# Math 454 Lecture 9: 7/11/2017

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## Contents



# 1 More nonhomogeneous recurrence relations

## 1.1 Solving nonhomogeneous recurrence relations with generating functions

We can also try to solve these types of problems with generating functions, which sometimes prevents us from having to guess particular solutions. This helps a great deal if you know a closed form for the generating function of the inhomogeneous part.

**Example 1.** Let's solve $h_n = 2h_{n-1} + 3^n$ for $n \geq 1$, and $h_0 = 2$.

**Guessing:** Let's first try this by guessing. The inhomogeneous part is $3^n$, and the homogeneous part is $h_n = 2h_{n-1}$. Using the guidelines from Lecture 8, I guess the particular solution $c3^n$. This implies

$$c3^n = c2 \cdot 3^{n-1} + 3^n,$$

or $3^n(3c - 2 \cdot 3c - 3) = 0$, or $c = 3$. Now our particular solution is $p_n = 3^{n+1}$. Using your favorite method of solving recurrences, we can also find a solution $g_n = 2^n$ for the homogeneous part. Hence the general solution is $c_1 g_n = c_1 2^n$. For

$$p_n + c_1 g_n$$

to satisfy the initial conditions (base cases), we must have $p_0+c_1g_0 = c_1 2^0+3^{0+1} = h_0$, or $c_1 = -1$. Finally, our solution is

$$p_n - g_n = 3^{n+1} - 2^n.$$

**Generating functions:** Again, we try to find an equation for the generating function $h(x) = \sum_{n\geq 0} h_n x^n$. This time, $h_n - 2hn - 1 - 3^n = 0$ for $n \geq 1$, so

$$\begin{array}{ccccccccc}
h(x) & & h_0 & + & h_1x & + & h_2x^2 & + & \dots \\
-2xh(x) & = & & + & (-2)h_0x & + & (-2)h_1x^2 & + & \dots \\
-\sum_{n\geq 0} 3^n x^n & & -1 & + & -3x & + & -3x^2 & + & \dots \\
 & = & h_0 - 1 & + & 0 & + & 0 & + & \dots \quad .
\end{array}$$

This gives us $(1-2x)h(x) - \sum_{n\geq 0} 3^n x^n = 1$; however, we can rewrite

$$\sum_{n\geq 0} 3^n x^n = \frac{1}{1-3x},$$

so that

$$h(x) = \frac{1}{1-2x} + \frac{1}{(1-3x)(1-2x)}.$$

With partial fractions, we can rewrite this as

$$h(x) = \frac{1}{1-2x} - \frac{2}{1-2x} + \frac{3}{1-3x} = \frac{3}{1-3x} - \frac{1}{1-2x},$$

or

$$h(x) = \sum_{i=1}^{n} \left(3 \cdot 3^n - 2^n\right),$$

so $h_n = 3 \cdot 3^n - 2^n$.

# 2 Exponential Generating functions

## 2.1 What is an exponential generating function

Exponential generating functions are the "order matters version" of generating functions. If generating functions help you count bags of fruit, exponential generating functions help you count ways to line up fruit.

**Definition 2.1.** If $a_0, a_1, a_2 \dots$ is a sequence, then the exponential generating function for $a_n$ is

$$a(x) = a_0 + \frac{a_1}{1!}x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \dots.$$

**Example 2.** The exponential generating function for the sequence $(1, 1, 1, 1, 1 \dots)$ is

$$a(x) = \sum_{i \geq 0} \frac{1}{i!} x^i = e^x.$$

The exponential generating function for the number of binary strings of length $n$ is

$$e^{2x} = \sum_{n \geq 0} \frac{(2x)^n}{n!} = \sum_{n \geq 0} 2^n \frac{x^n}{n!}.$$

**Theorem 2.1** (Multiplying exponential generating functions)**.** *If $a_i$ is 1 if $i$ copies of $A$ are allowed in a sequence and zero otherwise and $b_j$ is 1 if $j$ copies of $B$ are allowed in a sequence and zero otherwise, and*

$$a(x) = \sum_{i \geq 0} \frac{a_i}{i!} x^i$$

*and*

$$b(x) = \sum_{j \geq 0} \frac{b_j}{j!} x^j$$

*are the exponential generating functions for $a_n$ and $b_n$, then*

$$a(x)b(x) = \sum_{k \geq 0} \frac{1}{k!} \left( \sum_{i=0}^{k} \frac{k! a_i b_{k-i}}{i!(k-i)!} \right) x^k$$

is the exponential generating function for the number of strings of length $k$ with using allowed numbers of A's and allowed numbers of B's. Namely, $\sum_{i=0}^{k} \frac{k! a_i b_{k-i}}{i!(k-i)!}$.

*Proof.* The proof of the expression comes from the formula for multiplying generating functions along with multiplication and division by $k!$. The fact that $\sum_{i=0}^{k} \frac{k! a_i b_{k-i}}{i!(k-i)!}$ is the number of sequences of length $k$ with using allowed numbers of A's and allowed numbers of B's follows because the $a_i b_{k-i}$ will be one if and only if both $i$ and $k - i$ $A's$ and $B's$ are allowed in the string. □

**Example 3.** $e^x$ is the exponential generating function of the all-ones sequence, so it represents allowing any number of 0's in a string. $e^x$ can also represent allowing any number of 1's in a string. Hence

$$e^x e^x = e^{2x}$$

should be the exponential generating function for the number of strings of length $k$ with any number of 0's and 1's. This checks out, since

$$e^{2x} = \sum_{n \geq 0} \frac{(2x)^n}{n!} = \sum_{n \geq 0} 2^n \frac{x^n}{n!}.$$

## 2.2 $r$-permutations of multisets

This also extends for more factors.

**Example 4.** Suppose I wanted the exponential generating function $h(x)$ for the number of $k$-permutations of $\{2 \cdot A, 4 \cdot B, 3 \cdot C\}$. Since this is the number of strings of length $k$ containing at most 2 $A's$, and most 4 $B's$, and at most 3 $C's$, we can say that $h(x) = a(x)b(x)c(x)$ where each factor represents the allowable number of the corresponding letter. In this case,

$$h(x) = \left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)$$

is the exponential generating function for the number of $k$-permutations of the multiset $\{2 \cdot A, 4 \cdot B, 3 \cdot C\}$.

## 2.3 Finding closed forms

Sometimes we can use tricks to find closed forms for sequences by using their exponential generating functions.

**Example 5.** Suppose we wish to find the number of strings of length $k$ of 0's,1's, and 2's in which the number of $0's$ is even and the number of $1's$ is odd.

Let's try to find the exponential generating function $h(x)$. We know it should be of the form $h(x) = f_0(x)f_1(x)f_2(x)$, $f_0$ representing the allowable number of 0's, $f_1$ for $1's$ and $f_2$ for $2's$.

This means

$$f_0 = \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots\right)$$

,

$$f_1 = \left(\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} \dots\right),$$

and

$$f_2 = e^x.$$

Unfortunately, we cannot just cleverly substitute anything into $e^x$ to get closed forms for $f_0$ and $f_1$. However, we can use

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.$$

to get

$$f_0 = \frac{e^x + e^{-x}}{2}$$

and

$$f_1 = \frac{e^x - e^{-x}}{2}.$$

Now we have

$$\begin{aligned} h(x) &= \frac{1}{4}\left(e^x + e^{-x}\right)\left(e^x - e^{-x}\right)e^x \\ &= \frac{1}{4}\left(e^{2x} - e^{-2x}\right)e^x = \frac{1}{4}\left(e^{3x} - e^{-x}\right). \end{aligned}$$

Thus

$$h(x) = \frac{1}{4}\left(\sum_{n=0}^{\infty} \frac{3^n x^n}{n!} - \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}\right) = \sum_{n\geq 0} \frac{\frac{1}{4}(3^n - (-1)^n)}{n!} x^n,$$

so $h_n = \frac{1}{4}\left(3^n + (-1)^{n+1}\right)$.