Math 454 Lecture 3: 6/28/2017

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Contents

1 Multiset Permutations, Round 2

You can read more about this in Section 2.4 of Brualdi and Section 2.7 of Keller and Trotter.

Yesterday we saw that thenumber of permutations of the multiset

$$\{n_1 \cdot 1, n_2 \cdot 2, \ldots, n_k \cdot k\}$$

is

$$\frac{n!}{n_1! n_2! \ldots n_k!},$$

where $n = \sum_{i=1}^{k} n_i$. This was interpreted as the number of ways to write a sequence whose entries are in $[k]$, and with i appearing exactly n_i times. There is another way to interpret this - we can interpret multiset partitions as partitioning a set of size n into labeled parts.

Theorem 1.1. *If $n = \sum_{i=1}^{k} n_i$ and $|S| = n$, then the number of ways to partition S into k labeled parts where the ith part is of size n_i is also*

$$\frac{n!}{n_1! n_2! \ldots n_k!}.$$

Proof. We can assume the set is $[n]$. The number of partitions of $[n]$ into k labeled parts where the ith part has n_i elements is the number of ways to label $[n]$ with $[k]$ so that i appears n_i times, which is the same as the number of permutations of the multiset

$$\{n_1 \cdot 1, n_2 \cdot 2, \ldots, n_k \cdot k\}.$$

□

Example 1. Alice, Bob, Carol and Dale are about to play bridge. The game begins by dealing out the 52 (different) cards to the 4 people so that each person gets 13 cards. How many ways are there for the game to start?

We can think of this as partitioning the set of 52 cards into 4 parts (Alice, Bob, Carol, and Dale are the parts), where each is of size 13. *The parts are labeled because it matters who gets which hand.* For example, Alice getting all 13 clubs and Bob getting all 13 spades is different from *Bob* getting all 13 clubs and *Alice* getting all 13 spades.

Remark 1.1. This helps us see why $\binom{n}{i}$ appears so many ways. If we just want to partition a set into two parts, we can think of one of the parts as a subset and one as "the rest". If the subset should have i elements, then by Theorem 1.1, the number of ways to do this is

$$\frac{n!}{i!(n-i)!} = \binom{n}{i}.$$

By our previous theorem, the number of ways is

$$\frac{52!}{13!13!13!13!}.$$

Multiset permutations also give another way of seeing why $\binom{n}{i}$ is the number of binary strings with i ones, - it's because binary strings with i ones are exactly multiset permutations of $\{i \cdot 0, (n-i) \cdot 1\}$. Likewise for strings of any two symbols!

1.1 Unlabeled Parts (corrected!!!)

We can also count the number of ways to partition objects into unlabeled parts.

Theorem 1.2. *If $n = \sum_{i=1}^{k} n_i$ and $|S| = n$, then the number of ways to partition S into k unlabeled parts* ***of the same size*** *$n_1 = n_2 = \ldots n_k$ is.*

$$\frac{n!}{k!n_1!n_2!\ldots n_k!}.$$

Proof. Set up a $k!$-to-one function from labeled partitions to unlabeled partitions which maps a labeled partition to the corresponding one without labels. The division principle implies the theorem. □

1.2 Multiset r-permutations

You probably noticed that we defined multiset permutations but not multiset r-permutations. This is because the number of multiset r-permutations are pretty hard to compute. We can solve simple instances some simple instances by our four basic counting principles.

Definition 1.1. If

$$M = \{n_1 \cdot 1, n_2 \cdot 2, \ldots, n_k \cdot k\}$$

is a multiset with $\sum_{i=1}^{k} n_i = n$ and $r \leq n$, then an r-permutation of M is a sequence of length r with entries from $[k]$ in which i appears at most n_i times.

Example 2. By the addition principle, the number of 6-permutations of $\{3 \cdot a, 5 \cdot b\}$ is

$$\begin{aligned} &\text{\# of perms of } \{3 \cdot a, 3 \cdot b\} + \text{\# of perms of } \{1 \cdot a, 5 \cdot b\} + \text{\# of perms of } \{2 \cdot a, 4 \cdot b\} \\ &\qquad = \frac{5!}{3!2!} + \frac{5!}{1!5!} + \frac{5!}{2!4!}. \end{aligned}$$

2 Multinomial Theorem

Remember that the coefficient of $x^i y^{n-i}$ in $(x+y)^n$ is $\binom{n}{i}$. This is why sometimes the numbers $\binom{n}{i}$ are called *binomial coefficients*. Numbers of multiset permutations also appear in expansions of larger polynomials.

Example 3. Compute $(x+y+z)^3$, collect like terms and foil out.

Theorem 2.1.

$$(x_1 + \cdots + x_k)^n = \sum_{n_i \geq 0,\ n_1 + \cdots + n_k = n} \frac{n!}{n_1! n_2! \ldots n_k!} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}.$$

Proof. Just like the proof of the binomial theorem but with more variables! □

Definition 2.1. We define

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \ldots n_k!}.$$

The numbers

$$\binom{n}{n_1, n_2, \ldots, n_k}$$

are called *multinomial coefficients*.

This means we can rewrite the theorem:

Theorem 2.2.

$$(x_1 + \cdots + x_k)^n = \sum_{n_i \geq 0,\ n_1 + \cdots + n_k = n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}.$$

Remark 2.1. Binomial coefficients are also multinomial coefficients for $k = 2$; indeed,

$$\binom{n}{r} = \binom{n}{r, n-r}.$$

Example 4. Let's combinatorially prove

$$k^n = \sum_{n_i \geq 0,\ n_1 + \cdots + n_k = n} \binom{n}{n_1, n_2, \ldots, n_k}.$$

k^n is the total number of sequences of length n with entries of length k. However, we can count these another way. Each such sequence has some number n_1 of $1's$, n_2 of $2's$, ... n_k of k's. These numbers $n_1 \ldots n_k$ can be any list of nonnegative numbers $n_1 \ldots n_k$ with $n_1 + \cdots + n_k = n$. For a specific list $n_1, \ldots n_k$, there are $\binom{n}{n_1, n_2, \ldots, n_k}$ sequences with n_1 $1's$, n_2 $2's$, ... , and n_k k's. By the addition principle,

$$\sum_{n_i \geq 0,\ n_1 + \cdots + n_k = n} \binom{n}{n_1, n_2, \ldots, n_k}$$

is the total number of sequences of length $[n]$ with entries from k. Therefore,

$$k^n = \sum_{n_i \geq 0,\ n_1 + \cdots + n_k = n} \binom{n}{n_1, n_2, \ldots, n_k}.$$

Now let's prove this theorem by the Multinomial Theorem (easily). Just plugging in $x_1 = 1, x_2 = 1, \ldots x_k = 1$ gives

$$(x_1 + \cdots + x_k)^n = \sum_{n_i \geq 0,\ n_1 + \cdots + n_k = n} \binom{n}{n_1, n_2, \ldots, n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}.$$

3 Multiset Combinations

In addition to multiset permutations, we can also talk about multiset combinations.

Definition 3.1. If

$$M = \{n_1 \cdot 1, n_2 \cdot 2, \ldots, n_k \cdot k\}$$

is a multiset with $\sum_{i=1}^{k} n_i = n$ and $r \leq n$, then an *r-combination* of M is a multiset

$$\{r_1 \cdot 1, r_2 \cdot 2, \ldots, r_k \cdot k\}$$

with $0 \leq r_i \leq n_i$ for all i and $\sum_{i=1}^{k} r_i = r$. We could call this a submultiset of size r, but nobody says that, so unfortunately we have to be inconsistent.

Example 5. Let $S = \{2 \cdot a, 1 \cdot b, 3 \cdot c\}$. The 3-combinations of S are

$$\{2 \cdot a, 1 \cdot b\}, \{2 \cdot a, 1 \cdot c\}, \{1 \cdot a, 1 \cdot b, 1 \cdot c\}$$
$$\{1 \cdot a, 2 \cdot c\}, \{1 \cdot b, 2 \cdot c\}, \{3 \cdot c\}.$$

Example 6. You have 50 black beans, 50 pinto beans, and 500 grains of rice. How many burritos can you make with 400 objects?

Our burrito is a 400-combination of

$$\{50 \cdot \text{black beans}, 50 \cdot \text{pinto beans}, 500 \cdot \text{rice}\}.$$

In this case, we use the multiplication principle. We have 50 choices for how many black beans to use, 50 choices for how many pintos to use, and the rest must be rice.

Multiset combinations are really just integer solutions to linear equation with constraints.

3.1 Integer Solutions

Theorem 3.1. *If*

$$M = \{n_1 \cdot 1, n_2 \cdot 2, \ldots, n_k \cdot k\}$$

is a multiset with $\sum_{i=1}^{k} n_i = n$ *and* $r \leq n$*, then the number of* r-combinations *of* M *is the same as the number of integer solutions to the equation*

$$x_1 + \cdots + x_k = r$$
$$\textit{such that } 0 \leq x_i \leq n_i \textit{ for all } i.$$

Proof. The r-combination N is completely determined by the integers r_i, which satisfy

$$r_1 + \cdots + r_k = r$$
$$\text{such that } 0 \leq r_i \leq n_i \text{ for all } i,$$

and any r_i satisfying

$$r_1 + \cdots + r_k = r$$
$$\text{such that } 0 \leq r_i \leq n_i \text{ for all } i,$$

give rise to an r-combination. □

In many situations, there is no simple answer for the number of r-combinations of a multiset. In some cases we can hope to find a nice answer.

3.2 Infinite Multiplicity

We can also allow multisets to have infinite multiplicity. That is, some of the n_i can be infinite. For example,

$$\{\infty \cdot a, 2 \cdot b, 3 \cdot c\}$$

denotes the multiset where a has infinite multiplicity (we aren't going to need to worry about what "kind of infinity" this is; ∞ really means "unlimited".) An r-permutation or r-combination of a set where some of the elements have infinite multiplicities is allowed to have any number of those elements.

Example 7. The 4-permutations of

$$\{\infty \cdot a, \infty \cdot b, 1 \cdot c\}$$

are

$$\begin{aligned}
&\{(a,a,a,c),(a,a,b,c),(a,b,a,c),(b,a,a,c),(a,b,b,c),(b,a,b,c),(b,b,a,c),(b,b,b,c)\\
&(a,a,c,a),\ldots\\
&(a,c,a,a),\ldots\\
&(c,a,a,a),\ldots\}\\
&\cup\{\text{all sequences of a's and b's of length 4}\}
\end{aligned}$$

In this case there are $2^3 \cdot 4 + 2^4$, because there is either one c or no c. If there is one, there are 4 places to put the c and the rest can be any string of a and b. If there is no c, then we just have to choose a or b for each of the four entries of the sequence.
The 4-combinations are

$$\begin{aligned}
\{3 \cdot a, 1 \cdot c\}, \{2 \cdot a, 1 \cdot b, 1 \cdot c\}, \{1 \cdot a, 2 \cdot b, 1 \cdot c\}, \{3 \cdot b, 1 \cdot c\}\\
\{4 \cdot a\}, \{3 \cdot a, 1 \cdot b\}, \{2 \cdot a, 2 \cdot b\}, \{1 \cdot a, 3 \cdot b\}, \{4 \cdot b\}.
\end{aligned}$$

If all multiplicities are infinite, the number of r-permutations is just the number of sequences of length r with entries from the underlying set.

Example 8. The number of binary strings of length 6 is the number of 6-permutations of $\{\infty \cdot 0, \infty \cdot 1\}$.

3.3 r-combinations with infinite multiplicity

We know that the number of r-combinations of the multiset

$$M = \{\infty \cdot 1, \infty \cdot 2, \ldots, \infty \cdot k\}$$

is the number of integer solutions to

$$x_1 + \cdots + x_k = r$$
$$\text{such that } 0 \leq x_i \leq \infty \text{ for all } i,$$

but $x_i \leq \infty$ is not a constraint at all! This means we can forget about it, so the number of r-combinations of

$$M = \{\infty \cdot 1, \infty \cdot 2, \ldots, \infty \cdot k\}$$

is the number of integer solutions to

$$x_1 + \cdots + x_k = r$$
$$\text{such that } 0 \leq x_i \text{ for all } i.$$

In this situation, we actually can get a nice answer.

Theorem 3.2. *(stars and bars) The the number of r-combinations of*

$$M = \{\infty \cdot 1, \infty \cdot 2, \ldots, \infty \cdot k\},$$

which is also the number of integer solutions to

$$x_1 + \cdots + x_k = r$$
$$\textit{such that } 0 \leq x_i \textit{ for all } i,$$

is equal to

$$\binom{r+k-1}{r}.$$

Proof. The name of the theorem comes from the proof. Let's count the number of integer solutions. This is the number of partitions of r *indistinguishable* objects (stars) into k *distinguishable* parts. That is, only the *number* of stars in each part matter, because they are indistinguishable. This and the number of solutions are the same, because $(x_1, \ldots x_k)$ that are solutions to

$$x_1 + \cdots + x_k = r$$
$$\text{such that } 0 \leq x_i \text{ for all } i,$$

are in one to one correspondence partitions by the function mapping $(x_1, \ldots x_k)$ to a partition whose i^{th} part is of size x_i.

We can count the number of partitions of r indistinguishable stars into k distinguishable parts easily. As an example, suppose $r = 5$ and $k = 3$. We can make a diagram of the partition where part 1 has 3 stars, part 2 has 1 star, and part 3 has 1 star.

$$\star\star\star|\star|\star.$$

Let's also try the one where part 1 has 2 stars, part 2 has 0 stars, and part 3 has 3 stars.

$$\star\star \,||\, \star\star\star\,.$$

In general, partitions of the stars into k labeled parts are in one-to-one correspondence (meaning there is a one-to-one function between the two sets) with placements of $k-1$ *bars* indicating when each new part begins. The number of placements of $k-1$ in the spaces between r stars is the same as the number of sequences of length $r+k-1$ of which $k-1$ of the entries are bars and r of the entries are stars, which is

$$\binom{r+k-1}{k-1} = \binom{r+k-1}{r}.$$

□

Example 9. A box from Krispy Kreme holds 12 donuts, and they make 8 kinds (they probably have more, but I don't know how many). Assume they can make at least 12 of any kind. How many different combinations of donuts can we buy?

This is the number of 12-combinations of the multiset

$$\{12 \cdot (\text{donut } 1), \ldots, 12 \cdot (\text{donut } 8)\},$$

which is the same as the number of 12-combinations of

$$\{\infty \cdot (\text{donut } 1), \ldots, \infty \cdot (\text{donut } 8)\}.$$

This means we can use the stars and bars theorem to count the solutions, which will be

$$\binom{12+8-1}{8-1} = \binom{19}{7}.$$

Example 10. What if we are in the same situation as the preceding example, but we wish to ensure that we buy at least one donut of each kind? One way is to just note that we can first just buy one of every donut, and then we have to choose a 4-combination of

$$\{\infty \cdot (\text{donut } 1), \ldots, \infty \cdot (\text{donut } 8)\},$$

which can be done in

$$\binom{12+4-1}{4-1} = \binom{15}{3}$$

ways. Another way is to observe that this is the same as the number of integer solutions to the equation

$$x_1 + \cdots + x_8 = 12$$
$$\text{such that } 1 \leq x_i \text{ for all } i.$$

By making the change of variables $y_i = x_i - 1$, we have a new equation

$$\begin{gathered} y_1 + \cdots + y_8 = 4 \\ \text{such that } 0 \leq y_i \text{ for all } i. \end{gathered}$$

This has the same number of solutions because the function

$$f((x_1, \ldots, x_8)) = (x_1 - 1, \ldots, x_8 - 1)$$

is a one-to-one function. By the stars and bars theorem, the number of solutions is, of course,

$$\binom{12 + 4 - 1}{4 - 1} = \binom{15}{3}.$$