\left(\inf_{h\in \mathrm{GL}(V)_{F_\circ}} \frac{\prod_{j=1}^m (\det \nu_j T(h\gamma P h^\dagger)\nu_j^\dagger)^{\gamma(q_j-q_{j+1})}}{\prod_{i=1}^n (\det \eta_i h^\dagger h \eta_i^\dagger)^{\gamma(p_i-p_{i+1})}}\right)^{\frac{1}{\gamma}}\\
&= \gamma^{-\operatorname{Tr} Q}\left(\inf_{h\in \mathrm{GL}(V)_{F_\circ}} \gamma^{\gamma\sum_{i=1}^m i(q_i-q_{i+1})}\left(\frac{\det(Q,T(hPh^\dagger))}{\det(P,h^\dagger h)}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\\
&= \gamma^{-\operatorname{Tr} Q}\left(\inf_{h\in \mathrm{GL}(V)_{F_\circ}} \gamma^{\gamma\operatorname{Tr} Q}\left(\frac{\det(Q,T(hPh^\dagger))}{\det(P,h^\dagger h)}\right)^{\gamma}\right)^{\frac{1}{\gamma}}\\
&= \operatorname{cap}(T,P,Q).
\end{aligned}$$

The second item follows from the first item here and the first item of Theorem 4.3:

$$\begin{aligned}
\operatorname{cap}(T,P,Q) &= \gamma^{-\operatorname{Tr} Q}(\operatorname{cap}(T,\gamma P,\gamma Q))^{1/\gamma}\\
&= \gamma^{-\operatorname{Tr} Q}\left(\operatorname{cap}\operatorname{Trun}_{\gamma P,\gamma Q} T\right)^{1/\gamma}
\end{aligned}$$

□

Proof. Suppose $(p,q)\in\mathcal{K}'(T,E_\bullet,F_\bullet)\cap\mathbb{Z}^{m+n}$. By Claim 6.3, $\operatorname{cap}\operatorname{Trun}_{P,Q} T > 0 \iff \operatorname{cap}(T,p,q) > 0$, so $\operatorname{cap}\operatorname{Trun}_{P,Q} T > 0$. By Gurvits [**Gu04**], $\operatorname{cap}\operatorname{Trun}_{P,Q} T > 0$ if and only if $\operatorname{Trun}_{P,Q} T$ is rank non-decreasing. By Theorem 4.3, T is (P,Q)-rank-nondecreasing. By the first part of of Claim 6.3, $\mathcal{K}'(T,E_\bullet,F_\bullet)$ is a cone, so

$$\mathcal{K}'(T,E_\bullet,F_\bullet)\cap\mathbb{Q}^{m+n} = \mathbb{Q}(\mathcal{K}'(T,E_\bullet,F_\bullet)\cap\mathbb{Z}^{m+n}) = \mathcal{K}(T,E_\bullet,F_\bullet)\cap\mathbb{Q}^{m+n}.$$

The vertices of $\mathcal{K}(T,E_\bullet,F_\bullet)\cap\{(p,q):\sum p_i = 1\}$ are rational (each is the solution of a system of integer linear equations), so $\mathcal{K}(T,E_\bullet,F_\bullet)\cap\mathbb{Q}^{m+n}\cap\{(p,q):\sum p_i=1\}$ contains the vertices of $\mathcal{K}(T,E_\bullet,F_\bullet)\cap\{(p,q):\sum p_i = 1\}$. Because $\mathcal{K}'(T,E_\bullet,F_\bullet)$ is a convex cone, we have $\mathcal{K}'(T,E_\bullet,F_\bullet)\supset\mathcal{K}(T,E_\bullet,F_\bullet)$. □

7 Scalability implies rank-nondecreasingness

As our final step in the proof of Theorem 3.8, we show directly that T is approximately $(\mathrm{GL}(W)_{E_\circ},\mathrm{GL}(V)_{F_\circ})$-scalable to $(I_V\to Q, I_W\to P)$ only if T is (P,Q)-rank-nondecreasing. This suffices because

$$((\mathrm{GL}(W)_{E_\circ},\mathrm{GL}(V)_{F_\circ},P,Q,T)$$

is always block-diagonal.

The message of the next lemma is that if $T_{g,h}$ is very nearly a block-upper-triangular scaling that maps $(I_V\to Q, I_W\to P)$, then $T_{g,h}$ is very nearly (P,Q)-rank-nondecreasing.

Lemma 7.1. *Suppose P and Q are positive semidefinite operators and that there exists an ϵ-$(I_V \to Q, I_W \to P)$-scaling of T by $(\mathrm{GL}(W)_{E_\circ}, \mathrm{GL}(V)_{F_\circ})$. Then $|\operatorname{Tr} Q - \operatorname{Tr} P| \le (\sqrt{n} + \sqrt{m})\epsilon$ and*

$$\sum_{i \in \sigma(E_\circ)} \Delta q_i \dim E_i \cap L + \sum_{j \in \sigma(F_\circ)} \Delta p_i \dim F_j \cap R \le \operatorname{Tr} P + (\sqrt{n} + \sqrt{m})\epsilon \tag{33}$$

for every T-independent pair (L, R).

Corollary 7.2, the second part in particular, is the main result of this section.

Corollary 7.2. *If ϵ is smaller than the minimum nonzero number among*

$$\left\{ \frac{1}{\sqrt{m} + \sqrt{n}} \left| \operatorname{Tr} P - \sum_{i \in I} q_i - \sum_{j \in J} p_j . \right| : I \subset [m], J \subset [n] \right\}, \tag{34}$$

and there exists an ϵ-$(I_V \to Q, I_W \to P)$-scaling of T by $(\mathrm{GL}(W)_{E_\circ}, \mathrm{GL}(V)_{F_\circ})$, then T is (P,Q)-rank-nondecreasing.

In particular, if T is approximately $(\mathrm{GL}(V)_{F_\circ}, \mathrm{GL}(W)_{F_\circ})$-scalable to $(I_V \to Q, I_W \to P)$, then T is (P,Q)-rank-nondecreasing.

Lemma 7.1 and Corollary 7.2 are proved in 7.2.

7.1 Schubert cells and varieties

Definition 7.1 (Schubert cells and varieties)**.** If $F_\bullet$ is a complete flag and $I = \{i_1 < \cdots < i_k\}$ is a k-tuple, define the Schubert cell $\Omega_I^\circ(F_\bullet)$ by

$$\Omega_I^\circ(F_\bullet) = \{L : \dim L \cap F_{i_j} = j, \ \forall j \in [k]\} \tag{35}$$

and the Schubert variety $\Omega_I^\bullet(F_\bullet)$ by

$$\Omega_I^\bullet(F_\bullet) = \{L : \dim L = k \text{ and } \dim L \cap F_{i_j} \ge j, \ \forall j \in [k]\}. \tag{36}$$

One can think of I as the locations of the "jumps" in the sequence $(\dim L \cap F_j)_{j \in [n]}$. We collect a few standard facts about Schubert cells and varieties.

Fact 7.3.

1. $I = \{i : \dim L \cap F_i - \dim L \cap F_{i-1} > 0\}$ if and only if $L \in \Omega_I^\circ(F_\bullet)$. Hence, every subspace is in exactly one Schubert cell.
2. Each Schubert variety is partitioned by the Schubert cells contained within it.
3. $\Omega_I^\bullet(E_\bullet) \supset \Omega_{I'}^\circ(E_\bullet)$ if and only if $I = \{i_1, \ldots, i_k\}$ and $|I' \cap [i_j]| \ge j$ for all $j \in [k]$.
4. If $L \in \Omega_I^\circ(E_\bullet)$ and $E_\circ$ is a q-partial subflag of $E_\bullet$, then

$$\sum_{i \in I} q_i = \sum_{i \in \sigma(E_\circ)} \Delta q_i \dim L \cap E_i.$$

It will be convenient to move back and forth between expressions like $\sum_{i=1}^n \Delta p_i \dim F_i \cap L$ and expressions like $\sum_{i \in I} p_i$.

Lemma 7.4. *As throughout the rest of the paper $F_\bullet$ extends $F_\circ$ and $E_\bullet$ extends $E_\circ$. The quantities*

$$\max \sum_{i \in \sigma(E_\circ)} \Delta q_i \dim E_i \cap L + \sum_{j \in \sigma(F_\circ)} \Delta p_i \dim F_j \cap R \tag{37}$$

such that (L, R) is T-independent,

$$\max \sum_{i \in I} q_i + \sum_{j \in J} p_i \tag{38}$$

such that $\Omega^\circ_I(E_\bullet) \times \Omega^\circ_J(F_\bullet)$ contains a T-independent pair, and

$$\max \sum_{i \in I} q_i + \sum_{j \in J} p_i \tag{39}$$

such that $\Omega^\bullet_I(E_\bullet) \times \Omega^\bullet_J(F_\bullet)$ contains a T-independent pair are equal.

Proof. 37 and 38 are equal by 4 of Fact 7.3. 38 is trivially less than 39. It remains to show 39 is at most 38.

By 2 of Fact 7.3, the subsets $I \subset [m]$ and $J \subset [n]$ such that $\Omega^\bullet_I(E_\bullet) \times \Omega^\bullet_J(F_\bullet)$ contains a T-independent pair is exactly the set of $\Omega^\bullet_I(E_\bullet) \times \Omega^\bullet_J(F_\bullet)$ such that $\Omega^\bullet_I(E_\bullet) \supset \Omega^\circ_{I'}(E_\bullet)$ and $\Omega^\bullet_J(F_\bullet) \supset \Omega^\circ_{J'}(F_\bullet)$ for some I', J' such that $\Omega^\circ_{I'}(E_\bullet) \times \Omega^\circ_{J'}(F_\bullet)$ contains a T-independent pair. By 3 of Fact 7.3 and the fact that p and q are non-increasing, this implies

$$\sum_{i \in I} q_i + \sum_{j \in J} p_i \le \sum_{i \in I'} q_i + \sum_{j \in J'} p_i.$$

In other words, it is enough to look at Schubert cells containing an independent pair, and so 39 is at most 38. □

7.2 Proof of Lemma 7.1

A linear-algebraic tool known as Rayleigh trace will also come in handy.

Definition 7.2 (Rayleigh trace)**.** If A is a $n \times n$ Hermitian matrix and L a linear subspace of $\mathbb{C}^n$, define the *Rayleigh trace* of L by

$$R_A(L) := \sum_{i=1}^{k} \langle A u_i, u_i \rangle$$

where $\{u_1, \dots u_k\}$ is an orthonormal basis for L. Note that the value of the Rayleigh trace is independent of the choice of orthonormal basis; in fact, if π_L is an orthogonal projection to L, then

$$R_A(L) = \operatorname{Tr} A \pi_L.$$

Fact 7.5 (See the survey **F00**)**.** Suppose A is an $n \times n$ Hermitian operator with eigenvalues $\alpha_1 \ge \cdots \ge \alpha_n$. Suppose $I \subset [n]$. Then

$$\sum_{i \in I} \alpha_i = \min_{L \in \Omega^\bullet_I(F_\bullet(A))} R_A(L).$$

Finally we are ready to prove Lemma 7.1.

Proof of Lemma 7.1. Suppose there exists an ϵ-$(I_V \to Q, I_W \to P)$-scaling of T by $(\mathrm{GL}(W)_{E_\circ}, \mathrm{GL}(V)_{F_\circ})$. Then we wish to show $|\operatorname{Tr} Q - \operatorname{Tr} P| \le (\sqrt{n} + \sqrt{m})\epsilon$ and

$$\sum_{i\in\sigma(E_\circ)} \Delta q_i \dim E_i \cap L + \sum_{j\in\sigma(F_\circ)} \Delta p_i \dim F_j \cap R \le \operatorname{Tr} P + (\sqrt{n} + \sqrt{m})\epsilon \tag{40}$$

for every T-independent pair (L, R). By Lemma 7.4, it is enough to show that $|\operatorname{Tr} Q - \operatorname{Tr} P| \le (\sqrt{n} + \sqrt{m})\epsilon$ and

$$\sum_{i\in I} q_i + \sum_{j\in J} p_j \le \operatorname{Tr} P + (\sqrt{n} + \sqrt{m})\epsilon \tag{41}$$

for all $I \subset [m]$ and $J \subset [n]$ such that the set $\Omega_I^\circ(F_\bullet(Q)) \times \Omega_J^\circ(F_\bullet(P))$ contains a T-independent pair. Firstly,

$$\begin{aligned} &|\operatorname{Tr} Q - \operatorname{Tr} P| \\ \le &|\operatorname{Tr}(Q - T_{g,h}(I))| + |\operatorname{Tr}(P - T^*_{h,g}(I))| \\ \le &(\sqrt{n} + \sqrt{m})\epsilon. \end{aligned}$$

Now suppose $(L, R) \in \Omega_I^\circ(F_\bullet(P)) \times \Omega_J^\circ(F_\bullet(Q))$ contains a T-independent pair. Then $(L, R) \in \Omega_I^\bullet(F_\bullet(P)) \times \Omega_J^\bullet(F_\bullet(Q))$ contains a T-independent pair (L, R). Then

$$(\underline{L}, \underline{R}) = (g^{-1}L, h^{-1}R)$$

is a $T_{g,h}$-independent pair, and $(\underline{L}, \underline{R}) \in \Omega_I^\bullet(g^{-1}F_\bullet(Q)) \times \Omega_J^\bullet(h^{-1}F_\bullet(P))$. However, $g^{-1}F_\bullet(Q))$ and $\Omega_J^\bullet(h^{-1}F_\bullet(P)$ are still valid choices of $F_\bullet(Q)$ and $F_\bullet(P)$ because $(g, h) \in \mathrm{GL}(W)_{F_\circ(Q)} \times \mathrm{GL}(V)_{F_\circ(P)}$. Since $(\underline{L}, \underline{R})$ is T-independent, $T^*_{g,h}(\pi_{\underline{L}})\pi_{\underline{R}} = 0$.

$$\begin{aligned} R_Q(\underline{L}) &= \operatorname{Tr} Q\pi_{\underline{L}} \\ &\le \operatorname{Tr} T_{g,h}(I)\pi_{\underline{L}} + \epsilon\sqrt{m} \\ &= \operatorname{Tr} T^*_{h,g}(\pi_{\underline{L}}) + \epsilon\sqrt{m} \\ &= \operatorname{Tr} T^*_{h,g}(\pi_{\underline{L}})(I - \pi_{\underline{R}}) + \epsilon\sqrt{m} \\ &\le \operatorname{Tr} T^*_{h,g}(I)(I - \pi_{\underline{R}}) + \epsilon\sqrt{m} \\ &\le \operatorname{Tr} P(I - \pi_{\underline{R}}) + (\sqrt{n} + \sqrt{m})\epsilon \\ &= \operatorname{Tr} P - R_P(\underline{R}) + (\sqrt{n} + \sqrt{m})\epsilon. \end{aligned}$$

By Fact 7.5,

$$\sum_{i\in I} q_i + \sum_{j\in J} p_j \le R_P(\underline{R}) + R_Q(\underline{L}) \le \operatorname{Tr} P + (\sqrt{n} + \sqrt{m})\epsilon.$$

□

Proof of Corollary 7.2. Corollary 7.2 follows from Lemma 7.4 and Lemma 7.1. □

8 Non-upper triangular scalings: Proof of Theorem 3.9

Recall the proof overview for Theorem 3.9 in 3.3. There we suggested that the only difficult step in the proof of Theorem 3.9 is the Lemma 8.1 below.

Lemma 8.1. *The set of pairs* $(B^\dagger, C)$ *such that* $T_{B,C}$ *is* (P,Q)*-rank-nondecreasing is the complement of an affine variety* $V(T,P,Q)$ *in* $\mathrm{Mat}_m(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C})$*. Further, if* P *and* Q *have integral spectra, then* $V(T,P,Q)$ *is generated by finitely many polynomials of degree at most* $2(\operatorname{Tr} P)^2$*.*

We defer the proof of Lemma 8.1 to after the statement of Proposition 8.5. Before movng on, we show how Lemma 8.1 proves Theorem 3.9.

Proof of Theorem 3.9 modulo Lemma 8.1. We first prove 3 $\Longrightarrow$ 1. If T is approximately (G,H)-scalable to $(I_V \to Q, I_W \to P)$, then by Corollary 7.2, there exists $(g,h) \in G \times H$ such that $T_{g,h}$ is (P,Q)-rank-nondecreasing. By Lemma 8.1,

$$\{(g^\dagger, h) \in G \times H : T_{g,h} \text{ is } (P,Q)\text{-rank-nondecreasing}\}$$

is nonempty and the complement of an affine variety in a complex vector space isomorphic to $\oplus_i \mathrm{Mat}_{n_i}(\mathbb{C}) \times \oplus_j \mathrm{Mat}_{m_j}(\mathbb{C})$ (this is from our assumption that (G,H,P,Q,T) is block-diagonal, and $G \times H$ itself is the complement of zeroes of the polynomial $(x,y) \mapsto \det(x)\det(y)$. By definition of generic, we have that $T_{g,h}$ is (P,Q)-rank-nondecreasing for generic $(g^\dagger, h) \in G \times H$. 1 $\Longrightarrow$ 2 follows from Theorem 3.8. Next we show 2 $\Longrightarrow$ 3. Suppose $\operatorname{cap}(T_{g,h}, P, Q) > 0$ for generic $(g^\dagger, h) \in G \times H$. In particular, there exists $(g,h) \in G \times H$ such that $\operatorname{cap}(T_{g,h}, P, Q) > 0$. By Theorem 3.8, $T_{g,h}$ is approximately $(G_{E_\circ}, F_{F_\circ})$-scalable to $(I_V \to Q, I_W \to P)$, so T is approximately $(gG_{E_\circ}, hH_{F_\circ})$-scalable to $(I_V \to Q, I_W \to P)$. Because $gG_{E_\circ} \times hH_{F_\circ} \subset G \times H$, T is approximately (G,H)-scalable to $(I_V \to Q, I_W \to P)$. $\square$

8.1 Rank-nondecreasingness is a generic property

Before getting into the proofs, we review a few preliminaries from algebraic geometry.

Definition 8.1 (Irreducible)**.** Recall that an affine variety in $\mathbb{C}^n$ is the common zeroes of a set of finitely many n-variate polynomials. An affine variety $\mathcal{V}$ is said to be *irreducible* if it is not equal to a union of any two proper subvarieties.

Definition 8.2 (Constructible)**.** *Constructible* subsets of $\mathbb{C}^n$ are those of the form

$$\cup_{\alpha \in S} V_\alpha \setminus E_\alpha$$

for algebraic varieties $E_\alpha \subsetneq V_\alpha$ indexed by a finite set S.

Theorem 8.2 (Chevalley; see **M99**)**.** *If a subset of* $\mathbb{C}^n$ *is constructible, then the image of that set under the projection to the last* $l < n$ *coordinates is also constructible.*

Fact 8.3 (see Aside 9.8 **MilneAG**)**.** The closure of a constructible subset of $\mathbb{C}^n$ in the Euclidean topology is the smallest affine variety containing the subset.

Remark 8.4. There are two equivalent ways to view the action of G and H - they can either act on the completely positive map T, or they can act on the flags $E_\bullet$ and $F_\bullet$. By a change of variables, $\Omega_I(E_\bullet) \times \Omega_J(F_\bullet)$ contains a $T_{g,h}$-independent pair if and only if $\Omega_I(gE_\bullet) \times \Omega_J(hF_\bullet)$ contains a T-independent pair.

Definition 8.3. We define each constraint and the set of constraints imposed for $T_{g,h}$ to be (P,Q)-rank-nondecreasing.

1. If $I \subset [m]$ and $J \subset [n]$ let $*IJ$ denote the condition

$$\sum_{i\in[m]} q_i = \sum_{j\in[n]} p_j = N \text{ and } \sum_{i\in I} q_i + \sum_{j\in J} p_j \leq N. \tag{*IJ}$$

   It follows that $\mathcal{K}(T, E_\bullet, F_\bullet)$ is the set of (p,q) such that the linear inequalities $*IJ$ hold for all (I,J) such that $\Omega_I^\bullet(E_\bullet) \times \Omega_I^\bullet(F_\bullet)$ contains a T-independent pair.

2. If $E_\bullet$ is a flag of $\mathbb{C}^m$ and $F_\bullet$ a flag of $\mathbb{C}^n$, then define

$$P(T, gE_\bullet, hF_\bullet) := \{(I,J) : \Omega_I^\bullet(E_\bullet) \times \Omega_J^\bullet(F_\bullet) \text{ contains a } T_{g,h}\text{-independent pair}\}.$$

   By Remark 8.4, we are justified in using $gE_\bullet$ and $hF_\bullet$ in our notation. One should think of $P(T, gE_\bullet, hF_\bullet)$ as the set of (I,J) such that $T_{g,h}$ has to satisfy *IJ to be $(p,q,E_\bullet,F_\bullet)$-rank-nondecreasing.

We proceed with the statement and proof of the proposition which contains Lemma 8.1. The third item of the proposition means $P(T, gE_\bullet, hF_\bullet)$ is as small as possible, meaning the linear inequality *IJ is required to hold for the smallest possible number of pairs (I,J), for almost every $(g,h) \in G \times H$.

Proposition 8.5.

1. *Let $V(T, E_\bullet, F_\bullet, I, J)$ be the set of pairs $(B^\dagger, C) \in \mathrm{Mat}_m(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C})$ such that*

$$\Omega_I^\bullet(E_\bullet) \times \Omega_J^\bullet(F_\bullet)$$

   contains a $T_{B,C}$-independent pair. Then $V(T, E_\bullet, F_\bullet, I, J)$ is an affine variety in $\mathrm{Mat}_m(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C})$.

2. *The set of pairs $(B^\dagger, C)$ such that $T_{B,C}$ is (P,Q)-rank-decreasing is an affine variety in $\mathrm{Mat}_m(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C})$. Denote this variety*

$$V(T,P,Q).$$

3. *Suppose $H = \bigoplus_i \mathrm{GL}_{n_i}(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$ and $G = \bigoplus_j \mathrm{GL}_{m_j}(\mathbb{C}) \subset \mathrm{GL}_m(\mathbb{C})$. Let*

$$P(T, GE_\bullet, HF_\bullet) = \bigcap_{(g,h)\in G\times H} P(T, gE_\bullet, hF_\bullet).$$

   Then for a generic pair $(g^\dagger, h) \in G^\dagger \times H$,

$$P(T, gE_\bullet, hF_\bullet) = P(T, GE_\bullet, HF_\bullet).$$

   That is, the set of pairs $(g^\dagger, h)$ such that $P(T, gE_\bullet, hF_\bullet) = P(T, GE_\bullet, HF_\bullet)$ is Zariski-open and nonempty in $G \times H$.

Let's see how the proposition implies Lemma 8.1.

Proof of Lemma 8.1. The assertion that the scalings which cause T not to be (P,Q)-rank-nondecreasing is a variety is 2 of Proposition 8.5 in 8.1. The bound on the degree follows from Theorem 8.8 which uses the reduction and a result of Derksen. $\square$

This proof seems to be a standard application of the fact that if V is a variety in $X \times Y$ and X is actually a projective variety, then the projection of V to Y is also a variety.

Proof. First we prove 1. Let $\vec{A} = (A_1, \dots, A_r)$, the tuple of Kraus operators of T. Before proving 1, we prove a claim:

Claim 8.6. *If $I \subset [m]$, and $J \subset [n]$, the set*

$$W(I,J) := \{\vec{A} : \Omega_I^\bullet(E_\bullet) \times \Omega_J^\bullet(F_\bullet) \text{ contains a } T\text{-independent pair }\}$$

is an affine variety.

Proof of claim. Let $\vec{A} = (A_1, \dots, A_r)$. Suppose $|I| = k$ and $|J| = l$, and that L and R are given as $m \times k$, $n \times l$ matrices of full column rank. For a fixed I, J the set $\tilde{W}(I,J)$ of $(L^\dagger, R, \vec{A})$ such that (L,R) is T-independent, $L \in \Omega_I^\bullet(E_\bullet)$ and $R \in \Omega_J^\bullet(F_\bullet)$ is constructible. This is because the membership of $L \in \Omega_I^\bullet(E_\bullet)$ and $R \in \Omega_J^\bullet(F_\bullet)$ is determined by the vanishing or non-vanishing of certain determinants of submatrices of L and R determined by I and J and (L,R) is T-independent if and only if $L^\dagger A_i R = 0$ for all $i \in [r]$. This implies $\tilde{W}(I,J) = V \setminus E$, where $E \subsetneq V$ are algebraic varieties.

By Theorem 8.2, the projection of $\tilde{W}(I,J)$ to the coordinates corresponding to $\vec{A}$, which is exactly $W(I,J)$, is also constructible. However, we claim $W(I,J)$ is closed. Consider a sequence $\vec{A_s} \in W(I,J)$ converging to $\vec{A}$. This implies there exist L_s, R_s such that $(L_s, R_s, \vec{A_s}) \in \tilde{W}(I,J)$. However, we may assume L_s, R_s have orthonormal columns, since only the column space of L_s and R_s determines membership of $(L_s, R_s, \vec{A_s})$ in $\tilde{W}(I,J)$. By compactness, we may assume L_s and R_s converge to L, R, which will also be matrices with orthonormal columns. L and R are guaranteed to have full column rank, so $(L, R, \vec{A})$ must be in $\tilde{W}(I,J)$ by continuity.

Since $W(I,J)$ is a (Euclidean) closed constructible set, Fact 8.3 implies it is in fact an affine variety. ∎

Next recall that $(g^\dagger, h) \in V(T, E_\bullet, F_\bullet, I, J)$ if and only if

$$\Omega_I^\bullet(E_\bullet) \times \Omega_J^\bullet(F_\bullet)$$

contains a $T_{g,h}$-independent pair. Equivalently,

$$g^\dagger \vec{A} h := (g^\dagger A_i h : i \in [r]) \in W(I,J).$$

Since $f : (g^\dagger, h) \to g^\dagger \vec{A} h$ is a regular map and $W(I,J)$ is an affine variety, $f^{-1}W(I,J)$ is an affine variety and so $V(I,J) = f^{-1}W(I,J) \cap \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ is Zariski-closed in $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$. This proves 1.

8.5 follows from 1, because $T_{B^\dagger, C}$ is $(p, q, E_\bullet, F_\bullet)$-rank-decreasing if and only if $(B^\dagger, C)$ is contained in the union of $V(T, E_\bullet, F_\bullet, I, J)$ over pairs (I,J) such that *IJ is violated, which is a union of finitely many affine varieties and hence itself an affine variety. The second part follows similarly, except we take the union of $V(T, E_\bullet, F_\bullet, I, J)$ over pairs (I,J) such that $I \neq [m], J \neq [n]$ and *IJ does not hold strictly, and $([m], J)$ and $(I, [n])$ if *IJ is violated for those pairs.

3 follows from 1, because if S is a set of pairs of subsets, then

$$\{(g^\dagger, h) : P(T, gE_\bullet, hF_\bullet) \subset S\} = \bigcap_{(I,J) \notin S} (G^\dagger \times H \setminus V(T, E_\bullet, F_\bullet, I, J)).$$

In particular, it is Zariski-open in $G^\dagger \times H$. If we let $S = P(T, GE_\bullet, HF_\bullet)$, each of the open sets in the above intersection is nonempty, because if $(I,J) \notin P(T, GE_\bullet, HF_\bullet))$ then there is some $g^\dagger, h$

such that $(I,J) \notin P(T, gE_\bullet, hF_\bullet)$ which means $V(T, E_\bullet, F_\bullet, I, J)$ does not contain $G^\dagger \times H$. Because $G \times H$ is a nonempty Zariski-open subset of

$$\oplus_i \operatorname{Mat}_{n_i}(\mathbb{C}) \times \oplus_j \operatorname{Mat}_{m_j}(\mathbb{C}) \cong \mathbb{C}^{\sum n_i^2 + \sum m_j^2},$$

the intersection of a finite number of nonempty Zariski-open sets with $G \times H$ is nonempty and Zariski-open, proving 3. $\square$

8.1.1 Degree bounds

Much is known about the Kraus operators of rank-nondecreasing operators. In Lemma 8.5, we already showed that $(B^\dagger, C)$ such that $T_{B,C}$ is (P,Q)-rank-decreasing is an affine variety in $\operatorname{Mat}_m(\mathbb{C}) \times \operatorname{Mat}_n(\mathbb{C})$. Here we bound the degree of this variety in case P and Q have integral spectra. We use a theorem of Derksen:

Theorem 8.7 (**DM17**)**.** *The set*

$$\{(A_1, \ldots, A_r) \in \operatorname{Mat}_{n\times n}(\mathbb{C})^r : T : X \mapsto \sum_{i=1}^r A_i X A_i^\dagger \text{ is rank-decreasing}\}$$

is an affine algebraic variety generated by polynomials of degree at most $n^2 - n$. In particular, T is rank decreasing if and only if the polynomial $p : \operatorname{Mat}_{n-1}(\mathbb{C})^r \to \mathbb{C}$ given by

$$p(R_1, \ldots, R_r) = \det\left(\sum_{i=1}^r A_i \otimes R_i\right)$$

is identically zero.

Theorem 8.8. *Suppose P and Q have integral spectra. Then $V(T,P,Q)$ is an affine algebraic variety in $\operatorname{Mat}_m(\mathbb{C}) \times \operatorname{Mat}_n(\mathbb{C})$ generated by polynomials of degree at most $2(\operatorname{Tr} P)^2$.*

Proof. By Theorem 4.3, if P and Q have integral spectra,

$$V(T,P,Q) = \{(B^\dagger, C) : \operatorname{Trun}_{P,Q} T_{B,C} \text{ is rank-decreasing}\}.$$

By Theorem 8.7, $\operatorname{Trun}_{P,Q} T_{B,C}$ is rank decreasing if and only if the polynomial

$$p(B^\dagger, C, (R_{s,i,j})_{(s,i,j)\in[r]\times[p_1]\times[q_1]}) := \det\left(\sum_{s\in[r]} \sum_{i\in[p_1]} \sum_{j\in[q_1]} \tilde{A}_s^{ij} \otimes R_{s,i,j}\right)$$

is identically zero as a polynomial in $(R_{s,i,j})_{(s,i,j)\in[r]\times[p_1]\times[q_1]}$. As the entries of $\tilde{A}_s^{ij}$ are bilinear in $B^\dagger$ and C, p_j is a polynomial of degree at most $2(\operatorname{Tr} P)^2$ in $B^\dagger$ and C and $3(\operatorname{Tr} P)^2$ in $B^\dagger, C, (R_{s,i,j})_{s,i,j}^{l,p_1,q_1}$.

The theorem now follows because for each fixed tuple

$$(R_{s,i,j})_{(s,i,j)\in[r]\times[p_1]\times[q_1]},$$

$p|_{(R_{s,i,j})_{(s,i,j)\in[r]\times[p_1]\times[q_1]}}$ is a polynomial of degree at most $2(\operatorname{Tr} P)^2$ in B and C. $\square$

8.2 The polytope of marginals

When the flags $F_\circ(P)$ and $F_\circ(Q)$ are subflags of fixed flags $E_\bullet$ and $F_\bullet$, the possible spectra of marginals to which T can be approximately scaled fall in a certain convex polytope.

Definition 8.4. Suppose $H = \oplus_i \mathrm{GL}_{n_i}(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$ and $G \oplus_j \mathrm{GL}_{m_j}(\mathbb{C}) \subset \mathrm{GL}_m(\mathbb{C})$. Let

$$\mathcal{K}(T, G, H) := \{(p, q) : (p, q) \text{ satisfies } *IJ \text{ for all } (I, J) \in P(T, GE_\bullet, HF_\bullet)\}.$$

$\mathcal{K}(T, G, H)$ is clearly a convex polytope. Since $K(T_{g,h}, E_\bullet, F_\bullet) = K(T, gE_\bullet, hF_\bullet)$ is defined by the set of constraints $P(T, gE_\bullet, hF_\bullet)$ and $P(T, GE_\bullet, HF_\bullet) = \bigcap_{g\in G, h\in H} P(T, gE_\bullet, hF_\bullet)$, we have

$$\mathcal{K}(T, G, H) = \bigcup_{g,h} \mathcal{K}(T_{g,h}, E_\bullet, F_\bullet).$$

Definition 8.5. In case G, H act transitively on $E_\bullet, F_\bullet$, we abbreviate $\mathcal{K}(T) := \mathcal{K}(T, G, H)$. We will often assume that the trace of P and Q is one; so we write

$$\mathcal{K}_1(T, G, H) = \mathcal{K}(T, G, H) \cap \left\{(p, q) : \sum_{i=1}^{n} p_i = \sum_{i=1}^{m} q_i = 1\right\}.$$

Similarly for $\mathcal{K}_1(T)$.

Finally we can show that approximate (G, H)-scalability is equivalent to the membership of (p, q) in a convex polytope. That is, we can show Theorem 3.10, which we have restated equivalently below. Corollary 8.9 is an immediate consequence of Theorem 3.9 and item 3 of Proposition 8.5.

Corollary 8.9. *If (G, H, P, Q, T) is block-diagonal, then T is approximately (G, H)-scalable to (P, Q) if and only if*

$$(p, q) \in \mathcal{K}(T, G, H).$$

8.3 Algorithm GOSI

Next we discuss an algorithm for finding scalings in G and H rather than $G_{E_\circ}$, $H_{F_\circ}$. We first scale by random integer matrices. By the Schwarz-Zippel Lemma and the degree bound in Lemma 8.1, with high probability this random scaling yields a (P, Q)-rank-nondecreasing operator. We can then perform Algorithm TOSI on the resulting operator.

Algorithm GOSI. [Generic operator Sinkhorn iterations]

Input: T, P, Q as in 3.6 where P and Q have rational spectra and a real number $\epsilon > 0$.

Output: Left and right scalings $g \in G$ and $h \in H$ such that $T_{g,h}$ is an ϵ-$(I_V \to Q, I_W \to P)$-scaling of T, or ERROR.

1. Let $0 < \gamma \in \mathbb{Z}$ be such that γP and γQ have integral spectra. Choose each entry of $(g_0, h_0) \in \oplus_i \mathrm{Mat}_{n_i}(\mathbb{C}) \times \oplus_j \mathrm{Mat}_{m_j}(\mathbb{C})$ uniformly at random from $K := [3 \max\{2\gamma^2, n, m\}]$. If g_0 or h_0 is singular, output ERROR.

2. Let p_{min}, q_{min} be the least nonzero entries of p, q, respectively. Let $\underline{g}$ and $\underline{h}$ be the output of Algorithm TOSI with input $\underline{T_{g_0,h_0}}, \underline{P}, \underline{Q}, \frac{\epsilon}{4 \min\{p_{min}, q_{min}\}}$. If any step of Algorithm TOSI on this input cannot be performed, output ERROR.

3. Use $(\underline{g}, \underline{h})$ to compute (g, h) such that $T_{g_0 g, h_0 h}$ is an ϵ-$(I_V \to Q, I_W \to P)$-scaling of T_{g_0,h_0} as guaranteed by Corollary 3.20. Return $(g_0 g, h_0 h)$.

Claim 8.10. *If $(G, H, F_\circ(P), F_\circ(Q), T)$ is block diagonal and T is approximately (G, H)-scalable to $(I_V \to Q, I_W \to P)$, Algorithm GOSI outputs ERROR with probability at most 1/3, and otherwise is correct.*

Proof. Suppose T is approximately (G, H)-scalable to $(I_V \to Q, I_W \to P)$. By Theorem 3.9, T_{g_0,h_0} is (P, Q)-rank-nondecreasing for generic $(g_0^\dagger, h_0) \in G \times H$. In particular, by Lemma 8.1,

$$V' = V(T, P, Q) \cup \{B : \det B = 0\} \cup \{C : \det C = 0\}$$

is an affine algebraic variety in $\mathrm{Mat}_m(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C})$ generated by polynomials of degree at most $\max\{2\gamma^2, n, m\}$ that does not contain all of $\overline{G} \times \overline{H}$, which are just $\overline{H} = \oplus_i \mathrm{Mat}_{n_i}(\mathbb{C}) \subset \mathrm{Mat}_n(\mathbb{C})$ and $\overline{G} = \oplus_j \mathrm{Mat}_{m_j}(\mathbb{C}) \subset \mathrm{Mat}_m(\mathbb{C})$.

There must be some polynomial $p : \mathrm{Mat}_m(\mathbb{C}) \times \mathrm{Mat}_n(\mathbb{C}) \to \mathbb{C}$ of at degree at most $\max\{2\gamma^2, m, n\}$ that vanishes on V' but not on all of $\oplus_i \mathrm{Mat}_{m_i}(\mathbb{C}) \times \oplus_j \mathrm{Mat}_{n_j}(\mathbb{C})$. By the Schwarz-Zippel lemma, p vanishes on our random choice of $(g_0, h_0) \in \oplus_i \mathrm{Mat}_{m_i}(\mathbb{C}) \times \oplus_j \mathrm{Mat}_{n_j}(\mathbb{C})$ with probability at most 1/3. With probability at least 2/3 we have found g_0, h_0 such that T_{g_0,h_0} is (P, Q)-rank-nondecreasing.

Notice that $(G, H, F_\circ(P), F_\circ(Q), T_{g_0,h_0})$ is still block-diagonal so by Proposition 3.18,

$$(\underline{G}, \underline{H}, F_\circ(\underline{P}), F_\circ(\underline{Q}), \underline{T_{g_0,h_0}})$$

is block-diagonal. By Proposition 3.16, $\underline{T_{g_0,h_0}}$ is $(\underline{P}, \underline{Q})$-rank-nondecreasing. Theorem 5.1 shows every step of Algorithm TOSI works, so Algorithm GOSI does not output ERROR. By Corollary 3.20, step 3 is possible. □

Remark 8.11. Algorithms TOSI and GOSI terminate in a reasonable number of iterations; further, the parameter γ in Algorithm GOSI can be taken in $[2^{(m+n)b}]$ if the entries of p and q are $\leq b$-bit binary numbers. This does not inherently introduce issues. However, the numbers involved in the iterations of Algorithm TOSI might grow exponential. For this reason we must modify Algorithm TOSI by rounding in each step. The analysis of the Algorithm with rounding can be found in Appendix 12.4.

Remark 8.12. We can replace the condition "$(G, H, F_\bullet, E_\bullet, T)$-block-diagonal" by less restrictive conditions on $(G, H, F_\bullet, E_\bullet, T)$ and still obtain results like Theorem 3.9.

Theorem 8.13. *Suppose P and Q are nonsingular and*

1. *$G = G_0 G_\Delta$ where $G_\Delta \subset \mathrm{GL}(W)_{E_\bullet}$ and $G_0 = \mathcal{V} \cap \mathrm{GL}(W)$ for some irreducible variety $\mathcal{V}$,*
2. *H has the analogous decomposition $H_0 H_\Delta$,*
3. *$G^\dagger G = G_\Delta^\dagger G_\Delta$,*
4. *$H^\dagger H = H_\Delta^\dagger H_\Delta$, and*

5.

$$T(HPH^\dagger) \subset G^\dagger G$$
$$\text{and } T^*(GQG) \subset H^\dagger H. \tag{42}$$

Then T is approximately (G,H)-scalable to $(P \to I_W, Q \to I_V)$ if and only if $\mathrm{cap}(T_{g_0,h_0}, P, Q) > 0$*, or equivalently T_{g_0,h_0} is (P,Q)-rank-nondecreasing, for a generic $(g_0^\dagger, h_0) \in G_0^\dagger \times H_0$.*

Before showing an outline of the proof, we show that the first four assumptions are not as unnatural as they may appear.

Claim 8.14. *If G is a connected subgroup of* $\mathrm{GL}(W)$*, there is a choice of inner product $\langle\cdot,\cdot\rangle$ and flag $E_\bullet$ such that G satisfies 1 and 3.*

Proof. The proof is an assembly of standard facts about linear-algebraic groups and Lie groups. By the Levi-Malcev theorem (see Theorem 3.18.13 of [**Va13**]), $G = R \rtimes S$ where S is a connected semisimple Lie subgroup of G and R is a connected solvable Lie subgroup of G. Further, any semisimple Lie group S can be written KR' where K is a compact subgroup of S and R' is a simply connected solvable subgroup of S [**Ze73**]. Set $G_\Delta = R'R$ and $G_0 = S \supset K$. G_Δ is a solvable subgroup of G, and $G = KG_\Delta$.

1. Trivially, $G = G_0 G_\Delta \supset KG_\Delta$.

2. G_0 is a connected, semisimple complex algebraic group - thus, it is an irreducible algebraic variety.

3. Since K is compact, there is a choice of inner product $\langle\cdot,\cdot\rangle$ on W such that K is a subgroup of the unitary group (i.e. $\langle\cdot,\cdot\rangle$ is K invariant). This tells us $G_\Delta^\dagger G_\Delta = G^\dagger G$.

4. By Lie's theorem, because G_Δ is solvable, there is a flag $E_\bullet$ such that $G_\Delta \subset \mathrm{GL}(W)_{E_\bullet}$.

□

Next we outline the proof of Theorem 8.13. First let us treat the necessary condition. Corollary 7.2 still shows that if T is approximately (G,H)-scalable to $(P \to I_W, Q \to I_V)$, then there exists (g_0, h_0) such that T_{g_0,h_0} is (P,Q)-rank-nondecreasing. This implies T_{g_0,h_0} is (P,Q)-rank-nondecreasing for a generic $(g_0^\dagger, h_0)$ by Proposition 8.5. The only part of Proposition 8.5 that must change is 3, in which G and H must be replaced by G_0 and H_0. However, the proof of the modified 3 is similar because G_0 and H_0 are the intersections of irreducible varieties with the general linear groups.

Next we show the sufficient condition is still enough to guarantee scalings. By 42, each iteration of Algorithm TOSI can still be performed on T_{g_0,h_0} in G_Δ and H_Δ, so if there exists some g_0, h_0 such that T_{g_0,h_0} is (P,Q)-rank-nondecreasing then T_{g_0,h_0} is approximately (G_Δ, H_Δ)-scalable to $(P \to I_W, Q \to I_V)$. Algorithm GOSI still works, but one must still be able to sample a random element from G_0, H_0.

9 A sufficient condition for exact scalability

Here we state our partial results towards understanding the existence of $(P \to I_W, Q \to I_V)$-scalings, as opposed to ϵ-$(P \to I_W, Q \to I_V)$ scalings. Our scaling algorithm, Algorithm TOSI, is not in itself enough to guarantee the existence of exact scalings, because the scalings g and

h themselves need not converge to elements of G and H. Here we take a different route using local minima for the capacity function. First we establish that a local minimum for the infimum defining $\operatorname{cap}(T, P, Q)$ can be used to obtain $(P \to I_W, Q \to I_V)$-scalings. Next we show that if the (nontrivial) inequalities required for T to be (P, Q)-rank-nondecreasing hold with *strict* inequality, then such a local minimum exists. Doing so will require a limiting argument and a theorem of Gurvits.

9.1 Exact scalings from local minima for capacity

Throughout, A and B refer to $A = \sigma(F_\circ) \supset \sigma(p)$ and $B = \sigma(E_\circ) \supset \sigma(q)$.

Definition 9.1. Suppose $(G, H, E_\circ, F_\circ, T)$ is block-diagonal; recall that this states $V = \oplus_j V_j$ and $F_i = \oplus_j F_i \cap V_j$ for all $i \in A$. Say the tuple $(Y \succ 0 : Y_i : i \in A)$ *has the same block-structure as* H if $Y_i : F_i \to F_i$ *has the same block-structure as* H if $Y_i(F_i \cap V_j) = F_i \cap V_j$ for all j and all $i \in A$.

Let $\mathcal{K}(T, E_\circ, F_\circ)_{loc}$ be the set of $(p, q) \in \mathcal{K}(T, E_\circ, F_\circ)$ such that

$$f_{p,q}(Y_j : j \in A) := \frac{\prod_{i \in B}^m \left(\det \nu_i T \left(\sum_{j \in A} \Delta p_j \eta_j^\dagger Y_j \eta_j\right) \nu_i^\dagger\right)^{\Delta q_i}}{\prod_{j \in A} (\det Y_j)^{\Delta p_j}}$$

attains a *strict* local minimum Y with the same block-structure as H with value greater than 0 subject to $\prod_{j \in A} (\det Y_j)^{\Delta p_j} = 1$. Let $\mathcal{K}_1(T, E_\circ, F_\circ)_{loc} = \mathcal{K}(T, E_\circ, F_\circ)_{loc} \cap \mathcal{K}_1(T, E_\circ, F_\circ)$.

Recall Definition 3.16 for $\underline{T}$, which is a new completely positive map we define to handle singular P or Q. When P and Q are invertible, $\underline{T}$ is the same as T.

Lemma 9.1. *Suppose $(G, H, E_\circ, F_\circ, T)$ is block-diagonal. If $f_{p,q}$ has a local (not necessarily strict) minimum $0 \prec Y_j : F_j \to F_j$, $j \in A$ with value greater than 0, then $\underline{T}$ is exactly scalable to*

$$(\underline{P} \to I_{\operatorname{supp} Q}, \underline{Q} \to I_{\operatorname{supp} P})$$

by $(G_{E_\circ}, H_{F_\circ})$. In particular, $\underline{T}$ is exactly scalable to $(\underline{P} \to I_{\operatorname{supp} Q}, \underline{Q} \to I_{\operatorname{supp} P})$ by $(G_{E_\circ}, H_{F_\circ})$ if $(p, q) \in \mathcal{K}(T, E_\circ, F_\circ)_{loc}$.

Proof. This proof is tedious, but easy. We just examine consequences of $\nabla f_{p,q} = 0$. First of all, by Claim 4.6, we may assume the local minimum is obtained at

$$\begin{aligned} &Y = (\eta_j h \eta_j^\dagger \eta_j h^\dagger \eta_j^\dagger : j \in A) \\ &\text{for } \sum_{j \in A} \Delta p_j \eta_j^\dagger Y_j \eta_j = h P h^\dagger \end{aligned} \tag{43}$$

for some $h \in H$. h can be taken to be in H because Y has the same block-structure as H. Next, note that

$$\{(Y_j : j \in A) : 0 \prec Y_j : F_j \to F_j\} = \prod_{j \in A}^{1} \mathcal{S}_{++}(F_j)$$

is an open subset of $\prod_{i \in A} L(F_i) \cong \prod_{i \in A} \mathbb{R}^{i^2}$ and $f_{p,q} : \prod_{i \in A} \mathcal{S}_{++}(F_j) \to \mathbb{R}$ is a continuous, differentiable function. Thus, if $f_{p,q}$ has a local minimum at $Y = (Y_j : j \in A)$ with $f_{p,q}(Y) > 0$ then $\log f_{p,q}$ is continuous and differentiable in a neighborhood about Y and also achieves a local minimum at Y. Thus, $\nabla \log f_{p,q}|_Y = 0$. Let us compute the directional derivative of $\log f_{p,q}$ in the direction

$X = (X_j : j \in A)$. By the formula $\nabla_X \log\det A = \operatorname{Tr} A^{-1}X$ applied to both terms and the linearity of T, we have

$$\begin{aligned}
\nabla_X \log f_{p,q}|_Y &= \sum_{i\in B} \Delta q_i \nabla_X \log\det \nu_i T\left(\sum_{j=1}^n \Delta p_j \eta_j^\dagger Y_j \eta_j\right)\nu_i^\dagger - \sum_{j\in A}\Delta p_j \nabla_X \log\det Y_j \\
&= \sum_{i\in B}\Delta q_i \operatorname{Tr}\left[\left(\nu_i T\left(\sum_{j\in A}\Delta p_j \eta_j^\dagger Y_j \eta_j\right)\nu_i^\dagger\right)^{-1} \nu_i T\left(\sum_{j\in A}\Delta p_j \eta_j^\dagger X_j \eta_j\right)\nu_i^\dagger\right] \\
&\qquad - \sum_{j\in A}\Delta p_j \operatorname{Tr} Y_j^{-1} X_j.
\end{aligned}$$

Now we plug in Y from 43 and use Lemma 3.15 to obtain

$$\begin{aligned}
\nabla_X \log f_{p,q}|_Y = \sum_{i\in B}\Delta q_i \operatorname{Tr}\left[\left(\nu_i T\left(hPh^\dagger\right)\nu_i^\dagger\right)^{-1}\nu_i T\left(\sum_{j\in A}\Delta p_j \eta_j^\dagger X_j \eta_j\right)\nu_i^\dagger\right] \\
- \sum_{j\in A}\Delta p_j \operatorname{Tr} \eta_j h^{-\dagger}h^{-1}\eta_j^\dagger X_j.
\end{aligned}$$

Using the cyclic property of trace, we can deduce

$$\begin{aligned}
\nabla_X \log f_{p,q}|_Y = \\
\sum_{j\in A}\Delta p_j \operatorname{Tr}\left[\eta_j\left(T^*\left(\sum_{i\in B}\Delta q_i \nu_i^\dagger\left(\nu_i T\left(hPh^\dagger\right)\nu_i^\dagger\right)^{-1}\nu_i\right) - h^{-\dagger}h^{-1}\right)\eta_j^\dagger X_j\right].
\end{aligned}$$

By assumption, $\nabla_X \log f_{p,q}|_Y = 0$ for all $X = (X_j \in L(F_j(P) : j \in A)$. This implies that for all $j \in A$,

$$\eta_j\left(T^*\left(\sum_{i\in B}\Delta q_i \nu_i^\dagger\left(\nu_i T\left(hPh^\dagger\right)\nu_i^\dagger\right)^{-1}\nu_i\right) - h^{-\dagger}h^{-1}\right)\eta_j^\dagger = 0.$$

The above holds for all $j \in A$ if and only if it holds for $j = \max A$, and since η is defined to be $\eta_{\max A}$, in particular we have that it holds if and only if

$$\eta\left(T^*\left(\sum_{i\in B}\Delta q_i \nu_i^\dagger\left(\nu_i T\left(hPh^\dagger\right)\nu_i^\dagger\right)^{-1}\nu_i\right) - h^{-\dagger}h^{-1}\right)\eta^\dagger = 0,$$

Note that $\nu_i T\left(hPh^\dagger\right)\nu_i^\dagger$ must be nonsingular for $i \in B$ by the nonzeroness of $f_{p,q}(Y)$. In particular, $\nu T\left(hPh^\dagger\right)\nu^\dagger$ is nonsingular. Since $(G, H, F_\circ(P), F_\circ(Q), T)$ is block-diagonal, we can write $\nu T\left(hPh^\dagger\right)\nu^\dagger = \nu g^\dagger g \nu^\dagger$ for $g \in G_{F_\circ(P)}$. In particular, there exists g such that $\nu_i T\left(hPh^\dagger\right)\nu_i^\dagger = \nu_i g^\dagger g \nu_i$ for $i \in B$.

We now have

$$\begin{aligned}
0 &= \eta\left(T^*\left(\sum_{i\in B}\Delta q_i\nu_i^\dagger\left(\nu_i g^\dagger g\nu_i^\dagger\right)^{-1}\nu_i\right) - h^{-\dagger}h^{-1}\right)\eta^\dagger\\
&= \eta\left(T^*\left(\sum_{i\in B}\Delta q_i\nu_i^\dagger\left(\nu_i g^\dagger \nu_i^\dagger\nu_i g\nu_i^\dagger\right)^{-1}\nu_i\right) - h^{-\dagger}h^{-1}\right)\eta^\dagger\\
&= \eta\left(T^*\left(\sum_{i\in B}\Delta q_i\nu_i^\dagger\nu_i g^{-1}\nu_i^\dagger\nu_i g^{-\dagger}\nu_i^\dagger\nu_i\right) - h^{-\dagger}h^{-1}\right)\eta^\dagger\\
&= \eta\left(T^*\left(g^{-1}Qg^{-\dagger}\right) - h^{-\dagger}h^{-1}\right)\eta^\dagger.
\end{aligned}$$

Multiplying by $\nu h^\dagger\nu^\dagger$ on the left and $\nu h\nu^\dagger$ on the right, we obtain

$$\eta\left(h^\dagger T^*\left(g^{-1}Qg^{-\dagger}\right)h - I\right)\eta^\dagger = 0.$$

On the other hand, we have defined $\nu T\left(hPh^\dagger\right)\nu^\dagger = \nu g^\dagger g\nu^\dagger$. Combining these two, we have

$$\begin{aligned}
\eta T^*_{h,g^{-1}}(Q)\eta^\dagger &= I_{\operatorname{supp} P},\\
\nu T_{g^{-1},h}(P)\nu^\dagger &= I_{\operatorname{supp} Q}.
\end{aligned}$$

By Proposition 3.17, this says precisely that $\underline{T}_{\underline{g}^{-1},\underline{h}}$ is a $(\underline{P}\to I_{\operatorname{supp} Q}, \underline{Q}\to I_{\operatorname{supp} P})$ scaling of $\underline{T}$. $\square$

9.2 Indecomposable operators have local minima for capacity

Gurvits defined a completely positive operator $T: L(\mathbb{C}^n)\to L(\mathbb{C}^n)$ to be *indecomposable* if T is rank-nondecreasing and in addition $\operatorname{rk} T(X) > \operatorname{rk} X$ provided $X\succeq 0$ and $0\neq \operatorname{rk} X\neq n$. Equivalently, T is indecomposable if and only if $\dim L + \dim R < n$ for every T-independent pair such that $L\neq 0$ and $R\neq 0$.

Theorem 9.2 (Theorem 4.7, **Gu04**)**.** *The completely positive operator T is indecomposable if and only if*

$$\inf_{X\succeq 0, \det X = 1}\det T(X) \tag{44}$$

is positive and uniquely attained.

Gurvits also showed Lemma 9.1 for the setting $P = Q = I_V$; combined with the above theorem this shows indecomposable completely positive operators have doubly stochastic scalings. We now define a notion of indecomposability for completely positive maps and specified marginals.

Definition 9.2. (Indecomposability for specified marginals) Say T is $(p,q,E_\circ,F_\circ)$-*indecomposable* if $E_\circ$ is a q-partial flag, $F_\circ$ is a p-partial flag, and we have the *strict* inequality

$$\sum_{i\in\sigma(E_\circ)}\Delta q_i\dim E_i\cap L + \sum_{j\in\sigma(F_\circ)}\Delta p_j\dim F_j\cap R < \sum_{i=1}^n p_i \tag{45}$$

for all T-independent pairs (L,R) where $L\cap\operatorname{supp} Q\neq 0$ and $R\cap\operatorname{supp} P\neq 0$. Say T is (P,Q)-indecomposable if T is $(p,q,F_\circ(Q),F_\circ(P))$-indecomposable. Let

$$\mathcal{K}_{strict}(T,E_\circ,F_\circ) = \{(p,q) : T \text{ is } (p,q,E_\circ,F_\circ)\text{-indecomposable}\}.$$

For the special case $P = Q = I$ and $V = W = \mathbb{C}^n$, a completely positive map $T : L(\mathbb{C}^n) \to L(\mathbb{C}^n)$ is indecomposable if $\dim R + \dim L < n$ for every T-independent pair (L, R) such that $L \neq \{0\}$ and $R \neq \{0\}$.

Lemma 9.3. *T is (P, Q)-indecomposable if and only if $\operatorname{Trun}_{P,Q} T$ is indecomposable.*

Proof. The proof is identical to the proof of Lemma 4.7 with the term "rank-nondecreasing" replaced by "indecomposable." Also note that $W^1 = \operatorname{supp} Q$ and $V^1 = \operatorname{supp} P$. $\square$

We first prove an analogue of Theorem 9.2 for p and q rational, using our reduction to the doubly stochastic case. Recall $B := \sigma(E_\circ)$ and $A = \sigma(F_\circ)$.

Lemma 9.4. *Suppose $(p, q) \in \mathcal{K}(T, E_\circ, F_\circ)_{strict} \cap \mathbb{Q}^{n+m}$; then*

$$\inf_{Y_j : j \in A} f_{p,q}(Y_j : j \in A)$$

is positive subject to $0 \prec Y_j : F_j \to F_j,\ j \in A$ and is attained uniquely subject to $\prod_{j \in A} (\det Y_j)^{\Delta p_j} = 1$. In particular,

$$\mathcal{K}(T, E_\circ, F_\circ)_{strict} \cap \mathbb{Q}^{n+m} \subset \mathcal{K}(T, E_\circ, F_\circ)_{loc}.$$

Proof. This proof consists of the application of Theorem 9.2 to the reduction from Section 4. Suppose $(p, q) \in \mathcal{K}(T, E_\circ, F_\circ)_{strict} \cap \mathbb{Q}^{n+m}$. It is not hard to see that scaling (p, q) by a constant $\gamma > 0$ only affects $f_{p,q}$ by a positive multiplicative constant, so does not change positivity, attainment or unique attainment, of the infimum under consideration. Thus we may assume

$$(p, q) \in \mathcal{K}(T, E_\circ, F_\circ)_{strict} \cap \mathbb{Z}^{n+m}.$$

By Lemma 9.3, $\operatorname{Trun}_{P,Q} T$ is indecomposable. By Theorem 9.2,

$$\inf_{X \succeq 0, \det X = 1} \det \operatorname{Trun}_{P,Q} T(X)$$

is positive and uniquely attained. By Lemma 4.4, $\inf_{Y_j : j \in A} f_{p,q}(Y_j : j \in A)$ is positive subject to $0 \prec Y_j : F_j \to F_j,\ j \in A$ and is attained uniquely subject to $\prod_{j \in A} (\det Y_j)^{\Delta p_j} = 1$. $\square$

Next we need to show $\mathcal{K}_1(T, E_\circ, F_\circ)_{strict} \cap \mathbb{Q}^{n+m} \subset \mathcal{K}_1(T, E_\circ, F_\circ)_{loc}$ implies $\mathcal{K}_1(T, E_\circ, F_\circ)_{strict} \subset \mathcal{K}_1(T, E_\circ, F_\circ)_{loc}$. The next lemma would suffice because $\mathcal{K}_1(T, E_\circ, F_\circ)_{strict}$ is convex with rational vertices, and so $\mathcal{K}_1(T, E_\circ, F_\circ)_{strict} \cap \mathbb{Q}^{n+m}$ is dense in $\mathcal{K}_1(T, E_\circ, F_\circ)_{strict}$.

Lemma 9.5. *$\mathcal{K}_1(T, E_\circ, F_\circ)_{loc}$ is relatively open in $\mathcal{K}_1(T, E_\circ, F_\circ)$.*

Proof. We need to show that for every $(p, q) \in \mathcal{K}_1(T, E_\circ, F_\circ)_{loc}$ there is an Euclidean ball $B \subset \mathbb{R}^{m+n}$ centered at (p, q) such that $B \cap \mathcal{K}_1(T, E_\circ, F_\circ) \subset \mathcal{K}_1(T, E_\circ, F_\circ)_{loc}$. It is enough to show that for any point $(p, q) \in \mathcal{K}_1(T, E_\circ, F_\circ)_{loc}$ and subsequence $s = ((p(i), q(i)) \in \mathcal{K}_1(T, E_\circ, F_\circ))_{i>0}$ that converges to (p, q) in Euclidean distance, there is some n_0 such that if $i > n_0$ then $(p(i), q(i)) \in \mathcal{K}_1(T, E_\circ, F_\circ)_{loc}$.

Let $(p, q) \in \mathcal{K}_1(T, E_\circ, F_\circ)_{loc}$ and $s = ((p(i), q(i)) \in \mathcal{K}_1(T, E_\circ, F_\circ))_{i>0}$ be such a sequence converging to (p, q) in Euclidean distance. Let $Y = (Y_j : j \in A)$ be the promised strict local minimum of f with $f_{p,q}(Y) > 0$ and $0 \prec Y_j : F_j(P) \to F_j(P),\ j \in A$.

Claim 9.6. *There is a closed ϵ-ball about Y (in the trace norm)*

$$B_\epsilon(Y) \subset \{(Z_j : j \in A) : 0 \prec Z_j : F_j(P) \to F_j(P),\ j \in A\}$$

such that the sequence of functions $(f_{p(i),q(i)})_{i>0}$ converges uniformly to $f_{p,q}$ on $B_\epsilon(Y)$.

The claim implies the theorem via very broadly applicable reasoning: suppose $(f_{p(i),q(i)})_{i>0}$ converges uniformly to $f_{p,q}$ on $B_\epsilon(Y)$. Since Y is a strict local minimum of $f_{p,q}$, we can find a closed ball $B_{\epsilon'}(Y) \subsetneq B_\epsilon(Y)$ such that Y is a global minimizer of $f_{p,q}$ on $B_{\epsilon'}(Y)$. Since $f_{p,q}$ is continuous on $B_\epsilon(Y)$, and $\partial B_{\epsilon'}(Y)$ is compact,

$$\inf_{Z \in \partial B_{\epsilon'}(Y)} f_{p,q}(Z) - f_{p,q}(Y) = \min_{Z \in \partial B_{\epsilon'}(Y)} f_{p,q}(Z) - f_{p,q}(Y) = \delta > 0.$$

Now pick n_0 so that $|f_{p(i),q(i)}(Z) - f_{p,q}(Z)| < \delta/2$ for all $Z \in B_\epsilon(Y)$. If $i > n_0$, then $Z \in \partial B_{\epsilon'}(Y)$, then $f_{p(i),q(i)}(Z) > (f_{p,q}(Y) + \delta) - \delta/2 = f_{p,q}(Y) + \delta/2$, and $f_{p(i),q(i)}(Y) < f_{p,q}(Y) + \delta/2$. Hence, the global minimizer X of $f_{p(i),q(i)}$ on $B_{\epsilon'}(Y)$ cannot be on on the boundary of $B_{\epsilon'}(Y)$, so X is also the global minimizer on some open ball $B_{\epsilon''}(X) \subset B_{\epsilon'}(Y)$. Thus X is a local minimum of $f_{p(i),q(i)}$. By the remarks before the statement of the claim, this proves the lemma.

To finish, we prove Claim 9.6.

Proof of claim. If we can prove $f_{p',q'}$ is bounded above by a constant C on $B_\epsilon(Y)$ for $(p',q') \in \mathcal{K}_1(T, E_\circ, F_\circ)$, then it will be enough to show to prove $\log f_{p(i),q(i)}$ converges uniformly to $\log f_{p,q}$ on $B_\epsilon(Y)$, because $|f_{p,q} - f_{p(i),q(i)}| \le C|\log f_{p(i),q(i)} - \log f_{p,q}|$.

It is easy to find ϵ, C such that $f_{p',q'} \le C$ is bounded on $B_\epsilon(Y)$. Recall that $\sum_{j\in A} j\Delta p'_j = \sum_{i \in B} i \Delta q'_i = 1$. Indeed, take $1 > \epsilon = .5 \min_{j \in A} \min \lambda(Y_j)$; then if $Z \in B_\epsilon(Y)$ we have $\det_{Z_j} \ge (.5\epsilon)^j$ and

$$\det \nu_i T(\sum_{j \in A} \Delta p'_j Z_j)\nu_i^\dagger \le \det \nu_i T \left(\sum_{j\in A} (Y_j + I_{F_j}) \right) \nu_i^\dagger \le K^i$$

for some constant K. Then we may take $C = \prod_{i\in B} K^{i\Delta q'_i} / \prod_{j\in A} (.5\epsilon)^{j\Delta p'_j} = 2K/\epsilon$.

Next we must show the uniform convergence of $\log f_{p,q}$ to $\log f_{p(i),q(i)}$ on $B_\epsilon(Y)$. If $Z \in B_\epsilon(Y)$, we have

$$
\begin{aligned}
&|\log f_{p',q'}(Z) - \log f_{p,q}(Z)| \\
&= \Bigg| \sum_{i\in B} \Delta q_i \log\det \nu_i T \left(\sum_{j\in A} \Delta p_j \eta_j^\dagger Z_j \eta_j \right) \nu_i^\dagger - \sum_{j=1}^{n} \Delta p_j \log\det Z_j \\
&\quad - \sum_{i\in B} \Delta q_i' \log\det \nu_i T \left(\sum_{j\in A} \Delta p_j' \eta_j^\dagger Z_j \eta_j \right) \nu_i^\dagger + \sum_{j\in A} \Delta p_j' \log\det Z_j \Bigg| \\
&\le \left| \sum_{j\in A} (\Delta p_j' - \Delta p_j) \log\det Z_j \right| \\
&\quad + \left| \sum_{i\in B} (\Delta q_i - \Delta q_i') \log\det \nu_i T \left(\sum_{j\in A} \Delta p_j \eta_j^\dagger Z_j \eta_i \right) \nu_j^\dagger \right| \\
&\quad + \left| \sum_{i\in B} \Delta q_i' \left(\log\det \nu_i T \left(\sum_{j\in A} \Delta p_i \eta_j^\dagger Z_j \eta_i \right) \nu_j^\dagger - \log\det \nu_i T \left(\sum_{j\in A} \Delta p_i' \eta_j^\dagger Z_j \eta_j \right) \nu_i^\dagger \right) \right|
\end{aligned}
$$

The first two terms are clearly bounded by absolute constants times $\|\Delta p - \Delta p'\|$ and $\|\Delta q - \Delta q'\|$, respectively. The last term is a little more difficult. Since $f_{p',q'} \neq 0$ by $(p',q') \in \mathcal{K}_1(T, E_\circ, F_\circ)$, Theorem 3.8 and Lemma 4.5, we know that for all $i \in B$, $\det \nu_i T \left(\sum_{j\in A} \Delta p_i \eta_j^\dagger Y_j \eta_i \right) \nu_i^\dagger = 0$ implies $\Delta q_i' = 0$. Thus, we can assume the outer sum in the last term is over the smaller set $C = \{i : \det \nu_j T \left(\sum_{j\in A} \Delta p_i \eta_j^\dagger Y_j \eta_i \right) \nu_j^\dagger \neq 0\}$. Assume ϵ' is so small that $\nu_i T \left(\sum_{j\in A} \eta_j^\dagger X_j \eta_i \right) \nu_i^\dagger$ is nonsingular for all $i \in C$ and X in the *closed* ball $B_{\epsilon'}(a_j Y_j : j \in A)$. Now for all $i \in C$, $\log\det \nu_i T \left(\sum_{j=1}^n \eta_j^\dagger X_j \eta_j \right) \nu_i^\dagger$ is a continuous function on the compact set $B_{\epsilon'}(a_j Y_j : j \in A)$ and so must be uniformly continuous. Thus, for all $i \in C$,

$$
\left| \log\det \nu_i T \left(\sum_{j=1}^{n} \Delta p_i \eta_j^\dagger Z_j \eta_j \right) \nu_i^\dagger - \log\det \nu_i T \left(\sum_{j=1}^{n} \Delta p_i' \eta_j^\dagger Z_j \eta_j \right) \nu_i^\dagger \right|
$$

tends to zero uniformly in $\sum_{i\in A} \|(\Delta p_i - \Delta p_i') Z_j\|$ for $(\Delta p_j Z_j : j \in A), (\Delta p_j' Z_j : j \in A) \in B_{\epsilon'/2}(\Delta p_j Y_j : j \in A)$, which in turn tends to zero uniformly in $\|\Delta p - \Delta p'\|$ for $Z \in B_{\epsilon''}(Y)$. □

□

The next theorem is the main result of this section.

Theorem 9.7 (Exact scalability for block-triangulars)**.** *Suppose $(G, H, F_\circ, E_\circ, T)$ is block-diagonal and that T is (P,Q)-indecomposable. Then $\underline{T}$ is exactly scalable to*

$$
(\underline{P} \to I_{\operatorname{supp} Q}, \underline{Q} \to I_{\operatorname{supp} P})
$$

by $(G_{E_\circ},\ H_{F_\circ})$.

Proof. The polytope $\mathcal{K}_1(T, E_\circ, F_\circ)_{strict}$ is convex with rational vertices, and so $\mathcal{K}_1(T, E_\circ, F_\circ)_{loc} \cap \mathbb{Q}^{n+m}$ is dense in $\mathcal{K}_1(T, E_\circ, F_\circ)_{strict}$. By Lemma 9.5, $\mathcal{K}_1(T, E_\circ, F_\circ)_{loc}$ is relatively open in $\mathcal{K}_1(T, E_\circ, F_\circ)$ and thus relatively open in $\mathcal{K}_1(T, E_\circ, F_\circ)_{strict}$. Together the previous two sentences imply

$$\mathcal{K}_1(T, E_\circ, F_\circ)_{loc} \supset \mathcal{K}_1(T, E_\circ, F_\circ)_{strict}.$$

Lemma 9.1 now implies $\underline{T}$ is exactly scalable to

$$(\underline{P} \to I_{\operatorname{supp} Q}, \underline{Q} \to I_{\operatorname{supp} P})$$

by $(G_{E_\circ}, H_{F_\circ})$ for $(p, q) \in \mathcal{K}_1(T, E_\circ, F_\circ)_{strict}$; scalability does not change if we multiply P and Q by a positive constant, so $\underline{T}$ is exactly scalable to

$$(\underline{P} \to I_{\operatorname{supp} Q}, \underline{Q} \to I_{\operatorname{supp} P})$$

by $(G_{E_\circ}, H_{F_\circ})$ for $(p, q) \in \mathcal{K}(T, E_\circ, F_\circ)_{strict}$. □

10 Proofs for Applications

10.1 Matrix Scaling

Say X, Y is an ϵ-(r, c)-*scaling* if the row and column sum vectors of XAY are at most ϵ from r and c, respectively, in (say) Euclidean distance and that A is *almost* (r, c)-*scalable* if for every $\epsilon > 0$ there exists an ϵ-(r, c)-scaling of A. Given A, r, c, the (r, c)-scaling problem consists of deciding the existence of and finding ϵ-(r, c)-scalings. The (r, c)-scaling problem has practical applications such as statistics, numerical analysis, engineering, and image reconstruction, and theoretical uses such as strongly polynomial time algorithms for approximating the permanent [**Si64**, **RS89**,**LSW98**].

There is a simple criterion for almost-(r, c)-scalability.

Theorem 10.1 (**RS89**)**.** *A nonnegative matrix A is almost (r, c) scalable if and only if for every zero submatrix $L \times R$ of A,*

$$\sum_{i \in L} r_i \le \sum_{j \notin R} c_j.$$

We can reduce this to an instance of Question 1 as follows:

Definition 10.1. Suppose A is a nonnegative $m \times n$ matrix. For $i \in [m], j \in [n]$, define e_{ij} to be the $m \times n$ matrix with a one in the ij entry and zeros elsewhere. Let $T_A : L(\mathbb{C}^n) \to L(\mathbb{C}^m)$ be the completely positive map with Kraus operators $E_{ij} = \sqrt{A_{ij}} e_{ij}$, $i \in [m], j \in [n]$.

It is simple to check that A has row sums r and column sums c if and only if $T(I) = \operatorname{diag}(r)$ and $T^*(I) = \operatorname{diag}(c)$. Let G and H be the diagonal matrices in $\mathrm{GL}_m(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{C})$, respectively. If $gg^\dagger = D_1$ and $hh^\dagger = D_2$, then the row and column sums of $D_1 A D_2$ are exactly the diagonals of $T_{g,h}(I)$ and $T^*_{h,g}(I)$, respectively. That is,

Observation 10.2. *A is almost (r, c)-scalable if and only if T_A is approximately scalable to $(I_V \to \operatorname{diag}(r), I_W \to \operatorname{diag}(c))$ by (G, H).*

Here we show why Theorem 10.1 follows from Theorem 3.8.

Proof of Theorem 10.1. Let G and H be the subgroups of diagonal matrices in $\mathrm{GL}_m(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{C})$, respectively, and E and F the respective standard bases. We already observed that A is almost (r,c)-scalable if and only if the completely positive map $T_A : L(\mathbb{C}^n) \to L(\mathbb{C}^m)$ is approximately (G,H) scalable to $(I_V \to \mathrm{diag}(r), I_W \to \mathrm{diag}(c))$. Assume without loss of generality that $\sum r_i = \sum c_i = 1$.

Observe that $(G, H, F_\circ(\mathrm{diag}(r)), F_\bullet(\mathrm{diag}(c)), T_A)$, is block-diagonal with blocks $\langle e_i \rangle$, $i \in [m]$ for W and $\langle f_i \rangle, i \in [n]$ for V. Without loss of generality, we may we may assume r and c are non-increasing sequences so that $F_\circ(\mathrm{diag}(r)), F_\circ(\mathrm{diag}(c))$ are subflags of the standard flags $F_\bullet$ and $E_\bullet$ of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively, but $G \subset \mathrm{GL}(W)_{E_\bullet}$ and $H \subset \mathrm{GL}(V)_{F_\bullet}$. That is, diagonal matrices are upper-triangular in the standard basis.

By Theorem 3.8, $T_A : L(\mathbb{C}^n) \to L(\mathbb{C}^m)$ is approximately (G,H) scalable to (P,Q) if and only if T_A is $(\mathrm{diag}(r), \mathrm{diag}(c))$-rank-nondecreasing, or by Lemma 7.4

$$\sum_{i \in I} r_i + \sum_{j \in J} c_i \leq 1 \tag{46}$$

for all I, J such that $\Omega_I^\circ(E_\bullet) \times \Omega_J^\circ(F_\bullet)$ contains a T_A-independent pair.

Let us first describe the T_A-independent pairs (L,R): (L,R) is T_A-independent if and only if $l^\dagger E_{ij} r = \bar{l}_i \sqrt{a_{ij}} r_j = 0$ for every $i \in [m], j \in [n], l \in L, r \in R$. This tells us that (L,R) is T_A-independent if and only if $a_{ij} = 0$ for every i,j such that $L \not\subset e_i^\perp$ and $R \not\subset e_j^\perp$. If we replace L by $L' = \langle e_i : L \not\subset e_i^\perp \rangle \supset L$ and R by $R' = \langle e_i : R \not\subset e_i^\perp \rangle \supset R$, then (L', R') is still T_A-independent. We only need to check 46 for maximal T-independent pairs. Thus, we may assume $L = L_I = \langle e_i : i \in I \rangle$ and $R = R_J = \langle e_j : j \in J \rangle$ for some $I \subset [m]$, $J \subset [n]$. Further (L_I, R_J) is independent if and only if $I \times J$ is a zero submatrix of A, and $L_I \in \Omega_I^\circ(E_\bullet)$ and $R_J \in \Omega_J^\circ(F_\bullet)$. Hence, T_A is $(\mathrm{diag}(r), \mathrm{diag}(c))$-rank-nondecreasing if and only if

$$\sum_{i \in I} r_i + \sum_{j \in J} c_i \leq 1$$

or

$$\sum_{i \in I} r_i \leq \sum_{j \notin J} c_i$$

for all zero submatrices $I \times J$ of A, which is exactly the condition in Theorem 10.1. $\square$

10.2 Eigenvalues of sums of Hermitian matrices

Here is an old question in linear algebra, apparently originally due to Weyl.

Question 5 (Weyl)**.** Let α, β, γ be nonincreasing sequences of m real numbers. When are α, β, γ the spectra of some $m \times m$ Hermitian matrices A, B, C satisfying $A + B = C$?

This question essentially asks for a complete list of inequalities satisfied by the eigenvalues of sums of Hermitian matrices. Klyachko showed a relationship between the eigenvalues of sums of Hermitian operators and certain constants known as the *Littlewood-Richardson coefficients*.

Definition 10.2. If $I = \{i_1 < \cdots < i_k\} \subset [m]$, let $\rho(I)$ be the partition

$$\rho(I) = (i_k - k, \ldots, i_2 - 2, i_1 - 1).$$

The Littlewood-Richardson coefficient of the partitions λ, μ, and ν is a positive integer denoted $c^\lambda_{\mu,\nu}$. The Littlewood-Richardson coefficients arise in representation theory, Schubert calculus, and combinatorics. Deciding if $c^\lambda_{\mu,\nu} > 0$ arises naturally in Geometric Complexity Theory [**MNS12**]. Though computing $c^\nu_{\lambda,\mu}$ is $\#P$-hard, there exists an algorithm to decide if $c^\nu_{\lambda,\mu} > 0$ in strongly polynomial time [**MNS12**].

Theorem 10.3 ([**Kl98**])**.** *The three nonincreasing sequences α, β, γ of length m are the spectra of some $m \times m$ Hermitian matrices A, B, C satisfying $A + B = C$ if and only if $\sum_{i=1}^m \alpha_i + \beta_i - \gamma_i = 0$ and for all $n < m$,*

$$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \le \sum_{k \in K} \gamma_k$$

for all $|I| = |J| = |K| = n$ such that the Littlewood-Richardson coefficient $c^{\rho(K)}_{\rho(I),\rho(J)}$ is positive.

Combined with Theorem 10.3, Knutson and Tao's answer to the Saturation conjecture in the positive [**KT00**] and a different result of Klyachko [**Kl98**] show that the admissible spectra are described by a recursive system of inequalities originally conjectured by Alfred Horn [**H62**].

We show that Question 5 can be reduced to an instance of Question 1, after which Theorem 10.3 will be a corollary of our main theorem. This results in an algorithmic proof of Theorem 10.3.

Theorem 10.4. *There is a randomized algorithm $\mathcal{A}$ with success probability at least $2/3$ on the following input and output:*

Input: *A tuple (α, β, γ) of three length-m nonincreasing sequences satisfying $\sum_{i=1}^m \alpha_i + \beta_i - \gamma_i = 0$ and*

$$\sum_{i \in I} \alpha_i + \sum_{j \in J} \beta_j \le \sum_{k \in K} \gamma_k$$

for all I, J, K such that the Littlewood-Richardson coefficient $c^{\rho(K)}_{\rho(I),\rho(J)}$ is positive.

Output: *Matrices $A, B \in L(\mathbb{C}^m)$ with*

$$\lambda(A) = \alpha, \lambda(B) = \beta, \text{ and } \|A + B - \operatorname{diag}(\gamma)\| \le \epsilon.$$

Further, if the bit-complexity of (α, β, γ) is at most b, the complexity of $\mathcal{A}$ is $\operatorname{poly}(m, r, b, \frac{1}{\epsilon})$.

We can restate Question 5 for more than three matrices:

Definition 10.3. Define HE^r_m to be the set of $r+1$-length sequences of weakly decreasing m-length sequences of real numbers $(p(1), \ldots, p(r), q)$ such that there exist r Hermitian matrices A_i with $\lambda(A_i) = p(i)$ and

$$\lambda\left(\sum_{i \in [r]} A_i\right) = q.$$

We would like to understand the set HE^r_m, the set of sequences of $r+1$ spectra $p(1), \ldots p(r), q$ such that there exist Hermitian matrices H_i with $\lambda(H_i) = p(i)$ and $\lambda\left(\sum_{i=1}^r H_i\right) = q$. If $\tilde{q}$ is the spectrum of $\operatorname{diag}(-q)$, (that is, $-q$ but written backwards) we find that $(p(1), \ldots p(r), q) \in HE^r_m$ if and only if $(p(1), \ldots p(r), \tilde{q})$ appear as the spectra of Hermitian matrices summing to zero. Further, for any numbers $\lambda_1 + \cdots + \lambda_r = \lambda$, the $p(1) \ldots p(r)$ appear as spectra of Hermitian matrices summing to zero if and only if the spectra $p(1) + \lambda_1 \vec{1}, \ldots p(r) + \lambda_r \vec{1}$, appear as the spectra of

Hermitian matrices summing to $(\sum_{i=1}^r \lambda_i)I = \lambda I$. This can be seen by adding $\lambda_i I$ to each H_i. We can pick λ_i as large as we wish. In particular, pick them so large that every entry of $p(i)$ is larger than every entry of $p(i+1)$ and every entry of $p(r)$ is, say, bigger than

$$\frac{\sum_{i=1}^r \sum_{i=1}^m p_j(i)}{3mr}.$$

and then scale all $p(i)$ so that $\sum_{i=1}^r \sum_{i=1}^m p_j(i) = m$. Thus, we have reduced membership in HE_m^r to membership in $\mathcal{P}_m^{r+1}$, where $\mathcal{P}_m^r$ is defined as follows:

Definition 10.4. Define $\mathcal{P}_m^r$ to be the set of tuples $(p(1), \dots p(r)) \in (\mathbb{R}_+^m)^r$ such that

1. The rm entries of $(p(1), \dots p(r))$ form a weakly decreasing sequence,
2. $\sum_{s=1}^r \sum_{i=1}^m p_i(s) = m$,
3. and $p(r)_m > \frac{1}{3r}$, (The purpose of this assumption is to make the algorithm run faster; it's not necessary for Klyachko's theorem to hold true),
4. and there exist Hermitian matrices H_i, $i \in [r]$ such that $\lambda(H_i) = p(i)$ and

$$\sum_{i=1}^r H_i = I_m.$$

Klyachko's Theorem is a characterization of this set in terms of intersections of Schubert varieties with respect to generic flags, which can in turn be described by the combinatorially defined higher Littlewood-Richardson coefficients $c^{\lambda}_{\mu_1,\dots\mu_r}$. Here is the statement we wish to reprove via operator scaling. We do not show that it is the same as Theorem 10.3, but refer the reader to [**F00**].

Theorem 10.5 (Klyachko)**.** *A tuple of weakly decreasing sequences $(p(1), \dots p(r)) \in (\mathbb{R}_+^m)^r$ is in $\mathcal{P}_m^r$ if and only if it satisfies 1, 2, and 3 and and for each $1 \le k \le m$,*

$$\sum_{s=1}^r \sum_{i \in I(s)} p_i(s) \le k \tag{$*\mathcal{I}$}$$

for every tuple $\mathcal{I} = (I(1), \dots I(s)) \in \binom{[m]}{k}^r$ such that

$$\bigcap_{s\in[r]} \Omega^{\bullet}_{I(s)}(h(s)E_\bullet) \neq \emptyset$$

for generic $(h(1), \dots, h(s)) \in \mathrm{GL}_n(\mathbb{C})^r$. Here $E_\bullet$ is the standard flag of $\mathbb{C}^m$.

Definition 10.5. Define the set of tuples $S_k^m(r)$ to be

$$\left\{ (I(1), \dots, I(s)) \in \binom{[m]}{k}^r : \bigcap_{s\in[r]} \Omega^{\bullet}_{I(s)}(h(s)F_\bullet) \neq \emptyset \text{ for generic } (h(1), \dots, h(s)) \in \mathrm{GL}_m(\mathbb{C})^r \right\}.$$

With this definition in hand, it is enough to show the following:

Claim 10.6. *$p = (p(1), \dots, p(r))$ satisfying 1, 2, and 3 is in $\mathcal{P}_m^r$ if and only if p satisfies $*\mathcal{I}$ for all $\mathcal{I} \in S_k^m(r)$.*

We first define a class of completely positive maps T^r_m and show that the desired Hermitians H_i exist if and only if there exists a member of T^r_m with marginals we will specify.

Definition 10.6. Define $T^r_m : L(\mathbb{C}^{rm}) \to L(\mathbb{C}^m)$ to be the competely positive map with Kraus operators

$$A_i := \begin{bmatrix} 0_{m,mi-m} & I_m & 0_{m,mi} \end{bmatrix}$$

for $i \in [r]$. Let $P = \bigoplus_{s=1}^r \operatorname{diag}(p(s))$, $H = \oplus_{i=1}^r \mathrm{GL}_m(\mathbb{C}) \subset \mathrm{GL}_{mr}(\mathbb{C})$, and $G = \mathrm{GL}_m(\mathbb{C})$.

Clearly (G, H, P, I_m, T) is block-diagonal.

Claim 10.7. *$p \in \mathcal{P}^r_m$ if and only if T^r_m is approximately scalable to $(I_{rm} \to I_m, I_m \to P)$ by (G, H).*

Proof of claim. First we prove the "only if" statement. If $p \in \mathcal{P}^r_m$, then there exist $H_1, \ldots, H_r$ with $\lambda(H_i) = p(i)$ and $\sum_i H_i = I_m$. As $H_i \succeq 0$, we can write $H_i = B_i B_i^\dagger$ where $B_i^\dagger B_i = \operatorname{diag}(p(i))$. This is because $B_i B_i^\dagger$ and $B_i^\dagger B_i$ have the same spectrum and $B_i B_i^\dagger$ is invariant under $B_i \to B_i U_i$ for U_i unitary. Since B_i is invertible for $i \in [r]$, it follows that $h = \bigoplus_{i\in[r]} B_i \in H$ and

$$T_{h,I}(I_{rm}) = I_m \text{ and } T^*_{I,h}(I_m) = P,$$

so T is approximately (G, H) scalable to (P, I).

$$T(I_{rm}) = \sum_{i=1}^r B_i B_i^\dagger = \sum_{i=1}^r H_i^\dagger = I_m.$$

The "if" direction is also easy; suppose $(h(i), g(i))_{i\in\mathbb{N}}$ is a sequence of elements of $G \times H$ such that

$$T_{g(i),h(i)}(I_{rm}) \to I_m \text{ and } T^*_{h(i),g(i)}(I_m) \to P$$

as $i \to \infty$ and that $h(i) = \oplus_{s\in[r]} h_s(i)$. Set $B_s(i) = g^\dagger h_s(i)$; thus, we have

$$\sum_{s\in[r]} B_s(i) B_s(i)^\dagger \to I_m$$

and, for all $s \in [r]$,

$$B_s(i)^\dagger B_s(i) \to \operatorname{diag}(p(i)).$$

Since the $B_s(i)B_s(i)^\dagger$ are positive definite, eventually for all s $B_s(i)B_s(i)^\dagger$ is in the compact set $\{X : 0 \le X \le 2I_m\}$. Thus, we may pass to a subsequence such that for all s we have $B_s(i)B_s(i)^\dagger \to H_s$; by continuity the H_s satisfy

$$\sum_{s=1}^r H_s = I_m.$$

and $\lambda(H_s) = p(s)$ for $s \in [r]$. $\square$

Next we need to examine conditions under which T^r_m is approximately scalable to $(I_{rm} \to I_m, I_m \to P)$ by (G, H). Theorem 10.3 will follow from the next claim:

Claim 10.8. *For p satisfying 1, T^r_m is approximately scalable to $(I_{rm} \to I_m, I_m \to P)$ by (G, H) if and only if $\sum_{s=1}^r \sum_{i=1}^m p_i(s) = m$ and for each $1 < k \le m$,*

$$\sum_{s=1}^r \sum_{i\in I(s)} p_i(s) \le k \tag{$*\mathcal{I}$}$$

for every tuple $\mathcal{I} = (I(1), \dots I(s)) \in \binom{[m]}{k}^r$ *such that*

$$\bigcap_{s \in [r]} \Omega^{\bullet}_{I(s)}(h(s)E_\bullet) \neq \emptyset$$

for generic $(h(1), \dots, h(s)) \in \mathrm{GL}_n(\mathbb{C})^r$*. Here* $E_\bullet$ *is the standard flag of* $\mathbb{C}^m$*.*

Proof of claim. By Theorem 3.9, T^r_m is approximately scalable to marginals $(I_{rm} \to I_m, I_m \to P)$ by (G, H) if and only if

$$\sum_{s=1}^{r} \sum_{i=1}^{m} p_i(s) = m$$

and for all $1 < k \leq m$

$$\sum_{s=1}^{r} \sum_{i \in I(s)} p_i(s) + k \leq m$$

for every every T^r_m independent pair (L, R) such that $\dim L = k$ and

$$R \in \Omega^{\circ}_{\cup_{s=1}^r I(s) \times \{s\} \subset [m] \times [r]}(hF_\bullet),$$

for a generic $h = \oplus_{s \in [r]} h(s)$. Here $F_\bullet \supset F_\circ(P)$ can be taken to be the standard flag of $\mathbb{C}^{mr}$ because P is a diagonal matrix with non-increasing diagonal. Note that the generic element of G has disappeared; this is because the flag $F_\circ(I_m) = (\{0\}, \mathbb{C}^m)$ is invariant under $\mathrm{GL}_m(\mathbb{C})$. We may assume L and R are maximal T^r_m-independent pairs subject to the other being held fixed. Recall that, by definition, L and R are T^r_m-independent if and only if $A_i R \subset L^\perp$ for all $i \in [r]$. However, $A_i : \mathbb{C}^{mr} \to \mathbb{C}^m$ is the projection to the i^{th} summand of $\mathbb{C}^{mr} = \bigoplus_{i=1}^r \mathbb{C}^m$, so

$$R' = \sum_{i=1}^{r} A_i^\dagger L^\perp \supset R$$

and (L, R') are still T^r_m-independent. That is, we may replace R by the direct sum of r embedded copies of $L^\perp$. Abusing notation, we write

$$R' = \bigoplus_{i=1}^{r} L^\perp.$$

Let $E_\bullet$ be the standard flag of $\mathbb{C}^m$. Observe that

$$\bigoplus_{i=1}^{r} L^\perp \in \Omega^{\circ}_{\cup_{s=1}^r I(s) \times \{s\}}(hF_\bullet)$$

if and only if

$$L^\perp \in \bigcap_{s=1}^{r} \Omega^{\circ}_{I(s)}(h(s)E_\bullet),$$

because if $i = ms + k$, $k \in \{0, \dots, m-1\}$, then

$$\bigoplus_{i=1}^{r} L^\perp \cap F_i = \bigoplus_{i=1}^{r} L^\perp \cap (F_{sm} \oplus E_k) = L^\perp \cap E_k \oplus \bigoplus_{i=1}^{s} L^\perp.$$

Now T_m^r is approximately (G,H)-scalable to (P, I_m) if and only if

$$\sum_{s=1}^{r}\sum_{i=1}^{m} p_i(s) = m$$

and for all $1 < k < m$,

$$\sum_{s=1}^{r}\sum_{i\in I(s)} p_i(s) \leq m-k.$$

for all $(I(1), \dots I(r)) \in \binom{[m]}{m-k}^r$ such that $\bigcap_{s=1}^r \Omega^{\circ}_{I(s)}(h(s)E_\bullet)$ is nonempty for generic $(h(1), \dots, h(s)) \in \mathrm{GL}_m(\mathbb{C})^r$. The change of variables $m-k \to k$ yields the claim. □

Observation 10.9. *By the remarks preceding Definition 10.4 (which can clearly be implemented efficiently), we have shown that the Algorithm described in Theorem 10.4 exists, because Algorithm 12.10 efficiently finds ϵ-$(I_{rm} \to \frac{1}{m}I, I_m \to \frac{1}{m}P)$-scalings of T_m^r by (G,H) for $P = \mathrm{diag}(p)$ for $p \in \mathcal{P}_r^m$. That this algorithm has the desired complexity on such instances follows from Theorem 12.11 and the fact that the minimum entry of $\frac{1}{m}p$ is at least $1/3rm$.*

Further, Algorithm 12.10 actually finds real symmetric matrices because the initial random scalings are integer matrices and in each iteration the approximate Cholesky decompositions can be taken to be real matrices.

10.3 An extension of Barthe's Theorem and Schur-Horn theorem

Let $\mathcal{U} = (u_1, \dots, u_n) \in (\mathbb{C}^m)^n$ be an ordered tuple of complex m-vectors, and $p = (p_1, \dots, p_n) \in \mathbb{R}^n_{>0}$. Say a linear transformation $B : \mathbb{C}^m \to \mathbb{C}^m$ puts a collection of vectors $\mathcal{U}$ in *radial isotropic position* with respect to p if

$$\sum_{i=1}^{n} p_i \frac{Bv_i(Bv_i)^\dagger}{\|Bv_i\|^2} = I.$$

Question 6. Given $\mathcal{U}$ and p, when is there a linear transformation B that puts $\mathcal{U}$ in isotropic position with respect to p?

Forster **Fo02** showed that Question 6 has a positive answer if $p = \vec{1}$ and $\mathcal{U}$ is in general position, and used this fact to prove linear lower bounds on unbounded-error communication complexity by showing that explicit matrices have large sign-rank. Question 6 was completely solved by Barthe **B98** and applied to study the Brascamp-Lieb inequalities in analysis. Barthe's Theorem has also been applied in the unsupervised learning problem known as *subspace recovery* **AM13** and to prove upper bounds on the rate of locally correctable codes over the reals **ASZ14**. Barthe showed Question 6 has a positive answer if p lies in a certain polytope, which we now describe.

Definition 10.7. Let $\mathcal{U} = (u_1, \dots, u_n) \in (\mathbb{C}^m)^n$ be an ordered tuple of complex m-vectors. Let $\mathcal{B} \subset \binom{[n]}{k}$ be the collection of m-subsets S of $[n]$ such that $\{u_i : i \in S\}$ forms a basis of $\mathbb{C}^m$. If $q = (q_1, \dots, q_m)$ is a sequence of nonnegative numbers, define

$$\mathcal{K}_q(\mathcal{U}) = \mathrm{conv}\{(1_S(i)q_\sigma(i) : i \in [n]) : S \in \mathcal{B}, \sigma : S \leftrightarrow [m]\}$$

Informally, each vertex of the polytope is the indicator vector for each basis in $\mathcal{U}$ with the nonzero entries replaced by $q_1, \dots q_m$ in some order. If $\vec{1}$ is the all-ones vector, $\mathcal{B}(\mathcal{U}) := \mathcal{K}_{\vec{1}}(\mathcal{U})$ is known as the *basis polytope.*

Theorem 10.10 (**B98**). *$p \in \mathcal{B}(\mathcal{U})$ if and only if there are linear transformations B that put $\mathcal{U}$ arbitrarily close to radial isotropic position with respect to p.*

Further, if p is in the relative interior of $\mathcal{B}(\mathcal{U})$, then there are linear transformations B that put $\mathcal{U}$ in radial isotropic position with respect to p.

As a partial answer to Question 4, we prove a generalization of Barthe's Theorem.

Definition 10.8. Say $\mathcal{U}$ *can be approximately put in Q-isotropic position with respect to p* if for every $\epsilon > 0$ there exists an invertible linear transformation B such that

$$\left\| \sum_{i=1}^{n} p_i \frac{Bu_i(Bu_i)^\dagger}{\|Bu_i\|} - Q \right\| \leq \epsilon. \tag{47}$$

Theorem 10.11. *Suppose Q is a positive-definite matrix with spectrum $q = (q_1, \ldots, q_m)$. $\mathcal{U}$ can be approximately put in Q-isotropic position with respect to p if and only if $p \in \mathcal{K}_q(\mathcal{U})$.*

Before we prove Theorem 10.11, we observe that it implies the "if" direction of the classic Schur-Horn theorem relating the diagonal and spectra of a Hermitian matrix.

Theorem 10.12 (**H54**). *There is a Hermitian $n \times n$ matrix with diagonal $p_1 \geq \cdots \geq p_n$ and spectrum $q_1 \geq \cdots \geq q_n$ if and only if $q_1, \ldots, q_n$ majorizes $p_1, \ldots, p_n$. That is, for all $i \leq n$,*

$$\sum_{j=1}^{i} p_i \leq \sum_{j=1}^{i} q_i.$$

We do not use Theorem 10.11 for the "only if" direction because the "only if" direction is very easy, and the use of Theorem 10.11 would only overcomplicate things.

Proof of Theorem 10.12 from Theorem 10.11. We use an alternate description of majorization; see the textbook [**HJ90**]. The sequence $p_1, \ldots, p_n$ is majorized by $q_1, \ldots, q_n$ if and only if $p = (p_1, \ldots, p_n)$ is in the *permutohedron* of $q = (q_1, \ldots, q_n)$; that is,

$$p \in \operatorname{conv}\{(q_{\sigma(i)} : i \in [n]) : \sigma : [n] \leftrightarrow [n])\}.$$

However, if $\mathcal{U}$ is a basis for $\mathbb{C}^n$, then $\mathcal{K}_q(\mathcal{U})$ from Definition 10.7 is exactly the permutohedron of q. By Theorem 10.11, if $p \in \mathcal{K}_q(\mathcal{U})$, for every $\epsilon > 0$ there is an invertible linear transformation $B \in \mathrm{GL}_n(\mathbb{C})$ satisfying

$$\left\| \sum_{i=1}^{n} p_i \frac{Bu_i(Bu_i)^\dagger}{\|Bu_i\|} - Q \right\| \leq \epsilon.$$

If we let $\epsilon \to 0$, by compactness, we may pass to a subsequence in which for all i the vector $\frac{Bu_i}{\|u_i\|}$ converges to some unit vector v_i. Thus, we have

$$\sum_{i=1}^{n} p_i v_i (v_i)^\dagger = Q.$$

If V is a matrix with columns $\sqrt{p_i} v_i$, then $Q = VV^\dagger$. However, $V^\dagger V$ has the same spectrum as $VV^\dagger$ (i.e. $q_1, \ldots, q_n$), and the i^{th} diagonal entry of $V^\dagger V$ is $\langle \sqrt{p_i} v_i, \sqrt{p_i} v_i \rangle = p_i$. Thus, $V^\dagger V$ is the promised matrix. $\square$

Remark 10.13. Algorithm GOSI actually shows that the matrix $V^\dagger V$ in the proof can be taken to be a real, symmetric matrix because the random initial scaling is an integer matrix, and in each iteration the Cholesky decomposiitons can be taken to be real matrices. Note that if the entries of q are 0 except for $q_1, \ldots, q_m$ for $m < n$, we can actually say much more about the matrix V appearing in the proof. Theorem 10.11 tells us for which $m \times n$ matrices U (i.e. a matrix whose columns are the elements of $\mathcal{U}$) there exist matrices $B \in \mathrm{GL}_n(\mathbb{C})$ and $n \times n$ diagonal matrices D such that $D^\dagger U^\dagger B^\dagger BUD$ tends to a matrix with spectrum q and diagonal p. Alternatively, when there exists $m \times m$ invertible Hermitian matrices A and $n \times n$ diagonal matrices D such that $D^\dagger U^\dagger AUD$ tends to a matrix with spectrum q and diagonal p.

Let us begin the proof of Theorem 10.11. Our proof goes by a reduction to the problem of scaling an operator to specified marginals.

Proof of Theorem 10.11. Theorem 10.11 will be a straightforward consequence of Lemma 10.15, which shows that scalability is equivalent with membership in some other polytope, and Proposition 10.16 which shows that said other polytope is actually the same as $\mathcal{K}_q(\mathcal{U})$. First we need to form an instance of operator scaling.

Definition 10.9. If $\mathcal{U}$ is a tuple of n vectors in $\mathbb{R}^d$, let $T_\mathcal{U} : \mathrm{Mat}_{n\times n}(\mathbb{C}) \to \mathrm{Mat}_{m\times m}(\mathbb{C})$ be the completely positive map with Kraus operators

$$A_i := \left[\begin{array}{ccc} 0_{m,i-1} & u_i & 0_{m,n-i} \end{array} \right]$$

for $i \in [n]$. Here $0_{m,k}$ denotes an $m \times k$ zero submatrix.

Let P denote the matrix with diagonal $p = (p_1, \ldots, p_m)$, $G = \mathrm{GL}_m(\mathbb{C})$, and H the subgroup of $\mathrm{GL}_n(\mathbb{C})$ consisting of diagonal matrices. Question 4 now amounts to Question 1 for T, P, Q, G, H. Note that $(T, G, H, F_\circ(P), F_\circ(Q))$ is block-diagonal with blocks $\langle f_i \rangle, i \in [n]$ for V and one block for W.

Claim 10.14. *$\mathcal{U}$ can be approximately put in Q-isotropic position with respect to p if and only if $T_\mathcal{U}$ is approximately scalable to $(P \to Q, I_m \to I_n)$.*

Proof. Let $T = T_\mathcal{U}$. To see why, note that

$$T^*_{h,g}(I_m) = \mathrm{diag}(|h_i|^2 \|gu_i\|^2)$$

and

$$T_{g,h}(P) = \sum_{i=1}^n p_i |h_i|^2 gu_i (gu_i)^\dagger.$$

If $\|T_{g,h}(P) - Q\| \le \epsilon$ and $T^*_{h,g}(I_m) = I_n$, then $|h_i|^2 = \|gu_i\|^{-2}$ and so 47 holds for $B = g$. If instead $\|T^*_{h,g}(I_m) - I_n\| \le \epsilon$, one has to reset h to impose $T^*_{h,g}(I_m) = I_n$. However, that the change in $\|T_{g,h}(P) - Q\|$ from such a normalization is $O(\epsilon)$. Similarly, if B satisfies 47 then setting $g = B$ and $|h_i|^2 = \langle Bu_i, QBu_i \rangle^{-2}$ satisfies $\|T_{g,h}(P) - Q\| \le \epsilon$ and $T^*_{h,g}(I_m) = I_n$.

By reasoning analogous to that of Lemma 3.1, $T_\mathcal{U}$ is approximately (G, H)-scalable to $(P \to Q, I_m \to I_n)$ if and only if $T_\mathcal{U}$ is approximately (G, H)-scalable to $(I_n \to Q, I_m \to P)$ because P is nonsingular. ☐

In what follows, we use Theorem 3.9 to prove Theorem 10.11. Without loss of generality, we may assume Q is diagonal.

Lemma 10.15. *$\mathcal{U}$ can be approximately put in Q-isotropic position with respect to p if and only if*

$$\sum_{j=1}^{n} p_j = \sum_{i=1}^{m} q_i$$

and

$$\sum_{j\in J} p_j \le \sum_{i=1}^{\dim\langle u_j : j\in J\rangle} q_i$$

for all $J \subset [n]$.

Proof. Let $P = \operatorname{diag}(p)$. Without loss of generality, assume $\operatorname{Tr} Q = 1$. By Claim 10.14, $\mathcal{U}$ can be approximately put in q-isotropic position with respect to p if and only if $T_{\mathcal{U}}$ approximately (G,H)-scalable to $(I_n \to Q, I_m \to P)$ where $G = \mathrm{GL}_m(\mathbb{C})$ and H is the subgroup of diagonal matrices in $\mathrm{GL}_n(\mathbb{C})$.
Note that $F_\circ(P), F_\circ(Q)$ can be taken to be subflags of the standard flags $F_\bullet$ and $E_\bullet$ of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. By Theorem 3.9, $T_{\mathcal{U}}$ is approximately (G,H)-scalable to (P,Q) if and only if $T_{\mathcal{U}}$ is $(p,q,gE_\bullet,hF_\bullet)$-rank-nondecreasing for a generic $(g^\dagger,h)\in G\times H$. However, $hF_\bullet = F_\bullet$ because h is diagonal, so $T_{\mathcal{U}}$ is approximately (G,H)-scalable to (P,Q) if and only if $T_{\mathcal{U}}$ is $(p,q,gE_\bullet,F_\bullet)$-rank-nondecreasing for a generic $(g^\dagger,h)\in G\times H$. Equivalently, $(T_{\mathcal{U}})_{g,I}$ is (P,Q)-rank-nondecreasing for generic $g\in G$.

That is,

$$\sum_{i\in I} q_i + \sum_{j\in J} p_j \le 1$$

for all $I\subset[m]$, $J\subset[n]$ such that $\Omega_I^\circ(E_\bullet)\times\Omega_J^\circ(F_\bullet)$ contains a $(T_{\mathcal{U}})_{g,I}$-independent pair.

(L,R) is a $(T_{\mathcal{U}})_{g,I}$-independent pair if and only if

$$L\subset\langle gu_j : R\not\subset e_j^\perp\rangle^\perp.$$

As the pair (L_J,R_J) where

$$R' = \langle e_j : R\not\subset e_j^\perp\rangle\supset R$$

and

$$L' = \langle gu_j : R\not\subset e_j^\perp\rangle^\perp\supset L$$

is still independent, and we only need check the inequality for maximal independent pairs, we may assume $R = R_J := \langle e_j : j\in J\rangle$ and $L = gL_J := g\langle u_j : j\in J\rangle^\perp$ for some $J\subset[n]$. Note that $R_J\in\Omega_J^\circ(F_\bullet)$. Thus, $(T_{\mathcal{U}})_{g,I}$ is (P,Q)-rank-nondecreasing for generic $g\in\mathrm{GL}_n(\mathbb{C})$ if and only if

$$\sum_{i\in I} q_i + \sum_{j\in J} p_j \le 1$$

for all $I\subset[m]$, $J\subset[n]$ such that $gL_J\in\Omega_I^\circ(E_\bullet)$ for generic $g\in\mathrm{GL}_n(\mathbb{C})$. For fixed J and generic g, gL_J is simply a generic subspace of dimension $d=\dim L_J$. Such a subspace will satisfy $E_{n-d}\cap gL_J=\{0\}$. Equivalently, $gL_J\in\Omega^\circ_{\{n-d+1,\dots,n\}}(E_\bullet)$. Otherwise, the linear map $\pi_{E_{n-d}^\perp}\circ g : L_J\to E_{n-d}^\perp$ is singular, which implies a polynomial (namely, the determinant of the map) which is not identically 1 vanishes on g. This would be a contradiction to the genericity of g.

Since the number of J is finite, for generic g we must have $gL_J \in \Omega^\circ_{\{n-d+1,\dots,n\}}(E_\bullet)$ for $d = \dim L_J$ for all $J \subset [n]$. Thus,

$(T_{\mathcal{U}})_{g,I}$ is (P,Q)-rank-nondecreasing for generic $g \in \mathrm{GL}_n(\mathbb{C})$ if and only if

$$\sum_{i=n-\dim L_J+1}^{n} q_i + \sum_{j\in J} p_j \leq 1$$

for all $J \subset [n]$. Equivalently,

$$\sum_{j\in J} p_j \leq \sum_{i=1}^{\dim\langle u_j : j\in J\rangle} q_i$$

for all $J \subset [n]$.

□

Recall Definition 10.7 of the polytope $\mathcal{K}_q(\mathcal{U})$.

Proposition 10.16. *p is in $\mathcal{K}_q(\mathcal{U})$ if and only if*

$$\sum_{j=1}^{n} p_j = \sum_{i=1}^{m} q_i \tag{48}$$

and

$$\sum_{j\in J} p_j \leq \sum_{i=1}^{\dim\langle u_j : j\in J\rangle} q_i \tag{49}$$

for all $J \subset [n]$.

Proof. Polytopes of the form $\sum_{j\in J} p_j \leq f(J)$ for submodular set functions f are well-understood, so we first check that our constraints take this form.

Lemma 10.17. *The function*

$$f_q(J) = \begin{cases} \sum_{i=1}^{\dim\langle u_j : j\in J\rangle} q_j & J \neq \emptyset \\ f_q(J) = 0 & J = \emptyset \end{cases} \tag{50}$$

is a nonnegative, monotone, submodular function on the lattice of subsets of $[n]$.

Proof of Lemma 10.17. Nonnegativity and monotonicity are clear. To show that f_q is submodular, it is enough to show f_q gives decreasing marginal returns, that is, for $X \subset Y$ and $x \in [n] \setminus Y$,

$$f_q(X \cup \{j\}) - f_q(X) \geq f_q(Y \cup \{j\}) - f_q(Y).$$

Indeed,

$$\begin{aligned} f_q(X \cup \{j\}) - f_q(X) &= \sum_{i=1}^{\dim(\langle u_i : i\in X\rangle + u_j)} q_i - \sum_{i=1}^{\dim\langle u_i : i\in X\rangle} q_i \\ &= 1_{u_j \notin \langle u_i : i\in X\rangle} q_{\dim\langle u_i : i\in X\rangle+1} \geq 1_{u_j \notin \langle u_i : i\in Y\rangle} q_{\dim\langle u_i : i\in Y\rangle+1} \\ &= f_q(Y \cup \{j\}) - f_q(Y). \end{aligned}$$

□

Next we use a theorem of Edmonds.

Theorem 10.18 ([**Ed70**]). *If E is a finite set and L is an intersection-closed family of subsets of E, let*

$$P(E,f) = \{x \in \mathbb{R}^E_+ : \forall S \in L - \emptyset,\ \sum_{i \in S} x_i \le f(S)\}.$$

If f is a nonnegative, monotone function on $\mathcal{P}(E)$ with $f(\emptyset) = 0$, then each vertex x of $P(E,f)$ is given by

$$x_\sigma(i) = f(\{\sigma(j) : j \le i\}) - f(\{\sigma(j) : j < i\})$$

for some ordering $\sigma : [|E|] \leftrightarrow E$, and every ordering corresponds to such a vertex.

Next note that p satisfies the equality 48 and the inequality 49 if and only if

$$p \in \mathcal{K}'_q(\mathcal{U}) := P([n], f_q) \cap \left\{ p : \sum_{i=1}^n p_i = \sum_{i=1}^m q_i \right\}.$$

For $x \in P([n], f_q)$, $\sum_{i=1}^n x_i \le \sum_{i=1}^m q_i$ by the constraint when $J = [n]$. Thus, $\mathcal{K}'_q(\mathcal{U})$ is the convex hull of the vertices of $P([n], f_\lambda)$ that are contained in the hyperplane $\{p : \sum_{i=1}^n p_i = \sum_{i=1}^m q_i\}$. Recall from Definition 10.7 that these are exactly the vertices of $\mathcal{K}_q(\mathcal{U})$; hence $\mathcal{K}_q(\mathcal{U}) = \mathcal{K}'_q(\mathcal{U})$. The proposition is proved. □

□

10.4 Quantum Schrödinger bridges

The classical Schrödinger bridge problem seeks a suitable probability law for a diffusion process (e.g. Brownian motion) matching a given initial and a final marginal distribution. The discrete version of this problem for Markov chains seeks the transition matrix B satisfying $Bc = r$ of maximum relative entropy distance from A.

In [**GP15**] the authors show that the column-stochastic matrix B satisfying $Bc = r$ of maximum relative entropy from the prior transition matrix A is in fact a scaling XAY for X and Y diagonal. The change of variables $Y \leftarrow \operatorname{diag}(c)Y$ shows that a matrix is scalable to one satisfying the marginals r and c if and only if it is (r,c)-scalable. Without the relative entropy interpretation, it was understood that (r,c)-scalings arise as the solution of the same optimization problem as early as [**MO68**].

The analogue of a Markov transition matrix considered in [**GP15**] is a *unital quantum channel*, namely a completely positive map T satisfying $T^*(I) = I$. Given marginals P and Q, Georgieu and Pavon seek a suitable unital quantum channel T with $T(P) = Q$. Rather than posing a relative notion of entropy, they simply look for a quantum channel T that is a scaling of a "prior" quantum channel T' in some sense. Friedland [**Fr16**] points out that, under a change of variables, the channel Georgieu and Pavon seek is precisely a $(P \to Q, I_W \to I_V)$-scaling of T.

The authors of [**GP15**] conjecture that if $T(X) \succ 0$ for all $0 \ne X \succeq 0$ (a condition they refer to as *positivity improving*), then T can be scaled to a unital quantum channel T' satisfying $T'(P) = Q$. A proof of this conjecture for P and Q *nonsingular* appears in [**Fr16**] using contractive properties of a metric suited for use with fixed-point theorems. We extend the result of [**Fr16**] to possibly singular P and Q, all but proving the conjecture of Georgieu and Pavon.

Before proceeding, let us state their conjecture precisely. If Y is a vector space let $D(Y) = \{Y \in \mathcal{S}_{++}(Y) : \operatorname{Tr} Y = 1\}$. Elements of $D(Y)$ are called density matrices.

Conjecture 10.19 (**GP15**). *For any positivity-improving completely positive operator $T : L(V) \to L(W)$ and density matrices $P \in D(V), Q \in D(W)$, there exist operators $\phi_0, \hat{\phi}_T \in \mathcal{S}_{++}(V)$, $\phi_T, \hat{\phi}_0 \in \mathcal{S}_{++}(W)$ and linear transformations $\chi_0 \in L(W), \chi_T \in L(V)$ such that*

$$T(\phi_T) = \phi_0, \qquad T^*(\hat{\phi}_0) = \hat{\phi}_T \tag{51}$$

$$Q = \chi_0 \hat{\phi}_0 \chi_0^\dagger, \qquad P = \chi_T \hat{\phi}_T \chi_T^\dagger \tag{52}$$

$$\phi_0 = \chi_0^\dagger \chi_0, \qquad \phi_T = \chi_T^\dagger \chi_T. \tag{53}$$

Furthermore, χ_0 and χ_T can be taken to be Hermitian.

Our techniques can prove the conjecture, but, as in [**Fr16**], our transformations χ_0, χ_T are not guaranteed to be Hermitian. We state a stronger theorem, and as Corollary 10.23, show that the theorem implies Conjecture 10.19 (modulo the Hermitian part).

Theorem 10.20. *If there exists $(g, h) \in \mathrm{GL}(V) \times \mathrm{GL}(W)$ such that $T_{g,h}$ is (P, Q)-indecomposable, there exist positive-definite operators ϕ_0, ϕ_T, $\hat{\phi}_0, \hat{\phi}_T \in L(V)$ and linear transformations χ_0, χ_T satisfying 51, 52, and 53.*

Remark 10.21. The truth is much stronger. If there exists g_0, h_0 such that T_{g_0,h_0} is (P, Q)-indecomposable, then $T_{g,h}$ is (P, Q)-indecomposable for *generic* $(g^\dagger, h) \in \mathrm{GL}(W) \times \mathrm{GL}(V)$. We do not prove this because the proof is identical to that of Proposition 8.5.

Proof. We begin by showing how to proceed if P and Q are invertible. This shows how matrices 51, 52, and 53 should be obtained from a scaling by renaming variables. If P and Q are invertible, then by Theorem 9.7, the existence of $(g, h) \in \mathrm{GL}(V) \times \mathrm{GL}(W)$ such that $T_{g,h}$ is (P, Q)-indecomposable implies $T_{g,h}$ is scalable to $(P \to I_W, Q \to I_V)$ by $(\mathrm{GL}(V)_{F_\circ(P)}, \mathrm{GL}(W)_{F_\circ(Q)}$. By Lemma 3.1, this implies there exists $(g, h) \in \mathrm{GL}(V) \times \mathrm{GL}(W)$ such that

$$T(hh^\dagger) = g^{-\dagger} g^{-1}$$
$$\text{and } h^\dagger T^*(gQg^\dagger) h = P.$$

Then we can take $\phi_0 = g^{-\dagger} g^{-1}$, $\chi_0 = g^{-1}$, $\hat{\phi}_0 = gQg^\dagger$, $\hat{\phi}_T = T(gQg^\dagger)$, and $\chi_T = h^\dagger$.

However, P and Q may not be invertible. To handle this, we will use Proposition 3.16. Suppose there exists $(g_0, h_0) \in \mathrm{GL}(V) \times \mathrm{GL}(W)$ such that T_{g_0,h_0} is (P, Q)-indecomposable. Since $F_i(P) \subset \operatorname{supp} P$ and $F_i(Q) \subset \operatorname{supp} Q$ and the spectra of $\underline{P}$ and $\underline{Q}$ are the nonzero eigenvalues of P and Q, $\underline{T_{g_0,h_0}}$ is $(\underline{P}, \underline{Q})$-indecomposable. By Theorem 9.7, $\underline{T_{g_0,h_0}}$ is $(\mathrm{GL}(\operatorname{supp} Q)_{F_\circ(\underline{Q})}, \mathrm{GL}(\operatorname{supp} P)_{F_\circ(\underline{P})})$-scalable to $(\underline{P} \to I_{\operatorname{supp} Q}, \underline{Q} \to I_{\operatorname{supp} P})$. Write $\hat{T} := T_{g_0,h_0}$. By Lemma 3.1 applied to $\underline{\hat{T}}$, there exists $(\underline{g}, \underline{h}) \in \mathrm{GL}(\operatorname{supp}(Q)) \times \mathrm{GL}(\operatorname{supp}(P))$ such that

$$\nu \hat{T}(\eta^\dagger \underline{h}\underline{h}^\dagger \eta) \nu^\dagger = \underline{g}^{-\dagger} \underline{g}^{-1},$$
$$\underline{h}^\dagger \eta \hat{T}^*(\nu^\dagger \underline{g} \nu Q \nu^\dagger \underline{g}^\dagger \nu) \eta^\dagger \underline{h} = \eta P \eta^\dagger.$$

We need to extend $\underline{g}, \underline{h}$ to elements of $\mathrm{GL}(W), \mathrm{GL}(V)$, respectively, to obtain the desired matrices. That this is always possible is a result of the following elementary fact:

Fact 10.22. Let $A \in L(W)$ be positive semidefinite, ν a partial isometry to a subspace $L \subset W$, and suppose $\nu A \nu^\dagger = g^\dagger g$ for $g \in \mathrm{GL}(L)$. Then there exists a linear transformation $\tilde{g} : W \to W$ with

$$\begin{aligned}\tilde{g}^\dagger \tilde{g} &= A,\\ \tilde{g} L &\subset L, \text{ and}\\ \nu \tilde{g} \nu^\dagger &= g.\end{aligned}$$

Proof of Fact 10.22. Let $\tilde{g}$ be the block Cholesky decomposition of A with respect to the partial flag $(\{0\}, L, W)$. $\tilde{g}$ automatically satisfies the first two properties. Further, by Lemma 1,

$$\nu A \nu^\dagger = \nu \tilde{g}^\dagger \tilde{g} \nu^\dagger = \nu \tilde{g}^\dagger \nu^\dagger \nu \tilde{g} \nu^\dagger = g^\dagger g.$$

This implies $g = U \nu \tilde{g} \nu^\dagger$ for some unitary operator U on L. Now $(U \oplus I_{L^\perp})\tilde{g}$ satisfies all three properties. $\square$

We now use the claim to complete the proof. By Claim 10.22, there exists $\tilde{g} : W \to W$ such that

$$\begin{aligned}\hat{T}(\eta^\dagger \underline{h} \underline{h}^\dagger \eta) &= \tilde{g}^\dagger \tilde{g},\\ \tilde{g}\, \mathrm{supp}(Q) &\subset \mathrm{supp}(Q), \text{ and}\\ \nu \tilde{g} \nu^\dagger &= \underline{g}^{-1}.\end{aligned}$$

Since $\eta\eta^\dagger = I_{\mathrm{supp}\, P}$,

$$\hat{T}(\eta^\dagger \underline{h} \eta \eta^\dagger \underline{h}^\dagger \eta) = \tilde{g}^\dagger \tilde{g}. \tag{54}$$

We already know

$$\underline{h}^\dagger \eta \hat{T}^*(\nu^\dagger \underline{g} \nu Q \nu^\dagger \underline{g}^\dagger \nu) \eta^\dagger \underline{h} = \eta P \eta^\dagger,$$

so

$$\eta^\dagger \underline{h}^\dagger \eta \hat{T}^*(\nu^\dagger \underline{g} \nu Q \nu^\dagger \underline{g}^\dagger \nu) \eta^\dagger \underline{h} \eta = P. \tag{55}$$

Plugging in $\hat{T} = T_{g_0,h_0}$ to 54 and 55, we obtain

$$\begin{aligned}T(h_0 \eta^\dagger \underline{h} \eta \eta^\dagger \underline{h}^\dagger \eta h_0^\dagger) &= g_0^{-\dagger} \tilde{g}^\dagger \tilde{g} g_0^{-1}\\ \text{and } \eta^\dagger \underline{h}^\dagger \eta h_0^\dagger T^*(g_0 \nu^\dagger \underline{g} \nu Q \nu^\dagger \underline{g}^\dagger \nu g_0^\dagger) h \eta^\dagger \underline{h} \eta &= P.\end{aligned}$$

Take

$$\begin{aligned}\phi_0 &= g_0^{-\dagger} \tilde{g}^\dagger \tilde{g} g_0^{-1}, & \phi_T &= h \eta^\dagger \underline{h} \eta \eta^\dagger \underline{h}^\dagger \eta h^\dagger,\\ \hat{\phi}_0 &= g_0 \nu^\dagger \underline{g} \nu Q \nu^\dagger \underline{g}^\dagger \nu g_0^\dagger, & \hat{\phi}_T &= T^*(g_0 \nu^\dagger \underline{g} \nu Q \nu^\dagger \underline{g}^\dagger \nu g_0^\dagger),\\ \chi_0 &= \tilde{g} g_0^{-1}, & \text{and } \chi_T &= \eta^\dagger \underline{h}^\dagger \eta h^\dagger.\end{aligned}$$

The only nontrivial property to check for this assignment is $Q = \chi_0 \hat{\phi}_0 \chi_0^\dagger$. Expanding this, we find

$$\begin{aligned}&\chi_0 \hat{\phi}_0 \chi_0^\dagger\\ &= \tilde{g} g_0^{-1} g_0 \nu^\dagger \underline{g} \nu Q \nu^\dagger \underline{g}^\dagger \nu g_0^\dagger g_0^{-\dagger} \tilde{g}^\dagger\\ &= \tilde{g} \nu^\dagger \underline{g} \nu Q \nu^\dagger \underline{g}^\dagger \nu \tilde{g}^\dagger\end{aligned}$$

However, $\tilde{g}\,\mathrm{supp}(Q) = \mathrm{supp}(Q)$, so by Lemma 3.15, $\tilde{g}\nu^\dagger = \nu^\dagger\nu\tilde{g}\nu^\dagger = \nu^\dagger \underline{g}^{-1}$. This shows

$$\begin{aligned} &\tilde{g}\nu^\dagger \underline{g}\nu Q\nu^\dagger \underline{g}^\dagger \nu\tilde{g}^\dagger \\ &= \nu^\dagger \underline{g}^{-1} \underline{g}\nu Q\nu^\dagger \underline{g}^\dagger \nu \underline{g}^{-\dagger}\nu \\ &= \nu^\dagger\nu Q\nu^\dagger\nu = Q. \end{aligned}$$

This completes the proof. □

Corollary 10.23. *Conjecture 10.19 is true without the requirement that χ_0, χ_T are Hermitian.*

Proof. Suppose T is positivity-improving. We will show T is (P,Q)-indecomposable, so T, P, Q satisfy the hypotheses of Theorem 10.20 with $g = I_W, h = I_V$. Because T is positivity-improving, if $R \neq \{0\}$ then $T(\pi_R) \succ 0$ and so $T(\pi_R)\pi_L \neq 0$ for any $L \neq 0$. Thus (see Remark 3.5) the only T-independent pairs (L,R) have $L = \{0\}$ or $R = \{0\}$. Since $1 = \operatorname{Tr} P = \operatorname{Tr} Q$ the inequality 6 holds for any pairs (L,R) where L or R is $\{0\}$ and the inequality 45 is trivially satisfied for all T-independent pairs (L,R) such that neither L nor R is $\{0\}$. This implies T is (P,Q)-indecomposable. □

11 Future work

1. We wonder if there is an algorithm to find ϵ-scalings in time polynomial in $-\log(\epsilon)$ rather than ϵ^{-1}. As it is, our algorithm resembles alternating minimization; perhaps other optimization techniques could result in faster algorithms. The recent fast (r,c)-scaling algorithms [**Li17**, **M17**] give hope that this is possible. It has been indicated to the author that such a speed-up does exist in the case $P = Q = I$.

2. The algorithms herein, i.e. Algorithm 12.12, do not decide (P,Q)-rank-nondecreasingness in strongly polynomial time. In order to certify (P,Q)-rank-nondecreasingness one requires ϵ-$(I_V \to Q, I_W \to P)$-scalings for ϵ as small as a common denominator of the spectra p and q. However, our algorithm depends linearly on ϵ^{-1}. It would be nice to understand the complexity of determinining (P,Q)-rank-nondecreasingness from (p,q); at least for 3 there is a strongly polynomial time algorithm to decide if the reduction is (P,Q)-rank-nondecreasing that has nothing to do with operator scaling [**MNS12**].

3. If the completely positive map T^ρ and the bipartite mixed quantum state ρ are in correspondence via the Jamiołkowski isomorphism [**Ja74**], the scalings of T^ρ scale ρ in a very simple way that closely resembles the transformations allowed under the communication complexity class known as SLOCC [**DVC00**]. However, SLOCC equivalence of ρ, ρ' and interscalability of T^ρ and $T^{\rho'}$ are not the same. We hope to infer something meaningful about the relationship between ρ and ρ' when T^ρ can be scaled to $T^{\rho'}$.

4. The Littlewood-Richardson coefficients give combinatorial conditions for scalability. We wonder if this is possible for other operators than the one arising in the reduction for 3.

12 Appendix

12.1 Additional background

The following facts are standard and can be found in any textbook on matrix analysis, e.g. [**HJ90**].

Fact 12.1. If A and B are $n \times n$ matrices, then

$$\begin{aligned} \|A\| &\le \sqrt{n}\|A\|_2, \\ \|AB\| &\le \|A\|_2\|B\|, \\ \|AB\| &\le \sqrt{n}\|A\|_2\|B\|_2, \\ \text{and } \|AB\|_2 &\le \|A\|_2\|B\|_2. \end{aligned}$$

Fact 12.2. If A and B are matrices, then

$$\begin{aligned} \|A^{-1} - B^{-1}\| &\le \|A^{-1}\|_2\|B^{-1}\|_2\|A - B\| \\ \text{and } \|A^{-1} - B^{-1}\|_2 &\le \|A^{-1}\|_2\|B^{-1}\|_2\|A - B\|_2 \\ \text{and } \|A^{-1} - I\| &\le \frac{\|A - I\|}{1 - \|A - I\|} \text{ provided } \|A - I\| < 1. \end{aligned}$$

12.2 Additional proofs

Proof of Lemma 3.6. Recall that we wish to show

$$\det(P, Xh) = \det(P, X)\det(P, h), \tag{56}$$

$$\det(P, h^\dagger Xh) = \det(P, h^\dagger h)\det(P, X), \tag{57}$$

$$\text{and } \det(P, h^{-\dagger}h^{-1}) = \det(P, h^\dagger h)^{-1}. \tag{58}$$

First we prove 56.

$$\begin{aligned} \det(P, Xh) &= \prod_{i:\Delta p_i \neq 0} \det(\eta_i Xh\eta_i^\dagger)^{\Delta p_i} \\ &= \prod_{i:\Delta p_i \neq 0} \det(\eta_i X\eta_i^\dagger\eta_i h\eta_i^\dagger)^{\Delta p_i} \\ &= \det(P, X)\det(P, h). \end{aligned}$$

By taking two transposes of 56, we obtain

$$\begin{aligned} \det(P, h^\dagger Xh) &= \det(P, h^\dagger)\det(P, h)\det(P, X) \\ &= \prod_{i:\Delta p_i \neq 0} \det(\eta_i h^\dagger\eta_i^\dagger\eta_i h\eta_i^\dagger)^{\Delta p_i}\det(\eta_i X\eta_i^\dagger)^{\Delta p_i} \\ &= \prod_{i:\Delta p_i \neq 0} \det(\eta_i h^\dagger h\eta_i^\dagger)^{\Delta p_i}\det(\eta_i X\eta_i^\dagger)^{\Delta p_i} \\ &= \det(P, h^\dagger h)\det(P, X), \end{aligned}$$

proving 57. The identity 58 follows from 57 and $\det(P, I) = 1$. $\square$

Proof of Lemma 3.1. First prove that $1 \implies 2$. If $g \in G_{E_\circ}$ and $h \in H_{F_\circ}$ are such that $T_{g,h}$ is an ϵ-$(P \to I_V, Q \to I_V)$-scaling of T, then

$$\|g^\dagger T(hPh^\dagger)g - I_V\| \le \epsilon \text{ and } \|h^\dagger T^*(gQg^\dagger)h - I_V\| \le \epsilon.$$

Note that $\sqrt{P} \in H_{F_\circ}$ because H is block-diagonal and P is diagonal, and hence upper-triangular, in the basis F. Let $\tilde{h} = h\sqrt{P} \in H_{F_\circ}$ so that

$$\|T_{g,\tilde{h}}(I_W) - I_V\| \le \epsilon$$

and

$$\|T^*_{\tilde{h},g}(Q) - P\| \le \|\sqrt{P}T^*_{h,g}(Q)\sqrt{P} - P\| \le \|P\|_2^2 \|T^*_{h,g}(Q) - I_V\| \le \epsilon,$$

so T has an ϵ-$(I_V \to I_W, Q \to P)$-scaling by $(G_{E_\circ}, H_{F_\circ})$. A similar proof shows that if T has an ϵ-$(I_V \to I_W, Q \to P)$-scaling, then T has an ϵ-$(I_V \to Q, I_W \to P)$-scaling. The rest of the implications are similar. $\square$

Proof of 3.16. We first show 1. Recall

$$\operatorname{cap}(T, P, Q) = \inf_{h \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(Q, T(hPh^\dagger))}{\det(P, h^\dagger h)}.$$

However, $\det(Q, X)$ only depends on $\nu_i X \nu_i^\dagger$ for $i \in \sigma(F_\circ(Q))$, and $\nu_i \nu^\dagger \nu = \nu_i$. However, $\nu_i \nu^\dagger$ is a partial isometry $\operatorname{supp} Q \to F_i(Q)$. Thus,

$$\begin{aligned}
\det(Q, T(hPh^\dagger)) &= \prod_{i \in \sigma(F_\circ(Q))} \det(\nu_i T(hPh^\dagger)\nu_i^\dagger) \\
&= \prod_{i \in \sigma(F_\circ(\underline{Q})} \det(\nu_i \nu^\dagger \nu T(hPh^\dagger)\nu^\dagger \nu \nu_i^\dagger) \\
&= \det(\underline{Q}, \nu T(hPh^\dagger)\nu^\dagger).
\end{aligned}$$

A similar replacement can be made in the denominator. Now

$$\begin{aligned}
\operatorname{cap}(T, P, Q) &= \inf_{h \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(\underline{Q}, \nu T(hPh^\dagger)\nu^\dagger)}{\det(\underline{P}, \eta h^\dagger h \eta^\dagger)} \\
&= \inf_{h \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(\underline{Q}, \nu T(h\eta^\dagger \underline{P} \eta h^\dagger)\nu^\dagger)}{\det(\eta P \eta^\dagger, \underline{h}^\dagger \underline{h})} \\
&= \inf_{h \in \mathrm{GL}(V)_{F_\circ(P)}} \frac{\det(\underline{Q}, \nu T(\eta^\dagger \underline{hPh}^\dagger \eta)\nu^\dagger)}{\det(\underline{P}, \underline{h}^\dagger \underline{h})},
\end{aligned}$$

but in fact, $\underline{\mathrm{GL}(V)_{F\circ(P)}} = \mathrm{GL}(\operatorname{supp} P)_{F\circ(\underline{P})}$, so

$$\operatorname{cap}(T, P, Q) = \inf_{h \in \mathrm{GL}(\operatorname{supp} P)_{F_\circ(\underline{P})}} \frac{\det(\underline{Q}, \underline{T}(h\underline{P}h^\dagger)}{\det(\underline{P}, h^\dagger h)} = \operatorname{cap}(\underline{T}, \underline{P}, \underline{Q}).$$

We now prove 2. Let n', m' be the dimensions of $\operatorname{supp} Q$ and $\operatorname{supp} P$, respectively. Clearly,

$$\sum_{i=1}^m q_i = \sum_{j=1}^n p_j := N$$

if only if

$$\sum_{i=1}^{m'} q_i = \sum_{j=1}^{n'} p_j := N.$$

Further, the inequality

$$\sum_{i \in \sigma(F_\circ(Q))} \Delta q_i \dim F_i(P) \cap L + \sum_{j \in \sigma(F_\circ(P))} \Delta p_i \dim F_j(P) \cap R \le N \tag{59}$$

holds for all T-independent pairs (L, R) if and only if it holds for all T-independent pairs (L, R) where $L \subset \operatorname{supp} Q$, $R \subset \operatorname{supp} P$. The "only if" statement is clear, but the "if" statement follows because $\sigma(F_\circ(P)) \subset [n']$ and $\sigma(F_\circ(Q)) \subset [m']$, and in particular $F_{n'}(P) = \operatorname{supp} P$ and $F_{m'}(Q) = \operatorname{supp} Q$. Thus, replacing L by $L \cap \operatorname{supp} Q$ and R by $R \cap \operatorname{supp} P$ does not change the left-hand side of the inequality. However, if $L \subset \operatorname{supp} Q$, $R \subset \operatorname{supp} P$, then (L, R) is T-independent if and only if it is $\underline{T}$ independent, because the Kraus operators of $\underline{T}$ are $\nu A_i \eta^\dagger$, and for $l \in \operatorname{supp} Q$ and $r \in \operatorname{supp} P$, we have $l \nu A_i \eta^\dagger r = l^\dagger A_i r$. Thus, T is (P, Q)-rank-nondecreasing if and only if

$$\sum_{i=1}^{m'} q_i = \sum_{j=1}^{n'} p_j := N$$

and 59 holds for all $\underline{T}$ independent pairs (L, R); equivalently, $\underline{T}$ is $(\underline{P}, \underline{Q})$-rank-nondecreasing. ☐

12.3 Complexity assumptions

Here we prove the claim that the assumption that there are at most mn Kraus operators is without loss of generality. Alone, this is not difficult to prove. However, proving that T can essentially be represented by nm Kraus operators with small integer entries is slightly more difficult. The content of the following Lemma is that if we are interested in testing (P, Q)-rank-nondecreasingness or we are looking for ϵ-$(I_V \to Q, I_W \to P)$-scalings then we can efficiently replace T by a completely positive map $\tilde{T}$ represented by nm Kraus operators with small integer entries and run any algorthm on $\tilde{T}$ instead. Note that $\tilde{T}$ may depend on ϵ.

Lemma 12.3. *Suppose T, P, Q are as in 3.6 and the Kraus operators $A_1, \ldots A_r$ of T have b-bit binary complex entries with $r > mn$. Then for any $\epsilon > 0$, there is a completely positive operator $\tilde{T}$ with Kraus operators $B_1, \ldots B_{mn}$, each with $\operatorname{poly}(r, b)$-bit binary complex entries, such that for every $(g, h) \in \mathrm{GL}(W) \times \mathrm{GL}(V)$, $\tilde{T}_{g,h}$ is (P, Q)-rank-nondecreasing if and only if $T_{g,h}$ is (P, Q)-rank-nondecreasing and any $\frac{1}{3\sqrt{mn}}\epsilon$-$(I_V \to Q, I_W \to P)$-scaling of $\tilde{T}$ is an ϵ-$(I_V \to Q, I_W \to P)$-scaling of T.*

Further, the Kraus operators $B_1, \ldots B_{mn}$ can be computed in time $\operatorname{poly}(r, b, -\log \epsilon)$.

Proof. The strategy will be to find a completely positive map $\tilde{T}$ with Kraus operators $B_i = \sum_{j=1}^r U_{ji} A_i$ that approximates T, but has $B_i = 0$ for $i > mn$.

Claim 12.4. *If $B_i = \sum_{j=1}^r U_{ji} A_i$ for an invertible matrix $U \in \mathrm{GL}_r(\mathbb{C})$, then the completely positive map $\tilde{T} : X \mapsto \sum_{j=1}^r B_j X B_j^\dagger$ has $\tilde{T}_{g,h}$ rank-nondecreasing if and only if $T_{g,h}$ is rank-nondecreasing.*

Proof of claim. The Kraus operators of $\tilde{T}_{g,h}$ will be related to the Kraus operators of $T_{g,h}$ by the same matrix U. However, the T-independent pairs are precisely the $\tilde{T}$-independent pairs, because

$$x^\dagger \sum_{i=1}^r U_{ji} A_i y = 0$$

for all $j \in [r]$,$x \in L, y \in R$ if and only if $x^\dagger A_i y = 0$ for all $i \in [r]$, $x \in L, y \in R$ by the invertibility of U. ☐

Claim 12.5. *Suppose $\tilde{T}$ has the Kraus operators $B_1, \ldots, B_r$ where $B_j = \sum_{i=1}^r U_{ji} A_i$ for $i \in [r]$ and $\|UU^\dagger - I_r\| \leq \delta < .5$. Further suppose that $\|\tilde{T}(I) - P\| \leq \delta$. Then $\|T(X) - \tilde{T}(X)\| \leq 2\delta\sqrt{n} \operatorname{Tr} \tilde{T}(X)$.*

Proof of claim.

$$\begin{aligned}\tilde{T}(X) &= \sum_{j=1}^{r} B_j X B_j^\dagger \\ &= \sum_{j=1}^{r}\left(\sum_{i=1}^{r} U_{ji}A_i\right) X \left(\sum_{i'=1}^{r} U_{ji'}A_{i'}\right)^\dagger \\ &= \sum_{j=1}^{r}\sum_{i,i'=1}^{r} U_{ji}\overline{U_{ji'}} A_i X A_{i'}^\dagger \\ &= \sum_{i,i'=1}^{r} A_i X A_{i'}^\dagger \sum_{j=1}^{r} U_{ji}\overline{U_{ji'}} \\ &= \sum_{i,i'=1}^{r} A_i X A_{i'}^\dagger (UU^\dagger)_{ii'},\end{aligned}$$

so

$$\begin{aligned}\|\tilde{T}(X) - T(X)\| &\le \sum_{i,i'=1}^{r} \|A_i X A_{i'}^\dagger\| |(UU^\dagger - I_r)_{ii'}| \\ &\le \|UU^\dagger - I\| \sqrt{\sum_{i,i=1}^{r} \|A_i X A_{i'}^\dagger\|^2} \\ &\le \delta \sqrt{\sum_{i,i=1}^{r} \operatorname{Tr} A_i X A_{i'}^\dagger A_{i'} X A_i^\dagger} \\ &= \delta\sqrt{\operatorname{Tr} T^*(I) X T^*(I) X} \\ = \delta\sqrt{\|\sqrt{X}T^*(I)\sqrt{X}\|^2} &\le \delta\sqrt{n(\operatorname{Tr}\sqrt{X}T^*(I)\sqrt{X}\|)^2} \\ &= \delta\sqrt{n}\operatorname{Tr} T(X).\end{aligned}$$

Note that $A_j = \sum_{i=1}^r U_{ji}^{-1} B_i$, and by Fact 12.2,

$$\|U^{-1}U^{-\dagger} - I\| = \|U^{-\dagger}U^{-1} - I\| = \|(UU^\dagger)^{-1} - I\| \le \frac{\delta}{1-\delta} \le 2\delta.$$

Thus we can apply the previous bound with the roles of $\tilde{T}$ and T reversed to obtain $\|\tilde{T}(X) - T(X)\| \le 2\sqrt{n}\delta \operatorname{Tr}\tilde{T}(X) \le 2\sqrt{n}\delta(1+\sqrt{m}\delta) \le 4\delta\sqrt{mn}$. $\square$

If for every δ we can efficiently obtain U with $\|UU^\dagger - I\|_r \le \delta$, and $\operatorname{poly}(r, b, -\log\delta)$-bit complex binary entries with $B_j = 0$ for $j > mn$ we will be done: if $\tilde{T}_{g,h}$ is an ϵ-$(I_V \to Q, I_W \to P)$-scaling, by the second claim applied to $T_{g,h}, \tilde{T}_{g,h}$ and $T^*_{h,g}, \tilde{T}^*_{h,g}$ with $X = I$ and $\delta = \epsilon$ we have

$$\begin{aligned}\|T_{g,h}(I) - Q\| &\le \tilde{T}_{g,h}(I) - Q\| + \|\tilde{T}_{g,h}(I) - T_{g,h}(I)\| \\ &\le \epsilon + 2\sqrt{n}\epsilon \operatorname{Tr}\tilde{T}(I) \\ &\le \epsilon + 2\sqrt{n}\epsilon(1+\sqrt{m}\epsilon) \\ &\le 4\epsilon\sqrt{mn}.\end{aligned}$$

Such U exists and can be found efficiently by none other than finding an approximately orthonormal basis for some mn vectors. We first need

$$\sum_{i=1}^{r} U_{ji}(A_i)_{lk} = 0$$

for all $(l,k) \in [m] \times [n]$ and $j > mn$. That is, we need the vectors $(U_{ji})_{i\in[r]}$ for $j \in \{mn+1,\dots,r\}$ to be orthogonal to the mn vectors $((A_i)_{lk})_{i\in[r]}$ for $(l,k) \in [m] \times [n]$. Thus, an orthonormal basis for the orthogonal complement of $((A_i)_{lk})_{i\in[r]}$ for $(l,k) \in [m] \times [n]$ would suffice. The rest of the vectors $(U_{ji})_{i\in[r]}$ for $j \in [mn]$ could be taken to be an orthonormal basis for the span of $((A_i)_{lk})_{i\in[r]}$ for $(l,k) \in [m] \times [n]$. However, exact orthonormal bases may not have rational entries. Instead, we'll use the following standard fact which follows from the fact that Cholesky decompositions can be done in polynomial time.

Fact 12.6. Given a real number $\delta > 0$ and at most r vectors $u_1,\dots,u_k \in R^r$ where each entry of the u_k is a b-bit complex binary number, one can find a basis $v_1,\dots,v_k$ whose entries are $\operatorname{poly}(b,r,-\log\delta)$-bit complex binary numbers for the span of $u_1,\dots,u_k$ such that if V is the matrix whose rows are the v_i, $\|VV^\dagger - I\| \le \delta$ in time $\operatorname{poly}(b,r,-\log\delta)$.

This, we set the last $r - mn$ rows $(U_{ji})_{i\in[r]}$ for $j \in \{mn+1,\dots,r\}$ to be the basis guaranteed by Fact 12.6 applied to some basis of the orthogonal complement of $((A_i)_{lk})_{i\in[r]}$ for $(l,k) \in [m] \times [n]$ with $\operatorname{poly}(r,M)$-bit binary complex entries (which can be computed in time $\operatorname{poly}(r,M)$ using Gaussian elimination. For the first mn rows $(U_{ji})_{i\in[r]}$ for $j \in [mn]$ we do the same using the vectors $((A_i)_{lk})_{i\in[r]}$ for $(l,k) \in [m] \times [n]$. Then the first mn rows are automatically orthogonal to the last $r - mn$, and there will be a contribution of δ^2 from the first mn rows and δ^2 from the last $r - mn$ rows to the squared trace norm. Thus,

$$\|UU^\dagger - I\| \le \sqrt{2}\delta.$$

and U has $\operatorname{poly}(r,M,-\log\delta)$-bit complex binary entries. □

12.4 Algorithm TOSI with rounding

Superficially, the algorithm with rounding does not resemble Algorithm TOSI, but we will show that the steps of the new algorithm capture each step of the old.

Remark 12.7. The algorithms in this section actually compute ϵ-$(P \to I_W, Q \to I_V)$-scalings. However, by Corollary 3.20, ϵ-$(\underline{P} \to I_{\operatorname{supp} Q}, \underline{Q} \to I_{\operatorname{supp} P})$-scalings can be converted efficiently to 3ϵ-$(I_V \to Q, I_W \to P)$-scalings.

Definition 12.1. If A is a Hermitian matrix with entries represented as signed binary numbers and s a positive integer, let $\operatorname{Rnd}_s A$ be the Hermitian matrix obtained by dropping all but s bits of each entry after the decimal point.

Algorithm 12.8 (Algorithm TOSI with rounding)**.**

Input: T, P, Q, G, H such that $(G, H, F_\circ(P), F_\circ(Q), T)$ is block-diagonal, a real number $\epsilon > 0$, and an integer $s > 0$.

Output: $g \in G$ and $h \in H$ such that $\operatorname{ds}_{P,Q}(T_{g,h}) \le \epsilon$.

Let s be a nonnegative integer, and set $\tilde{S}_0^i = I_{F_i(P)}$ for $i \in \sigma(F_\circ(P))$.

1. Increment j.
2. **If j is odd:** For $i \in \sigma(F_\circ(Q))$, set

$$\tilde{S}_j^i = \left(\nu_i T \left(\sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger \operatorname{Rnd}_s\left(\tilde{S}_{j-1}^k\right)\eta_k\right)\nu_i^\dagger\right)^{-1}.$$

   Find $g \in G_{F_\circ(Q)}$ and $h \in H_{F_\circ(P)}$ such that

$$\|gg^\dagger - \tilde{S}_j^i\|_2 \leq 2^{-s}$$

   and

$$\|hh^\dagger - \tilde{S}_{j-1}^i\|_2 \leq 2^{-s}$$

   If j is even: For $i \in \sigma(F_\circ(P))$, set

$$\tilde{S}_j^i = \left(\eta_i T^* \left(\sum_{k\in\sigma(F_\circ(Q))} \Delta q_k \nu_k^\dagger \operatorname{Rnd}_s\left(\tilde{S}_{j-1}^k\right)\nu_k\right)\eta_i^\dagger\right)^{-1}.$$

   Find $g \in G_{F_\circ(Q)}$ and $h \in H_{F_\circ(P)}$ such that

$$\|gg^\dagger - \tilde{S}_{j-1}^i\|_2 \leq 2^{-s}$$

   and

$$\|hh^\dagger - \tilde{S}_j^i\|_2 \leq 2^{-s}.$$

3. If $\operatorname{ds}_{P,Q} T_{g,h} < \epsilon$,

   Return: g, h.

Assumption 2. The entries of p and q are binary $\leq \log M$-bit numbers.

Theorem 12.9. *Suppose $(G, H, F_\circ(P), F_\circ(Q), T)$ is block-diagonal and T, P, Q satisfy satisfy Assumptions 1 and 2. Define*

$$s = \lceil 2(t+4)^2(\lg\alpha + \lg\beta)\rceil$$

and

$$t = \frac{112(m+n)\log(m+n) + 56\log M}{\min\{\epsilon, p_n\}, + \min\{\epsilon, q_m\}} + 2.$$

If T is (P,Q)-rank-nondecreasing, then Algorithm 12.8 terminates in at most t steps and can be performed in time

$$\operatorname{poly}(\epsilon^{-1}, p_n^{-1}, q_m^{-1}, n, m, \log M).$$

We delay the proof of Theorem 12.9 until after some analysis of the effect of rounding errors. We now show how to incorporate Algorithm 12.8 into Algorithm GOSI. The algorithm with rounding is identical to Algorithm GOSI except instead of calling Algorithm TOSI after scaling randomly, we call Algorithm 12.8.

Algorithm 12.10 (GOSI with rounding)**.**

Input: T, P, Q as in 3.6 where P and Q have rational spectra and a real number $\epsilon > 0$.

Output: Left and right scalings $g \in G$ and $h \in H$ such that $T_{g,h}$ is an ϵ-$(I_V \to Q, I_W \to P)$-scaling of T, or ERROR.

1. Let $0 < \gamma \in \mathbb{Z}$ be such that γP and γQ have integral spectra. Choose each entry of $(g_0, h_0) \in \oplus_i \operatorname{Mat}_{n_i}(\mathbb{C}) \times \oplus_j \operatorname{Mat}_{m_j}(\mathbb{C})$ uniformly at random from $K := [3\max\{2\gamma^2, n, m\}]$. If g_0 or h_0 is singular, output ERROR.

2. Let p_{min}, q_{min} be the least nonzero entries of p, q, respectively. Let $\underline{g}$ and $\underline{h}$ be the output of Algorithm 12.8 with input $\underline{T_{g_0,h_0}}, \underline{P}, \underline{Q}, \frac{\epsilon}{4\min\{p_{min}, q_{min}\}}$. If any step of Algorithm 12.8 on this input cannot be performed, output ERROR.

3. Use $(\underline{g}, \underline{h})$ to compute (g, h) such that T_{g_0g,h_0h} is an ϵ-$(I_V \to Q, I_W \to P)$-scaling of T_{g_0,h_0} as guaranteed by Corollary 3.20. Return (g_0g, h_0h).

Theorem 12.9 and Claim 8.10 lead to good complexity bounds for Algorithm 12.10.

Theorem 12.11. *Suppose $(G, H, F_\circ(P), F_\circ(Q), T)$ is block-diagonal and T, P, Q satisfy Assumptions 1 and 2. Define*

$$s = \lceil 2(t+4)^2(\lg\alpha + \lg\beta)\rceil.$$

If T is approximately (G, H)-scalable to (P, Q), then Algorithm 12.10 on input $T, P, Q, G, H, \epsilon, s$ outputs ERROR with probability at most 1/3, and can be performed in time

$$\operatorname{poly}(\epsilon^{-1}, p_{min}^{-1}, q_{min}^{-1}, n, m, \log M).$$

In particular, Algorithm 12.10 terminates.

Proof. The first step succeeds in producing $g \in G_0$ and $h \in H_0$ such that $T_{g,h}$ is (P, Q)-rank-nondecreasing with probability at least $2/3$ by Claim 8.10. By Assumption 2, the parameter γ in Algorithm GOSI can be taken in $[2^{(m+n)\log M}]$. Thus, the new Kraus operators can be computed in polynomial time and have bit-complexity $\log M' = \operatorname{poly}(n, m, \log M)$.

Thus, $T_{g,h}$ satisfies Assumption 1 for $\log M' = \operatorname{poly}(n, m, \log M)$, and Assumptions 3.14 and 2 are unchanged. By Theorem 12.9, Algorithm 12.8 outputs correct scalings with probability at least $2/3$ and can be performed in time $\operatorname{poly}(\epsilon^{-1}, p_{min}^{-1}, q_{min}^{-1}, n, m, \log M)$ By Corollary 3.20, step 3 is possible and since the output of Algorithm 12.8 must have had bit-complexity $\operatorname{poly}(s, m, n)$, can be done in time $\operatorname{poly}(\epsilon^{-1}, p_{min}^{-1}, q_{min}^{-1}, n, m, \log M)$. $\square$

Now we can obtain an exponential time algorithm for membership in $K(T, G, H)$, proving Theorem 3.13. Along with Algorithm 12.10 this implies an exponential time algorithm to decide approximate scalability of T to $(I_V \to Q, I_W \to P)$.

Algorithm 12.12.

Input: T, P, Q, G, H such that $(G, H, F_\circ(P), F_\circ(Q), T)$ is block-diagonal and P and Q have rational spectra with denominator d.

Output: YES with probability at least $2/3$ if $(p, q) \in K(T, G, H)$ and NO if $(p, q) \notin K(T, G, H)$.

1. Run Algorithm 12.10 with parameters as in Theorem 12.11 with some $\epsilon < \frac{1}{(\sqrt{n}+\sqrt{m})d}$. If ERROR is output, **return** NO. If ERROR is not output, **return** YES.

Corollary 12.13. *There is a $\operatorname{poly}(n, m, d)$-time algorithm that outputs YES with probability 2/3 if $x \in K(T, G, H)$ and NO with probability 1 if $x \notin K(T, G, H)$.*

Since d can be taken to be at most the total bit-length of the descriptions of p and q, Theorem 3.13 is proved.

Proof. By Theorem 12.11, Algorithm 12.10 outputs an ϵ-$(I_V \to Q, I_W \to P)$-scaling with probability at least $2/3$ if $(p,q) \in \mathcal{K}(T,G,H)$. Note that Algorithm 12.10 outputs ERROR if it does not output an ϵ-$(I_V \to Q, I_W \to P)$-scaling.

By Corollary 7.2, if $T_{g,h}$ is an ϵ-$(I_V \to Q, I_W \to P)$-scaling for

$$\epsilon < \min \left\{ \frac{1}{\sqrt{m}+\sqrt{n}} \left| \operatorname{Tr} P - \sum_{i \in I} q_i - \sum_{j \in J} p_j . \right| : I \subset [m], J \subset [n] \right\} \setminus \{0\}$$

then $T_{g,h}$ is (P,Q)-rank-nondecreasing, or $(p,q) \in \mathcal{K}(T_{g,h}, F_\circ, E_\circ) \subset \mathcal{K}(T,G,H)$. Note that the above inequality holds if we take $\epsilon < \frac{1}{(\sqrt{n}+\sqrt{m})d}$. Thus, Algorithm 12.12 outputs YES with probability at least $2/3$ if $(p,q) \in \mathcal{K}(T,G,H)$ and always outputs NO otherwise. $\square$

12.4.1 A recurrence from Sinkhorn scaling

We define a matrix recurrence that captures the scalings from Algorithm G for arbitrary marginals.

Definition 12.2. Define $S_0^i = I_{F_i(P)}$ for $i \in \sigma(F_\circ(P))$, and

If j is odd: For $i \in \sigma(F_\circ(Q))$,

$$S_j^i = \left(\nu_i T \left(\sum_{k \in \sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger S_{j-1}^k \eta_k \right) \nu_i^\dagger \right)^{-1}.$$

If j is even: For $i \in \sigma(F_\circ(P))$,

$$S_j^i = \left(\eta_i T^* \left(\sum_{k \in \sigma(F_\circ(Q))} \Delta q_k \nu_k^\dagger S_{j-1}^k \nu_k \right) \eta_i^\dagger \right)^{-1}.$$

Observe that the above recurrence is the same recurrence as that in Algorithm 12.8 without rounding. We analyse Algorithm 12.8 by showing that the above recurrence contains $g_j g_j^\dagger$ and $h_j h_j^\dagger$ from Algorithm TOSI in a precise way.

First, we show several useful relationships between scalings g and h and the operators $gg^\dagger$ and $hh^\dagger$ which will be used to analyse the recurrence.

Lemma 12.14. *Suppose $g \in \mathrm{GL}(W)_{F_\circ(Q)}$ and $h \in \mathrm{GL}(V)_{F_\circ(P)}$. Let $S_1^m = gg^\dagger$ and $S_0^n = hh^\dagger$, and define operators S_0^i and S_1^j by*

$$(S_0^j)^{-1} = \eta_j (S_0^n)^{-1} \eta_j^\dagger.$$

for $j \in \sigma(F_\circ(P))$ and

$$(S_1^i)^{-1} = \nu_i (S_1^m)^{-1} \nu_i^\dagger$$

for $i \in \sigma(F_\circ(Q))$, Then

$$S_0^j = \eta_j h \eta_j^\dagger \eta_j h^\dagger \eta_j^\dagger, \; S_1^i = \nu_i g \nu_i^\dagger \nu_i g^\dagger \nu_i^\dagger, \tag{60}$$

$$hPh^\dagger = \sum_{k \in \sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger S_0^k \eta_k, \; gQg^\dagger = \sum_{k \in \sigma(F_\circ(Q))} \Delta q_k \nu_k^\dagger S_1^k \nu_k \tag{61}$$

and

$$\mathrm{ds}_{P,Q}\, T_{g,h} = \sum_{i\in\sigma(F_\circ(P))} \Delta p_i \operatorname{Tr}\left(S_0^i \eta_i T^* \left(\sum_{k\in\sigma(F_\circ(Q))} \Delta q_k \nu_k^\dagger S_1^k \nu_k \right) \eta_i^\dagger - I_{F_i(P)} \right)^2$$
$$+ \sum_{i\in\sigma(F_\circ(Q))} \Delta q_i \operatorname{Tr}\left(S_1^i \nu_i T \left(\sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger S_0^k \eta_k \right) \nu_i^\dagger - I_{F_i(Q)} \right)^2 \tag{62}$$

Before proving the above Lemma, we use it to show how the S_j^i determine the iterates of Algorithm TOSI.

Corollary 12.15. *For $j \geq 0$, define g_j and h_j as in Algorithm TOSI.*

For $j \geq 0$ odd: *For $i \in \sigma_{i\in F_\circ(Q)}$,*

$$\nu_i g_j \nu_i^\dagger \nu_i g_j^\dagger \nu_i^\dagger = S_j^i.$$

In particular,

$$g_j g_j^\dagger = S_j^m,$$

For $j \geq 0$ even: *For $i \in \sigma_{i\in F_\circ(P)}$.*

$$\eta_i h_j \eta_i^\dagger \eta_i h_j^\dagger \eta_i^\dagger = S_j^i$$

In particular,

$$h_j h_j^\dagger = S_j^n.$$

Proof of Corollary 12.15. We prove the first two claims by induction on j; 62 will follow from those. For $j = 0$, clearly the claims hold. We only perform the induction step in the case where j is odd, because the proof for j even is analogous.

If j odd, in Algorithm TOSI we defined g_j by

$$g_j g_j^\dagger = T(h_{j-1} P h_{j-1}^\dagger)^{-1}.$$

By induction, $h_{j-1}h_{j-1}^\dagger = S_{j-1}^n$. By 61,

$$g_j g_j^\dagger = T\left(\sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger S_{j-1}^k \eta_k \right)^{-1} = S_j^m.$$

The claim for j odd now follows from 60. ∎

Now we prove Lemma 12.14.

Proof of Lemma 12.14. First we show 60. We only show $S_0^j = \eta_j h \eta_j^\dagger \eta_j h^\dagger \eta_j^\dagger$; the other equation is analogous. By $h \in \mathrm{GL}(V)_{F_\circ(P)}$ and Lemma 3.15.

$$(S_0^j)^{-1} = \eta_j (h^\dagger)^{-1} h^{-1} \eta_j^\dagger = \eta_j (h^\dagger)^{-1} \eta_j^\dagger \eta_j h^{-1} \eta_j^\dagger.$$

By Lemma 3.15, $(S_0^j)^{-1} = (\eta_j h \eta_j^\dagger \eta_j h^\dagger \eta_j^\dagger)^{-1}$, proving 60.

Next we show 62. By Definition 5.1,

$$\mathrm{ds}_{P,Q}\, T_{g,h} = \sum_{i \in \sigma(F_\circ(P))} \Delta p_i \operatorname{Tr}\left(\eta_i (h^\dagger T^*(gQg^\dagger)h - I)\eta_i^\dagger\right)^2 + \sum_{i \in \sigma(F_\circ(Q))} \Delta q_i \operatorname{Tr}\left(\nu_i (g^\dagger T(hPh^\dagger)g - I)\nu_i^\dagger\right)^2$$

Using the cyclic properties of trace, we can rewrite the first term in the above sum:

$$\begin{aligned} &\sum_{i \in \sigma(F_\circ(P))} \Delta p_i \operatorname{Tr}\left(\eta_j (h^\dagger T^*(gQg^\dagger)h - I)\eta_j^\dagger\right)^2 \\ &= \sum_{i \in \sigma(F_\circ(P))} \Delta p_i \operatorname{Tr}\left(\eta_i h \eta_i^\dagger \eta_i h^\dagger \eta_i^\dagger \eta_i T^*(gQg^\dagger)\eta_i - I_{F_i(P)}\right)^2. \end{aligned}$$

Applying 60 and 61 to the above line, we obtain

$$\begin{aligned} &\sum_{i \in \sigma(F_\circ(P))} \Delta p_i \operatorname{Tr}\left(\eta_j (h^\dagger T^*(gQg^\dagger)h - I)\eta_j^\dagger\right)^2 \\ &= \sum_{i \in \sigma(F_\circ(P))} \Delta p_i \operatorname{Tr}\left(S_0^i \eta_i T^*\left(\sum_{k \in \sigma(F_\circ(Q))} \Delta q_k \nu_k^\dagger S_1^k \nu_k\right)\eta_i^\dagger - I_{F_i(P)}\right)^2. \end{aligned}$$

The second term can be rewritten analogously, proving the claim. □

12.4.2 Perturbations of Sinkhorn iterates

Next we'll need to analyse sequences $\tilde{S}_j^i$ that are very close to S_j^i, e.g. the $\tilde{S}_j^i$ in Algorithm 12.8.

Definition 12.3. Say a sequence of sets of matrices $\{\tilde{S}_j^i : i \in \sigma(F_\circ(Q))\}$ for j odd and $\{\tilde{S}_j^i : i \in \sigma(F_\circ(P))\}$ for j even is a δ-sequence if

$\tilde{S}_0^i = I_{F_i(P)}$ for $i \in \sigma(F_\circ(P))$ and

If j is odd: For $i \in \sigma(F_\circ(Q))$,

$$\tilde{S}_j^i = \left(\nu_i T\left(\Delta_j + \sum_{k \in \sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger \tilde{S}_{j-1}^k \eta_k\right)\nu_i^\dagger\right)^{-1}$$

If j is even: For $i \in \sigma(F_\circ(P))$,

$$\tilde{S}_j^i = \left(\eta_i T^*\left(\Delta_j + \sum_{k \in \sigma(F_\circ(Q))} \Delta q_k \nu_k^\dagger \tilde{S}_{j-1}^k \nu_k\right)\eta_i^\dagger\right)^{-1}$$

where Δ_j are Hermitian operators with $\Delta_j \prec \delta P$ if j is odd and $\Delta_j \prec \delta Q$ if j is even. The Δ_j should be thought of as small perturbations.

For small enough δ, a δ-sequence does not stray far from the actual S_j^i's. To prove this we will need some eigenvalue bounds.

Lemma 12.16. *Suppose $P \in \mathcal{S}_{++}(V), Q \in \mathcal{S}_{++}(W)$ and $(p,q) \in \mathcal{K}_1(T, F_\bullet(Q), F_\bullet(P))$ where T satisfies Assumption 1. Then*

1. $\|T(P)\|_2 \le (mnM)^2$ *and* $\|T^*(Q)\|_2 \le (mnM)^2$.

2. $$\|T(P)^{-1}\|_2 \le \|T(P)\|_2^{\frac{1}{q_m}} e^{\frac{5(n+m)\log(n+m)+2\log M}{q_m}}$$

   and
   $$\|T^*(Q)^{-1}\|_2 \le \|T^*(Q)\|_2^{\frac{1}{p_n}} e^{\frac{5(n+m)\log(n+m)+2\log M}{p_n}}.$$

Proof. First we show 1. We prove $\|T(P)\|_2 \le (mnM)^2$; the other claim of 1 follows similarly. Since $\operatorname{Tr} P = 1$, $p_1 \le 1$, so
$$T(P) \preceq T(I).$$
However, $\|T(I)\|_2 \le (mnM)^2$. The last inequality follows because each entry of $T(I)$ is of magnitude at most $(mn)nM^2$ and since $T(I)$ is $m \times m$, $\|T(I)\|_2 \le (mnM)^2$.

Next we show 2. If $(p,q) \in \mathcal{K}_1(T, F_\bullet(Q), F_\bullet(P))$, then by Lemma 5.10,
$$\det(Q, T(P)) \ge \operatorname{cap}(T, P, Q) \ge e^{-5(n+m)\log(n+m)-2\log M}.$$
But
$$\det(Q, T(P)) \le \|T(P)\|_2 \lambda_{min}(T(P))^{q_m},$$
so
$$\|T(P)^{-1}\|_2 = \lambda_{min}(T(P))^{-1} \le \|T(P)\|_2^{\frac{1}{q_m}} e^{\frac{5(n+m)\log(n+m)+2\log M}{q_m}}.$$
2 can be proved analogously because
$$(p,q) \in \mathcal{K}_1(T, F_\bullet(Q), F_\bullet(P)) \iff (q,p) \in \mathcal{K}_1(T^*, F_\bullet(P), F_\bullet(Q)),$$
by symmetry of the definition of $\mathcal{K}(T, F_\bullet(Q), F_\bullet(P))$ from rank-nondecreasingness. □

Lemma 12.17. *Suppose $(p,q) \in \mathcal{K}_1(T, F_\bullet(Q), F_\bullet(P))$ and T satisfies Assumption 1. Let*
$$\alpha = e^{\frac{8(m+n)\log(m+n)+4\log M}{q_m}}.$$
and
$$\beta = e^{\frac{8(m+n)\log(m+n)+4\log M}{p_n}}.$$
Suppose $t \ge 2$ even, and that $\tilde{S}_j$ is a δ-sequence with $\delta \le (\alpha\beta)^{-t/2}$.

1. *If $j < t$ is odd,*
   $$\|\tilde{S}^i_j\|_2, \|(\tilde{S}^i_j)^{-1}\|_2 \le \alpha^{(j+1)/2}\beta^{(j-1)/2},$$
   and if $j < t$ is even,
   $$\|\tilde{S}^i_j\|_2, \|(\tilde{S}^i_j)^{-1}\|_2 \le (\alpha\beta)^{j/2}.$$

2. *If $j < t$, $\|\tilde{S}^i_j - S^i_j\|_2 \le \delta(\alpha\beta)^{\binom{j+3}{2}}$.*

3. *For odd $j < t-1$, suppose $\tilde{g}_j \in \mathrm{GL}(W)_{F_\circ(Q)}$ is such that $\|\tilde{g}_j\tilde{g}_j^\dagger - \tilde{S}_j^m\|_2 \le \delta$, and if j is even, suppose $\tilde{h}_j \in \mathrm{GL}(V)_{F_\circ(P)}$ is such that $\|\tilde{h}_j\tilde{h}_j^\dagger - \tilde{S}_j^n\|_2 \le \delta$. Then*
$$\left|\mathrm{ds}_{P,Q}\, T_{\tilde{g}_j,\tilde{h}_{j-1}} - \mathrm{ds}_{P,Q}\, T_{g_j,h_j}\right| \le \delta(\alpha\beta)^{(j+4)^2}.$$

Proof. We prove the first two items by induction on j. We prove 1 first. Clearly 1 holds for $j = 0$. We only perform the induction step in case j is odd, since j even is analogous. Suppose $t-1 \ge j \ge 1$ is odd. Then the Leowner ordering is reversed under inversion, so

$$\begin{aligned}
\|\tilde{S}_j^i\|_2 &= \left\|\left(\nu_i T\left(\sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger \tilde{S}_{j-1}^k \eta_k + \Delta_j\right)\nu_i^\dagger\right)^{-1}\right\|_2 \\
&\le \left\|\left(\nu_i T\left(\sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger (\alpha\beta)^{(j-1)/2} I \eta_k - \delta P\right)\nu_i^\dagger\right)^{-1}\right\|_2 \\
&\le \left\|\left(\nu_i T\left(((\alpha\beta)^{(j-1)/2} - \delta)P\right)\nu_i^\dagger\right)^{-1}\right\|_2
\end{aligned}$$

By Lemma 12.16, and $\delta \le (\alpha\beta)^{-t/2} \le \frac{1}{2}(\alpha\beta)^{-(j-1)/2}$, the last line is at most

$$\begin{aligned}
&((\alpha\beta)^{-(j-1)/2} - \delta)^{-1}(m^2n^2M^2)^{\frac{1}{qm}} e^{\frac{5(m+n)\log(m+n)+2\log M}{qm}} \\
&\le 2(\alpha\beta)^{(j-1)/2}(m^2n^2M^2)^{\frac{1}{qm}} e^{\frac{5(m+n)\log(m+n)+2\log M}{qm}} \\
&\le \alpha^{(j+1)/2}\beta^{(j-1)/2}.
\end{aligned}$$

Next, by our induction hypothesis,

$$\begin{aligned}
\|(\tilde{S}_j^i)^{-1}\|_2 &= \left\|\nu_i T\left(\sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger \tilde{S}_{j-1}^k \eta_k + \Delta_j\right)\nu_i^\dagger\right\|_2 \\
&\le \left\|\nu_i T\left((\alpha\beta)^{(j-1)/2}P + \delta P\right)\nu_i^\dagger\right\|_2 \le ((\alpha\beta)^{(j-1)/2} + \delta)m^2n^2M^2 \le \alpha^{(j+1)/2}\beta^{(j-1)/2}
\end{aligned}$$

with a great deal of slack. Next we prove 2. By Fact 12.2 and 1,

$$\begin{aligned}
&\|\tilde{S}_j^i - S_j^i\|_2 \\
&\le \|(\tilde{S}_j^i)^{-1}\|_2 \|(S_j^i)^{-1}\|_2 \|(\tilde{S}_j^i)^{-1} - (S_j^i)^{-1}\|_2 \\
&\le (\alpha\beta)^{j+1}\left\|\nu_i T\left(\Delta_j + \sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger\left(\tilde{S}_{j-1}^k - S_{j-1}^k\right)\eta_k\right)\nu_i^\dagger\right\|_2.
\end{aligned}$$

By our induction hypothesis,

$$\begin{aligned}
&(\alpha\beta)^j\left\|\nu_i T\left(\Delta_j + \sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger\left(\tilde{S}_{j-1}^k - S_{j-1}^k\right)\eta_k\right)\nu_i^\dagger\right\|_2 \\
&\le \alpha^{2j}\left\|\nu_i T\left(\delta P + (\alpha\beta)^{\binom{j+2}{2}}\delta P\right)\nu_i^\dagger\right\|_2 \\
&\le (\alpha\beta)^{j+1}\left(\delta + (\alpha\beta)^{\binom{j+2}{2}}\delta\right)m^2n^2M^2 \\
&\le \delta(\alpha\beta)^{\binom{j+2}{2}+j+2} = \delta(\alpha\beta)^{\binom{j+3}{2}}.
\end{aligned}$$

again with room to spare.

Finally, we prove 3, again only in case j is odd. Let $\hat{S}^m_j = \tilde{g}_j\tilde{g}_j^\dagger$ and $\hat{S}^n_{j-1} = \tilde{h}_{j-1}\tilde{h}^\dagger_{j-1}$, and define matrices $\hat{S}^i_{j-1}$ and $\hat{S}^k_j$ by

$$(\hat{S}^i_{j-1})^{-1} = \eta_j(\hat{S}^n_{j-1})^{-1}\eta_j^\dagger.$$

for $j \in \sigma(F_\circ(P))$ and

$$(\hat{S}^k_j)^{-1} = \nu_k(\hat{S}^m_j)^{-1}\nu_k^\dagger$$

for $k \in \sigma(F_\circ(Q))$. By Lemma 12.14 and 1

$$\|\hat{S}^k_j\|_2 = \|\nu_k\tilde{g}_j\nu_k^\dagger\nu_k\tilde{g}_j^\dagger\nu_k^\dagger\|_2 \le \|\hat{S}^m_j\|_2 + \delta \le 2(\alpha\beta)^{(j+1)/2} \tag{63}$$

and

$$\begin{aligned}\|(\hat{S}^k_j)^{-1}\|_2 &\le \|(\hat{S}^m_j)^{-1}\|_2 \\ &\le \|(\tilde{S}^m_j - \delta I)^{-1}\|_2 \\ &\le 2(\alpha\beta)^{(j+1)/2}.\end{aligned} \tag{64}$$

Applying Fact 12.2 twice implies

$$\begin{aligned}\|\hat{S}^k_j - \tilde{S}^k_j\|_2 &\le \|\hat{S}^k_j\|_2\|\tilde{S}^k_j\|_2\|(\hat{S}^k_j)^{-1} - (\tilde{S}^k_j)^{-1}\|_2 \\ &= \|\hat{S}^k_j\|_2\|\tilde{S}^k_j\|_2\|\nu_k((\tilde{g}_j\tilde{g}_j^\dagger)^{-1} - (\tilde{S}^m_j)^{-1})\nu_k^\dagger\|_2 \\ &\le \|\hat{S}^k_j\|_2\|\tilde{S}^k_j\|_2\|(\tilde{g}_j\tilde{g}_j^\dagger)^{-1}\|_2\|(\tilde{S}^m_j)^{-1}\|_2\|\tilde{g}_j\tilde{g}_j^\dagger - \tilde{S}^m_j\|_2 \\ &\le 4(\alpha\beta)^{2(j+1)}\delta.\end{aligned} \tag{65}$$

By the triangle inequality,

$$\|\hat{S}^k_j - S^k_j\|_2 \le 4(\alpha\beta)^{2(j+1)}\delta + (\alpha\beta)^{\left(\frac{j+3}{2}\right)}\delta \le 2(\alpha\beta)^{\left(\frac{j+3}{2}\right)}\delta \tag{66}$$

By similar proofs, the same bound holds for $\hat{S}^i_{j-1}, (\hat{S}^k_{j-1})^{-1}$.

Note that Lemma 12.14 applies to $\tilde{g}_j$, $\hat{S}^i_j$ and $\tilde{h}_{j-1}$, $\hat{S}^i_{j-1}$ for j odd. By Corollary 12.15, Lemma 12.14 applies for g_j, S^i_j and h_j, S^i_{j-1} as well. Define $\hat{S}^i_{j+1}$ via (the even step of) the recurrence in Definition 12.2 applied to $\hat{S}^i_j$. Then

$$\begin{aligned}&\left|\mathrm{ds}_{P,Q}\, T_{\tilde{g}_j,\tilde{h}_{j-1}} - \mathrm{ds}_{P,Q}\, T_{g_j,h_j}\right| \\ &\le \sum_{i\in\sigma(F_\circ(P))} \Delta p_i \left|\mathrm{Tr}\left(S^i_{j-1}\left(S^i_{j+1}\right)^{-1} - I_{F_i(P)}\right)^2 - \left(\hat{S}^i_{j-1}\left(\hat{S}^i_{j+1}\right)^{-1} - I_{F_i(P)}\right)^2\right| \\ &+ \sum_{i\in\sigma(F_\circ(Q))} \Delta q_i \left|\mathrm{Tr}\left(S^i_j\left(S^i_j\right)^{-1} - I_{F_i(Q)}\right)^2 - \left(\hat{S}^i_j\nu_i T\left(\sum_{k\in\sigma(F_\circ(P))}\Delta p_k\eta_k^\dagger\hat{S}^k_{j-1}\eta_k\right)\nu_i^\dagger - I_{F_i(Q)}\right)^2\right|,\end{aligned}$$

so

$$
\begin{aligned}
&\left|\mathrm{ds}_{P,Q}\, T_{\tilde{g}_j,\tilde{h}_{j-1}} - \mathrm{ds}_{P,Q}\, T_{g_j,h_j}\right| \\
&\le \sum_{i\in\sigma(F_\circ(P))} \Delta p_i \left|\operatorname{Tr}\left(S^i_{j-1}\left(S^i_{j+1}\right)^{-1} - I_{F_i(P)}\right)^2 - \left(\hat{S}^i_{j-1}\left(\hat{S}^i_{j+1}\right)^{-1} - I_{F_i(P)}\right)^2\right| \quad (67)\\
&\quad + \sum_{i\in\sigma(F_\circ(Q))} \Delta q_i \left\|\hat{S}^i_j \nu_i T\left(\sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger \hat{S}^k_{j-1}\eta_k\right)\nu_i^\dagger - I_{F_i(Q)}\right\|^2. \quad (68)
\end{aligned}
$$

We first bound 68. Using the triangle inequality, the i^{th} term can be bounded as

$$
\begin{aligned}
&\left\|\hat{S}^i_j \nu_i T\left(\sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger \hat{S}^k_{j-1}\eta_k\right)\nu_i^\dagger - I_{F_i(Q)}\right\|^2 \le \\
&\left(\left\|\left(\hat{S}^i_j - S^i_j\right)\nu_i T\left(\sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger \hat{S}^k_{j-1}\eta_k\right)\nu_i^\dagger\right\|\right.\\
&\left.+\left\|S^i_j \nu_i T\left(\sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger \left(\hat{S}^k_{j-1} - S^k_{j-1}\right)\eta_k\right)\nu_i^\dagger\right\|\right)^2.
\end{aligned}
$$

By applying the Fact 12.1 and the inequalities 65 and 63 to the two terms in the parentheses, the squared quantity is at most

$$
\begin{aligned}
m&\left(2((\alpha\beta)^{\binom{j+3}{2}}\delta)(2(\alpha\beta)^{(j+1)/2})\|T(P)\|_2\right)^2\\
&= 2^8(\alpha\beta)^{j^2+6j+7}\delta^2 m(mnM)^4\\
&\le (\alpha\beta)^{j^2+6j+9}\delta^2 = (\alpha\beta)^{(j+3)^2}\delta^2.
\end{aligned}
$$

Since $\sum \Delta q_i \le p_1 \le 1$,

$$
68 \le (\alpha\beta)^{(j+3)^2}\delta^2.
$$

Next we bound 67. On each term we can use the identity $(A-I)^2-(B-I)^2 = (A+B-2I)(A-B)$, and then $|\operatorname{Tr}(A+B-2I)(A-B)| \le n\|A+B-2I\|_2\|A-B\|_2$ which, on the i^{th} term, will yield

$$
\begin{aligned}
&\left|\operatorname{Tr}\left(S^i_{j-1}\left(S^i_{j+1}\right)^{-1} - I_{F_i(P)}\right)^2 - \left(\hat{S}^i_{j-1}\left(\hat{S}^i_{j+1}\right)^{-1} - I_{F_i(P)}\right)^2\right|\\
&\le \left\|S^i_{j-1}\left(S^i_{j+1}\right)^{-1} + \hat{S}^i_{j-1}\left(\hat{S}^i_{j+1}\right)^{-1} - 2I_{F_i(P)}\right\|_2 \left\|S^i_{j-1}\left(S^i_{j+1}\right)^{-1} - \hat{S}^i_{j-1}\left(\hat{S}^i_{j+1}\right)^{-1}\right\|_2\\
&\le 3(\alpha\beta)^{j+2}\left\|S^i_{j-1}\left(S^i_{j+1}\right)^{-1} - \hat{S}^i_{j-1}\left(\hat{S}^i_{j+1}\right)^{-1}\right\|_2 \quad (69)
\end{aligned}
$$

because 1 and $j < t-1$ implies

$$
\|S^i_{j-1}\left(S^i_{j+1}\right)^{-1}\|_2 \le \|S^i_{j-1}\|_2\|\left(S^i_{j+1}\right)^{-1}\|_2 \le (\alpha\beta)^j
$$

and 63 implies

$$
\left\|\left(\hat{S}^i_{j+1}\right)^{-1}\right\|_2 = \left\|\eta_i T^*\left(\sum_{k\in\sigma(F_\circ(Q))} \Delta q_k \nu_k^\dagger \hat{S}^k_j \nu_k\right)\eta_i^\dagger\right\|_2 \le 2(\alpha\beta)^{(j+1)/2}(mnM)^2, \quad (70)
$$

so

$$\|\hat{S}^i_{j-1}\left(\hat{S}^i_{j+1}\right)^{-1}\|_2$$
$$\leq 4(\alpha\beta)^{(j+1)}(mnM)^2 \leq (\alpha\beta)^{j+2}.$$

Next we apply the identity $\|AB - CD\|_2 \leq \|B\|_2\|A - C\|_2 + \|C\|_2\|B - D\|_2$ to 69 and take the sum to obtain

$$67 \leq$$
$$3(\alpha\beta)^{j+2} \sum_{i\in\sigma(F_\circ(P))} \Delta p_i \left(\left\| S^i_{j-1} - \hat{S}^i_{j-1} \right\|_2 \left\| \left(\hat{S}^i_{j+1}\right)^{-1} \right\|_2 \right.$$
$$\left. + \left\| \hat{S}^i_{j-1} \right\|_2 \left\| \eta_i T^* \left(\sum_{k\in\sigma(F_\circ(Q))} \Delta q_k \nu_k^\dagger \left(S^k_j - \hat{S}^k_j\right) \nu_k \right) \eta_i^\dagger \right\|_2 \right).$$

We now use the bounds 63, 66 and 70 and $\sum \Delta p_i \leq 1$ to obtain

$$67 \leq 3(\alpha\beta)^{j+2}\left(4(\alpha\beta)^{\binom{j+3}{2}+(j+1)/2}(mnM)^2\delta\right)$$
$$= 12(\alpha\beta)^{(j^2+8j+13)/2}(mnM)^2\delta \leq \frac{1}{2}(\alpha\beta)^{(j+4)^2/2}.$$

Combining our bounds on 67 and 68 yields 3 with plenty of room. □

12.4.3 Proof of Theorem 12.9

Proof of Theorem 12.9. Suppose Algorithm 12.8 terminates in at most t steps where t is defined as in Theorem 12.9; since

$$t \leq \operatorname{poly}(\epsilon^{-1}, p_n^{-1}, q_m^{-1}, n, m, \log M)$$

and

$$s \leq (20(t+4)^2(m+n)\log(m+n) + 8\log M)(p_n^{-1} + q_m^{-1}),$$

rounding ensures the bit-complexity of the matrices $\tilde{S}^i_j$ remains polynomial. This ensures that computing the $\tilde{S}^i_j$ can be done in polynomial time. Finding g, h can be done in polynomial time by Fact 3.14.

We now show Algorithm 12.8 terminates in at most $t-2$ steps. By Lemma 12.17, it is enough to show that for some $j \leq t-2$,

$$\operatorname{ds}_{P,Q} T_{g_j,h_j} \leq \frac{\epsilon}{2} \tag{71}$$

and that $\tilde{S}^i_j$ is a δ-sequence with

$$\delta(\alpha\beta)^{(t+4)^2} \leq \min\left\{\frac{\epsilon}{2}, 1\right\}. \tag{72}$$

(72 always implies the hypothesis $\delta < (\alpha\beta)^{-t/2}$ for Lemma 12.17.) The inequality 71 holds for some $j \leq t-2$ by Corollary 12.15 and Corollary 5.12.

Now we show $\tilde{S}^i_j$ is a δ-sequence satisfying 72.Indeed, suppose j is odd. The proof for the even case is analogous. If

$$s > 2(t+4)^2(\lg\alpha + \lg\beta),$$

then $\mathrm{Rnd}_s\, \tilde{S}^k_j = \tilde{S}^k_j + \Delta^i_j$ with

$$\|\Delta^i_j\|_2 \le (n+m)(\alpha\beta)^{2(t+4)^2}$$

so

$$\sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger \,\mathrm{Rnd}_s\left(\tilde{S}^k_{j-1}\right)\eta_k$$
$$= \sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger \tilde{S}^k_{j-1}\eta_k + \sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger \Delta^i_j \eta_k$$

where

$$\sum_{k\in\sigma(F_\circ(P))} \Delta p_k \eta_k^\dagger \Delta^i_j \eta_k \preceq (n+m)(\alpha\beta)^{-2(t+4)^2} P.$$

As

$$(n+m)(\alpha\beta)^{-2(t+4)^2} \le \min\{1, \frac{\epsilon}{2}\}(\alpha\beta)^{-(t+4)^2},$$

we are done. □

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