Math 454 Lecture 8: 7/10/2017

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July 18, 2017

Contents

1 Generating functions and recurrences

This follows the second half of Section 7.4 in Brualdi. Generating functions are especially suited to help us solve recurrences. Suppose

Example 1. Let's solve the recurrence

$$h_n = 5h_{n-1} - 6h_{n-2}$$

for $n \geq 2$ with $h_0 = 1$ and $h_1 = -2$. Consider the generating function

$$h(x) = \sum_{n \geq 0} h_n x^n.$$

This just carries the information about the sequence h_n as coefficients of a power series. The main thing to notice is that multiplying h by x shifts the generating function as follows:

$$\begin{array}{rcccccccccc} h(x) & = & h_0 & + & h_1 x & + & h_2 x^2 & + & \dots & h_n x^n \\ xh(x) & = & & & h_0 x & + & h_1 x^2 & + & \dots & h_{n-1} x^n \\ x^2 h(x) & = & & & & & h_0 x^2 & + & \dots & h_{n-2} x^n. \end{array}$$

Using this and the recurrence helps us get a nice equation for the generating function. Our recurrence relation can also be written

$$h_n - 5h_{n-1} + 6h_{n-2} = 0.$$

If we multiply the second line of the above equation by -5, the third by 6, and add them together, we get

$$\begin{array}{rcccccccc} h(x) & & h_0 & + & h_1 x & + & h_2 x^2 & + & \ldots & h_n x^n \\ -5xh(x) & = & & + & (-5)h_0 x & + & (-5)h_1 x^2 & + & \ldots & (-5)h_{n-1}x^n \\ +6x^2 h(x) & & & & & + & 6h_0 x^2 & + & \ldots & 6h_{n-2}x^n. \\ & = & h_0 & + & (h_1 - 5h_0)x & + & 0 & + & \ldots & 0. \end{array}$$

Thus,

$$(1 - 5x + 6x^2)h(x) = 1 - 7x,$$

or

$$h(x) = \frac{(1 - 7x)}{1 - 5x + 6x^2}.$$

This is all well and good, but unless it helps us find the coefficients, it is ultimately useless. To deal with this, we need to express it in a way that allows us to use generating functions whose coefficients we *already know* to find the coefficients of $h(x)$. This is the partial fraction decomposition. First, write $1 - 5x + 6x^2 = (1 - 2x)(1 - 3x)$. Then, using the partial fraction decomposition, we can write

$$h(x) = \frac{(1 - 7x)}{1 - 5x + 6x^2} = \frac{(1 - 7x)}{(1 - 2x)(1 - 3x)} = \frac{c_1}{1 - 2x} + \frac{c_2}{1 - 3x}$$

for some constants c_1 and c_2. To solve for these, we clear the denominator, and find

$$(1 - 3x)c_1 + (1 - 2x)c_2 = 1 - 7x,$$

which implies

$$\begin{aligned} c_1 + c_2 &= 1 \\ -3c_1 - 2c_2 &= -7. \end{aligned}$$

or (adding thrice the first row to the second)

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_2 &= -4, \end{aligned}$$

or $c_2 = -4, c_1 = 5$. Now, we know

$$f(x) = \frac{5}{1 - 2x} - \frac{4}{1 - 3x}.$$

Now we can use Newton's binomial theorem with $n = -1$ (really, this is just the famililar Taylor series for $1/(1-x)$) to obtain

$$f(x) = \sum_{n \geq 0} (5 \cdot 2^n - 4 \cdot 3^n) x^n.$$

Thus, $h_n = 5 \cdot 2^n - 4 \cdot 3^n$ for all $n \geq 0$.

Here's the idea - because of the shifting idea above, if we start with a recurrence relation $f_n = a_1 f_{n-1} + \cdots + a_k f_{n-k}$, the generating function $f(x)$ satisfies

$$(1 - a_1 x - a_2 x^2 - \cdots - a_k x^k) f(x) = b_0 + b_1 x + \cdots + b_{k-1} x^{k-1}.$$

$b_0, b_1, \ldots, b_{k-1}$ can easily be found from the first few terms of f_n. This means

$$f(x) = \frac{b_0 + b_1 x + \cdots + b_{k-1} x^{k-1}}{1 - a_1 x - a_2 x^2 - \cdots - a_k x^k},$$

where we know the b_i's and the $a_i's$ come from the recurrence. Next, we factor the denominator as

$$1 - a_1 x - a_2 x^2 - \cdots - a_k x^k = (1 - r_1 x)^{n_1} (r_2 - x)^{n_2} \ldots (r_t - x)^{n_t}.$$

Note that this is a bit different than the usual factoring - here the roots will be $x = \frac{1}{r_i}$. Using the partial fraction decomposition, express

$$\begin{aligned} f(x) &= \frac{b_0 + b_1 x + \cdots + b_{k-1} x^{k-1}}{1 - a_1 x - a_2 x^2 - \cdots - a_k x^k} \\ &= \frac{c(1)_1}{(1 - r_1 x)} + \cdots + \frac{c(1)_{n_1}}{(1 - r_1 x)^{n_1}} \\ &\qquad + \ldots \\ &+ \frac{c(l)_1}{(1 - r_t x)} + \cdots + \frac{c(l)_{n_t}}{(1 - r_t x)^{n_t}}. \end{aligned}$$

Next, use your favorite method to solve for the constants $c(i)_j$ (you can clear the denominators or just plug in different values for x, each of which will give you a $k \times k$ linear system to solve). Finally, each thing of the form

$$\frac{1}{(1 - rx)^n}$$

can be rewritten using newton's binomial theorem to become

$$\sum_{k \geq 0} \binom{n + k - 1}{k} r^k x^k.$$

From this, we can extract the coefficients. Let's see another example.

1.1 Repeated roots.

This works even if we have repeated roots.

Example 2. Suppose h_n satisfies the recurrence $h_n = -h_{n-1} + 16h_{n-2} - 20h_{n-3}$ for $n \geq 3$ and $h_0 = 0, h_1 = 1, h_2 = -1$.

We can use our shifting trick again to solve this recurrence. This time

$$h_n + h_{n-1} - 16h_{n-2} + 20h_{n-3} = 0,$$

so

$$\begin{array}{ccccccccccc}
h(x) & & h_0 & + & h_1 x & + & h_2 x^2 & + & h_3 x^3 & \dots \\
+xh(x) & = & & + & h_0 x & + & h_1 x^2 & + & h_2 x^3 & \dots \\
-16x^2 h(x) & & & & & + & (-16)h_0 x^2 & + & (-16)h_1 x^3 & \dots \\
+20x^3 h(x) & & & & & & & + & 20h_0 x^3 & \dots \\
& = & h_0 & + & (h_1 + h_0)x & + & (h_2 + h_1 - 16h_0) & + & 0 & \dots. \\
& = & 0 & + & 1x & + & 0 & + & 0 &
\end{array}$$

Hence, $(1 + x - 16x^2 + 20x^3)h(x) = x$, or

$$h(x) = \frac{x}{1 + x - 16x^2 + 20x^3}.$$

We need to factor $1 + x - 16x^2 + 20x^3$. Though there exist formulae for roots of a cubic, they are messy. Fortunately we can get lucky in this case by guessing roots; recall that if p/q is a rational root of this polynomial in reduced form then p divides 1 and q divides 20. Trying $\pm 1, \pm 1/2, \pm 1/4, \pm 1/5, \pm 1/10, \pm 1/20$, we find that $1/2$ is a root. This means $(2x - 1)$ divides the above polynomial. Dividing $(2x - 1)$ into $1 + x - 16x^2 + 20x^3$, we find

$$(2x - 1)(10x^2 + 3x - 1) = 1 + x - 16x^2 + 20x^3,$$

and so $1 + x - 16x^2 + 20x^3 = (2x - 1)^2(5x + 1)$. This tells us our generating function can be written

$$h(x) = \frac{x}{(1 - 2x)^2(5x + 1)},$$

and by decomposing into partial fractions,

$$h(x) = \frac{x}{(1 - 2x)^2(5x + 1)} = \frac{c_1}{1 - 2x} + \frac{c_2}{(1 - 2x)^2} + \frac{c_3}{1 + 5x}.$$

Clearing the denominators, we get

$$x = c_1(1 - 2x)(1 + 5x) + c_2(1 + 5x) + c_3(1 - 2x)^2$$

or

$$\begin{aligned} c_1+c_2+c_3&=0\\ 3c_1+5c_2-4c_3&=1\\ -10c_1+0+4c_3&, \end{aligned}$$

or

$$\begin{bmatrix} 1 & 1 & 1\\ 3 & 5 & -4\\ -10 & 0 & 4 \end{bmatrix}\begin{bmatrix} c_1\\ c_2\\ c_3 \end{bmatrix}=\begin{bmatrix} 0\\ 1\\ 0 \end{bmatrix}.$$

Subtracting thrice the first row from the second and adding ten times the first row to the third, this we have

$$\begin{bmatrix} 1 & 1 & 1\\ 0 & 2 & -7\\ 0 & 10 & 14 \end{bmatrix}\begin{bmatrix} c_1\\ c_2\\ c_3 \end{bmatrix}=\begin{bmatrix} 0\\ 1\\ 0 \end{bmatrix}.$$

Next, subtract 5 times the second row from the third.

$$\begin{bmatrix} 1 & 1 & 1\\ 0 & 2 & -7\\ 0 & 0 & 49 \end{bmatrix}\begin{bmatrix} c_1\\ c_2\\ c_3 \end{bmatrix}=\begin{bmatrix} 0\\ 1\\ -5 \end{bmatrix}.$$

This gives us $c_3=-5/49$, and substituting c_3 back in to the next lines, we find $2c_2+35/49=1$, or $c_2=\frac{7}{49}=\frac{1}{7}$, and finally $c_1=-c_2-c_3=-2/49$. Our generating function is

$$h(x)=-\frac{2}{49}\frac{1}{1-2x}+\frac{1}{7}\frac{1}{(1-2x)^2}-\frac{5}{49}\frac{1}{1+5x};$$

Newton's binomial theorem allows us to rewrite this

$$\begin{aligned} h(x)=-\frac{2}{49}\frac{1}{1-2x}+\frac{1}{7}\frac{1}{(1-2x)^2}-\frac{5}{49}\frac{1}{1-(5x)}\\ -\frac{2}{49}\sum_{k\geq 0}2^kx^k+\frac{1}{7}\sum_{k\geq 0}\binom{k+2-1}{k}2^kx^k-\frac{5}{49}\sum_{k\geq 0}(-5)^kx^k. \end{aligned}$$

Hence,

$$h_n=-\frac{2}{49}2^n+\frac{1}{7}(n+1)2^n-\frac{5}{49}(-5)^nx^n.$$

1.2 Characteristic roots for repeated roots

If you find that your characteristic equation has some root repeated, you will not have enough solutions to solve for the initial values. However, in that case we also get some other solutions.

Theorem 1.1. *If $q \neq 0$ is a repeated root of the polynomial $p(x) = x^k - a_1x^{k-1} - a_2x^{k-2} - \cdots - a_k$, then nq^k is also a solution of the recurrence $f_n - a_1f_{n-1} + a_2f_{n-2} + \cdots + a_kf_{n-k} = 0$.*

Proof. If q is a repeated root of $p(x)$, then it is also a root of the derivative $p'(x)$ (check this by factoring $p(x)$). However, the recurrence relation with q^n plugged in is

$$\begin{aligned} &nq^n - a_1(n-1)q^{n-1} - a_2(n-2)q^{n-2} - \cdots - a_k(n-k)q^{n-k} \\ &= q^{n-k}(nq^k - a_1(n-1)q^{k-1} - a_2(n-2)q^{k-2} - \cdots - (n-k)a_k), \end{aligned}$$

which is zero whenever

$$nq^k - a_1(n-1)q^{k-1} - a_2(n-2)q^{k-2} - \cdots - (n-k)a_k$$

is zero. We can rewrite this as

$$\begin{aligned} &(n-k+k)q^k - a_1(n-k+k-1)q^{k-1} - a_2(n-k+k-2)q^{k-2} - \cdots - (n-k+k-k)a_k \\ &= (n-k)p(q) + \left(kq^k - a_1(k-1)q^{k-1} - a_2(n-k+k-2)q^{k-2} - \cdots - 0 \cdot a_k\right) \\ &= (n-k)p(q) + qp'(q) = 0, \end{aligned}$$

so nq^n satisfies the recurrence relation. □

2 Nonhomogeneous Recurrences

Sometimes our equations may not be homogeneous, such as

Example 3.

$$\begin{aligned} h_n &= 3h_{n-1} - 4n \\ h_0 &= 2. \end{aligned}$$

The *homogeneous part* of this recurrence relation is $h_n = 3h_{n-1}$, and the inhomogeneous part is $-4n$.

2.1 Guessing a particular solution

One way to solve this type of problem is by guessing a solution to the recurrence relation, *including the inhomogeneous part, but ignoring the initial values.* This is called the *particular solution.* Then we find enough solutions of the homogeneous version (the equation but with the inhomogeneous part dropped out) *ignoring the initial conditions.* All linear combinations of these solutions of the homogeneous part are called the *general solution.* Next we find which instance of the general solution to add to the particular solution to recover the initial values.

Remark 2.1. How do we know when we have enough solutions that all their linear combinations form the general solution? What we need is that the solutions are linearly independent. In the case of order one generating functions, the solution just needs to be nonzero, and for order two generating functions, they just cannot be multiples of one another. If the solutions are not linearly independent, you may not be able to recover the initial values.

Suppose p_n satisfies $p_n = 3p_{n-1} - 4n$, and g_n satisfies $g_n = 3g_{n-1}$ (the homogeneous part). Here p_n is a particular solution, and $c_1 g_n$ is the general solution. (the phrase general solution means an expression representing all possible solutions of the homogeneous part of the equation). Then

$$p_n + c_1 g_n = 3p_{n-1} - 4n + 3c_1 g_{n-1} = 3(p_{n-1} + c_1 g_{n-1}) - 4n,$$

so $p_n + c_1 g_n$ is also a solution. The key insight here is that adding solutions of the homogeneous part to the particular solution gives new solutions to the recurrence.

If we guess our particular solution p_n to be a polynomial of degree 1 in n, such as $an+b$, then at least $3p_{n-1} - 4n$ will be another polynomial of degree 1 in n. This is encouraging, so let's continue. To solve for a and b, we plug $an + b$ into the recurrence relation. This gives us

$$an + b = 3(a(n-1) + b) - 4n,$$

or $(-2a+4)n + (-2b+3a) = 0$, which holds for all n if and only if $a = 2$ and $b = 3$. Thus, our particular solution is

$$p_n = 2n + 3.$$

Next, a solution to the homogeneous part is

$$g_n = 3^n,$$

which we can find by our favorite method of solving homogeneous, linear recurrences with constant coefficients. Thus, the general solution is

$$c_1 3^n.$$

For the solution $h_n = p_n + c_1 g_n = c_1 3^n + 2n + 3$ to satisfy the initial conditions $h_0 = 2$, $3 + c_1 = 2$, so $c_1 = (-1)$. Finally, our solution is

$$h_n = -3^n + 2n + 3.$$

Remark 2.2 (Guidelines for guessing a particular solution)**.** In general, one should guess functions that look like the inhomogeneous part. If your equation looks like

$$f_n = a_1 f_{n-1} + \cdots + a_k f_{n-k} + b_n,$$

then b_n is called the *inhomogeneous part* of the recurrence relation, and

$$f_n = a_1 f_{n-1} + \cdots + a_k f_{n-k}$$

is called the *homogeneous part*. If

- b_n is a polynomial of degree k (for example, $3n^2 - 2n + 1$ is a polynomial of degree 2), try to guess a polynomial of degree k for f. (for the example here, we would guess $an^2 + bn + c$).

- b_n is an exponential d^n, try to guess an exponential $c \cdot d^n$. (for example, if b_n were 4^n, we would guess $c \cdot 4^n$.

2.2 Generating functions

We can also try to solve these types of problems with generating functions, which sometimes prevents us from having to guess particular solutions. This helps a great deal if you know a closed form for the generating function of the inhomogeneous part.

Example 4. Let's solve $h_n = 2h_{n-1} + 3^n$ for $n \geq 1$, and $h_0 = 2$.

Again, we try to find an equation for the generating function $h(x) = \sum_{n \geq 0} h_n x^n$. This time, $h_n - 2hn - 1 - 3^n = 0$ for $n \geq 1$, so

$$\begin{array}{ccccccccc}
h(x) & & h_0 & + & h_1 x & + & h_2 x^2 & + & \dots \\
-2xh(x) & = & & + & (-2)h_0 x & + & (-2)h_1 x^2 & + & \dots \\
-\sum_{n\geq 0} 3^n x^n & & -1 & + & -3x & + & -3x^2 & + & \dots \\
 & = & h_0 - 1 & + & 0 & + & 0 & + & \dots \quad .
\end{array}$$

This gives us $(1-2x)h(x) - \sum_{n \geq 0} 3^n x^n = 1$; however, we can rewrite

$$\sum_{n \geq 0} 3^n x^n = \frac{1}{1-3x},$$

so that

$$h(x) = \frac{1}{1-2x} + \frac{1}{(1-3x)(1-2x)}.$$

With partial fractions, we can rewrite this as

$$h(x) = \frac{1}{1-2x} - \frac{2}{1-2x} + \frac{3}{1-3x} = \frac{3}{1-3x} - \frac{1}{1-2x},$$

or

$$h(x) = \sum_{i=1}^{n} \left(3 \cdot 3^n - 2^n\right),$$

so $h_n = 3 \cdot 3^n - 2^n$.

3 Catalan Numbers

Generating functions can also help with non-linear recurrences. This is the case for the solution to the following problem:

Question 1. How many ways C_n are there to triangulate a convex $(n+2)$-gon?

A *triangulation* is a division of the $(n+2)$-gon into triangles by nonintersecting lines between vertices.
We consider the 2-gon to be a sensible thing that can be triangulated in exactly one way, so $C_0 = 1$.

Theorem 3.1. *C_n satisfies the recurrence*

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

for $n \geq 0$.

Proof. Order the vertices of the $n+3$-gon counterclockwise from 0 to $n+2$, and pick the triangle containing the two adjacent vertices $n+1$ and $n+2$. There are $n+1$ different choices for the other vertex of this triangle, namely the other vertices $0 \ldots n$). Each is a different triangulation (lines do not intersect), so these cases are disjoint. Suppose we picked vertex i. The polygons created by the vertices $n+2, 0, 1, \ldots, i$ and $i, i+1, \ldots, n+1$ are convex $(i+2)$ and $n-i+2$-gons, respectively (possibly 2-gons). The number of ways to triangulate each of these is, by definition, C_i and C_{n-i}, so

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}.$$

□

None of the usual techniques we know work on this recurrence, but it is amenable to generating functions!

Let $h(x) = \sum_{n \geq 0} C_n$ be the generating function for the sequence $(C_n)_{n \geq 0}$. The sum

$$\sum_{i=0}^{n} C_i C_{n-i}$$

should look familiar from our formula for multiplying generating functions. If $f(x) = \sum_{i \geq 0} a_i$, and $g(x) = \sum_{j \geq 0} b_j$, then

$$f(x)g(x) = \sum_{n \geq 0} \left(\sum_{i=0}^{n} a_i b_{n-i} \right) x^n.$$

This means if we multiply $h(x)$ by itself, we get

$$h(x)h(x) = \sum_{n\geq 0} \left(\sum_{i=0}^{n} C_i C_{n-i} \right) x^n = \sum_{n\geq 0} C_{n+1} x^n = \frac{1}{x} \left(\sum_{n\geq 1} C_n x^n \right).$$

Since $C_0 = 1$,

$$\frac{1}{x} \left(\sum_{n\geq 1} C_n x^n \right) = \frac{h(x) - 1}{x},$$

so

$$h(x) = 1 + xh(x)^2.$$

This says $h(x)$ is a solution of the equation $xh(x)^2 - h(x) + 1$, or $h(x) = \frac{1\pm\sqrt{1-4x}}{2x}$. By our definition, we should have $h(0) = 1$, but $\frac{1+\sqrt{1-4x}}{2x}$ tends to infinity as $x \to 0$, let's try the other root $\frac{1+\sqrt{1-4x}}{2x}$. Since this has a power series expansion, its coefficients must satisfy the same recurrence as $h(x)$, and it has first coefficient 1 in its power series, so the coefficients must actually match x. We can apply Newton's binomial theorem to obtain

$$h(x) = \frac{1}{2x} - \frac{1}{2x}(1-4x)^{1/2} = \frac{1}{2x} - \frac{1}{2x} \sum_{k\geq 0} \binom{1/2}{k} (-4)^k x^k = -\frac{1}{2} \sum_{k\geq 1} \binom{1/2}{k} (-4)^k x^{k-1}.$$

The reason for the last equality is that $\binom{1/2}{0}$ is just 1, so the $1/2x$ terms cancel out. For $k \geq 1$, we can rewrite

$$\begin{aligned}
\binom{1/2}{k} &= \frac{(1/2)(1/2-1)\dots(1/2-k+1)}{k!} \\
&= \frac{(-1)^{k-1}}{2^k} \frac{(1)(1)(3)\dots(2k-3)}{k!} \\
&= \frac{(-1)^{k-1}}{2^k} \frac{(2k-2)!}{(2k-2)(2k-4)\dots 4\cdot 2\cdot k!} \\
&= \frac{(-1)^{k-1}}{2^{2k-1}} \frac{(2k-2)!}{(k-1)!k!} \\
&= \frac{(-1)^{k-1}}{k\cdot 2^{2k-1}} \binom{2k-2}{k-1}
\end{aligned}$$

For us, everything miraculously cancels:

$$\begin{aligned}
h(x) &= -\frac{1}{2}\sum_{k\geq 1}\binom{1/2}{k}(-4)^k x^{k-1} \\
&= -\frac{1}{2}\sum_{k\geq 1}\frac{(-1)^{k-1}}{k\cdot 2^{2k-1}}\binom{2k-2}{k-1}(-4)^k x^{k-1} \\
&= \sum_{k\geq 1}\frac{(-1)^k}{k\cdot 2^{2k}}\binom{2k-2}{k-1}(-4)^k x^{k-1} \\
&= \sum_{k\geq 1}\frac{1}{k}\binom{2k-2}{k-1}x^{k-1}, \\
&= \sum_{k\geq 0}\frac{1}{k+1}\binom{2k}{k}x^k,
\end{aligned}$$

So

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

These are called the *Catalan numbers*, and they count lots of other quantities in combinatorics.