# Posets: Math 454 Lecture 17 (7/26/2017)

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July 26, 2017

Keller and Trotter Poset chapter:

http://www.rellek.net/book/ch_posets.html.

I recommend Sections 6.1-6.3 and 6.5.

## Contents



## 1 Definitions

**Definition 1.1.** A *poset* $\mathcal{P} = (X, \preceq)$ is a pair consisting of a set $X$ called the *ground set* and a partial order $\preceq$ on $X$. A partial order is any relation that is

**Reflexive:** For all $x \in X$,

$$x \preceq x.$$

**Antisymmetric:** For all $x, y \in X$,

$$x \preceq y \text{ and } y \preceq x \text{ implies } y = x.$$

**Transitive:** For all $x, y, z \in X$,

$$x \preceq y \text{ and } y \preceq z \text{ implies } x \preceq z.$$

**Example 1.** $\mathcal{P} = (\{a, b, c, d\}, \preceq)$ is a poset if $\preceq$ is defined by

$$a \preceq a, b \preceq b, c \preceq c, d \preceq d$$
$$a \preceq b, a \preceq c, b \preceq d, c \preceq d, a \preceq d.$$

I forgot to write parts of this down in class. Apologies. In order to define a relation, you need to describe the entire thing.

**Remark 1.1** (Notation clarification:)**.** A *binary relation* on a set $X$ is actually a set of pairs of elements of $X$. A partial order is a special kind of binary relation, so $\preceq$ is actually a set of pairs. The meaning of $x \preceq y$ is really $(x, y) \in \preceq$. The second notation is very cumbersome, and this is why the first way of writing it is usually used. I use $\preceq$ to be a placeholder for partial orders that need to be defined, such as in the previous example.

**Example 2.** If $\preceq$ is the partial order from the previous example, then

$$\preceq = \{(a, a), (b, b), (c, c), (d, d), (a, b), (a, c), (b, d), (c, d), (a, d)\}.$$

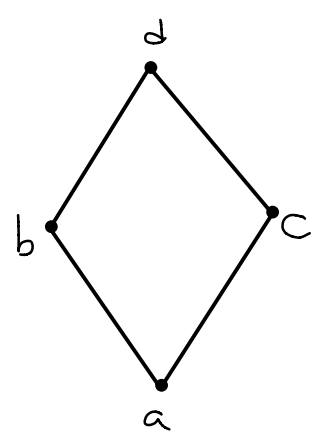
**Definition 1.2.** $x$ is *covered by* $y$ if $x \preceq y$ and if $x \preceq z \preceq y$ then $z = y$ or $z = x$. That is, there is no element of $X$ "in between" $x$ and $y$.

**Example 3.** In the previous example, $b$ covers $a$, $c$ covers $a$, and $d$ covers $b$. However, $d$ does not cover $a$, because $a \preceq b \preceq d$.

The notion of cover lets us draw posets.

**Definition 1.3.** A *Hasse diagram* is a drawing of a poset in the plane where the points are a ground set, and if $x$ covers $y$ then $x$ is drawn somewhere above $y$ and they are connected by a line.

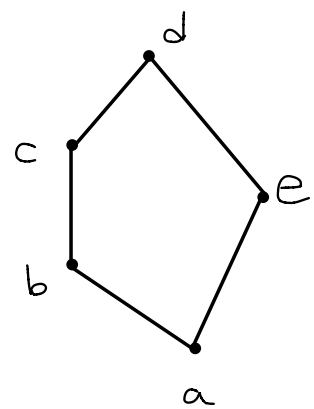
**Example 4.** The poset from the previous example has Hasse diagram



and the poset $\mathcal{P} = (X, \preceq)$ where $\preceq$ is defined by

$$\begin{aligned} a \preceq a, b \preceq b, c \preceq c, d \preceq d, e \preceq e \\ a \preceq b, a \preceq c, a \preceq d, b \preceq c, b \preceq d, c \preceq d \\ a \preceq e, e \preceq d \end{aligned}$$

has Hasse diagram

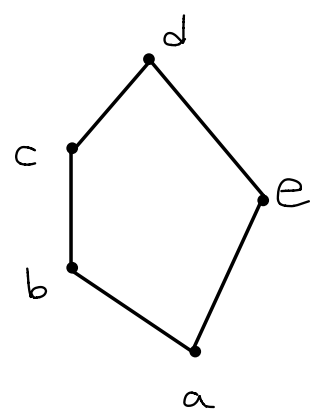



Once you know how to define posets by Hasse diagrams, you don't need to write down the entire relation anymore.

**Definition 1.4** (Shorthands:)**.** Commonly $y \succeq x$ is written to mean $x \preceq y$, and $x \prec y$ means $x \preceq y$ but $x \neq y$. Similarly, $y \succ x$ means $x \prec y$.

**Definition 1.5.** Two elements $x$ and $y$ of $X$ are *comparable* in $\mathcal{P} = (X, \preceq)$ if $x \preceq y$ or $y \preceq x$.

**Example 5.** In



$c$ and $e$ are incomparable and $b$ and $e$ are incomparable, but
The pairs $a$ and $a$, $b$ and $b$, $c$ and $c$, $d$ and $d$, $e$ and $e$
$a$ and $b$, $a$ and $c$, $a$ and $d$, $b$ and $c$, $b$ and $d$, $c$ and $d$,
$a$ and $e$, $e$ and $d$ are all *comparable.*

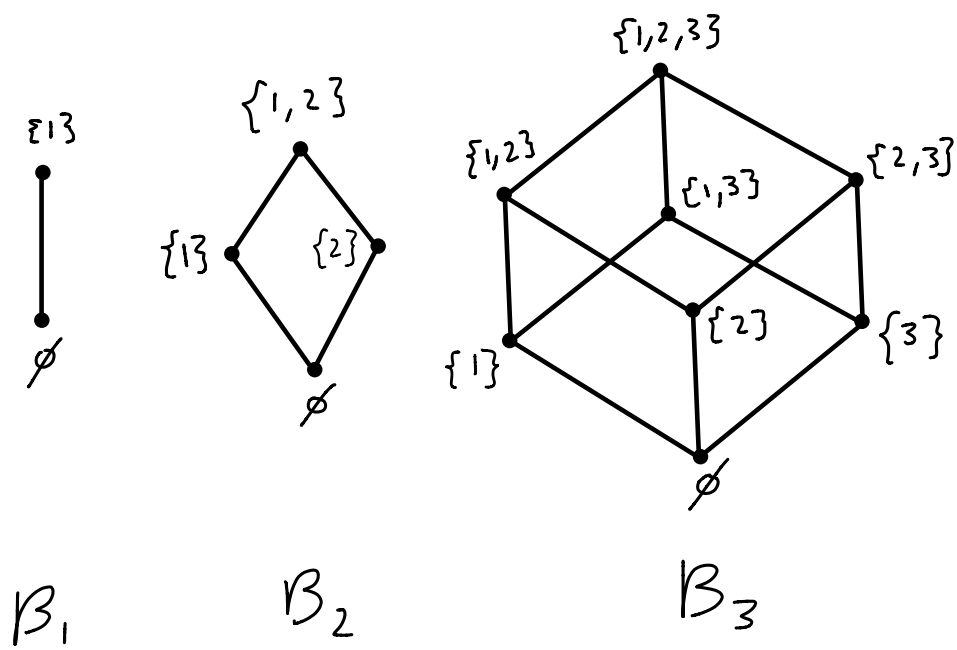
## 1.1 Examples

**Example 6** (2-D integer lattice)**.** $\mathbb{Z} \times \mathbb{Z}$ can be made into a poset $\mathcal{P} = (\mathbb{Z} \times \mathbb{Z}, \preceq)$, where $\preceq$ is defined by $(x_1, y_1) \preceq (x_2, y_2)$ if $x_1 \leq x_2$ and $y_1 \leq y_2$.

**Example 7** (Any family of subsets)**.** If $\mathcal{F}$ denotes a family of subsets of $[n]$, then $\mathcal{P} = (\mathcal{F}, \subseteq)$ is a poset, because $\subseteq$ satisfies the properties of a partial order.

**Example 8** (The Boolean lattice $\mathcal{B}_n$)**.** This is the example above if we set $\mathcal{F} = 2^{[n]}$. Here $2^{[n]}$ denotes the *power set* of $[n]$, or the set of all subsets of $[n]$. We define the Boolean lattice to be the poset

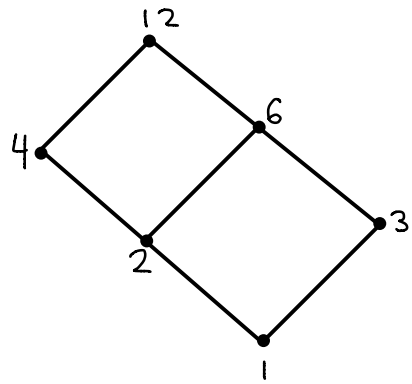
$$\mathcal{B}_n = (2^{[n]}, \subseteq).$$




**Example 9.** If $n$ is a number, the set $D$ of divisors of $n$ can be made into a poset $\mathcal{P} = (D, \preceq)$, where $\preceq$ is defined by

$$a \preceq b \text{ if and only if } a|b.$$

This is because divisibility relation satisfies the properties of a partial order.




## 2 Chains, Antichains

**Definition 2.1.** If $(X, \preceq)$ is a poset, a *chain* $C$ is a subset of $X$ such that any two points of $C$ are comparable.

**Definition 2.2.** If $(X, \preceq)$ is a poset, an *antichain* $A$ is a subset of $X$ such that *no* two points of $C$ are comparable.

**Definition 2.3.** If $\mathcal{P}$ is a finite poset, the *height* of $\mathcal{P}$, denoted

$$\text{height}(\mathcal{P}),$$

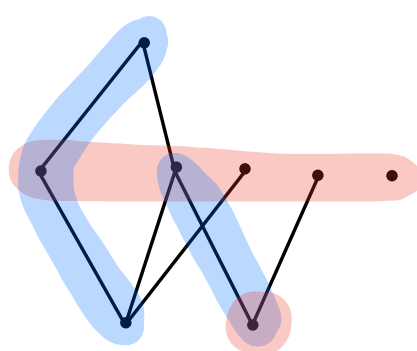
is the size of its longest chain.

**Definition 2.4.** If $\mathcal{P}$ is a finite poset, the the *width* of $\mathcal{P}$

$$\text{width}(\mathcal{P}),$$

is the size of its largest antichain.

**Example 10.** A chain of size 2 and of size 3, and an antichain of size 5 and of size 1.



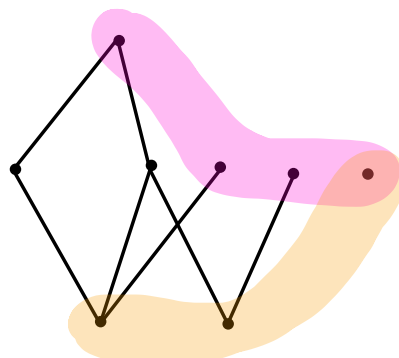
As you can see, this poset has height 3 and width 5.

**Definition 2.5.** If $\mathcal{P} = (X, \preceq)$ is a poset, say $x \in X$ is *minimal* if there are no elements $y \in X$ with $y \prec x$. The set of minimal elements is denoted by $\min(\mathcal{P})$.

If $\mathcal{P} = (X, \preceq)$ is a poset, say $x \in X$ is *maximal* if there are no elements $y \in X$ with $y \succ x$. The set of minimal elements is denoted by $\max(\mathcal{P})$.

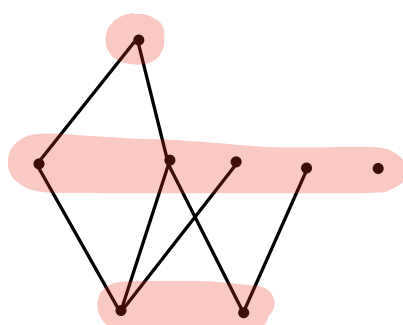
**Fact 2.0.1.** $\max(\mathcal{P})$ and $\min(\mathcal{P})$ are both antichains.

**Example 11.** The poset $(\mathbb{R}^+, \leq)$, that is the *positive* real numbers with the usual ordering, has no minimal elements. The maximal elements of the following poset are highlighted in pink, and the minimal in orange:



**Theorem 2.1** (Anti-Dilworth[1])**.** *It is possible to decompose a finite poset $\mathcal{P}$ into* $\text{height}(\mathcal{P})$*, and no fewer, antichains.*

**Example 12.** A partition of a poset of height 3 into 3 antichains:



[1]This isn't a real name. You'll understand why I named it this way soon enough.

*Proof.* Remove $\min(\mathcal{P})$, repeat on the poset $(X \setminus \min(\mathcal{P}), \preceq)$. Removes one element from the longest chain each time. You can draw the Hasse diagram by doing this, and the "height" of the Hasse Diagram will be $\text{height}(\mathcal{P})$.

**Fact 2.0.2.** If $A$ is an antichain and $C$ is a chain, then $|A \cap C| \leq 1$.

If fewer than $\text{height}(P)$ antichains, some two elements of the longest chain would be in one antichain; impossible by Fact 2.0.2. □

**Corollary 2.1** (Large is tall or wide)**.**

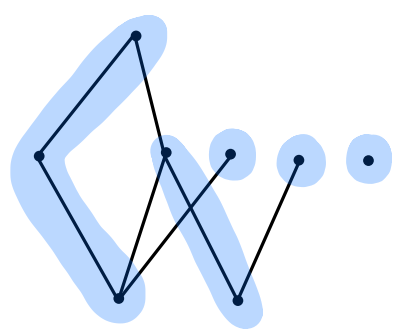
$$|\mathcal{P}| \leq \text{height}(\mathcal{P})\,\text{width}(\mathcal{P}).$$

## 2.1 Dilworth's Theorem

It is much harder to find a decomposition into the least possible number of chains, however, it pays off because chains are so nice.

**Theorem 2.2** (Dilworth's Theorem)**.** *It is possible to decompose a finite poset $\mathcal{P}$ into* $\text{width}(\mathcal{P})$*, and no fewer, chains.*

**Example 13.** A partition of a poset of width 5 into 5 chains:



*Proof.* First, note that there is no way to decompose $\mathcal{P}$ into fewer than $\text{width}(\mathcal{P})$ chains because of Fact 2.0.2. This is induction on the size of the ground set $X$. If $|X| = 1$, obviously this is true because $\text{height}(\mathcal{P}) = \text{width}(\mathcal{P}) = 1$, and $X$ is a chain. Next, assume $|X| = k > 2$, and assume the hypothesis of Dilworth's Theorem holds if $|X| < k$. If $X \geq 2$, there are two cases:

**Case 1. There is an antichain $A$ of size** $\text{width}(\mathcal{P})$ **that is neither** $\min(\mathcal{P})$ **nor** $\max(\mathcal{P})$**:** Let

$$A_+ = \{x : \text{ there is } y \in A \text{ with } y \preceq x\}.$$

In words, $A_+$ is everything *above* $A$. Let

$$A_- = \{x : \text{ there is } y \in A \text{ with } y \succeq x\}.$$

$A_-$ is everything *below* $A$.

**Claim 2.1.** *We need to make a few claims to handle this case:*

1. $|A_+| < k$, *because otherwise* $A = \min(\mathcal{P})$.
2. $|A_-| < k$, *because otherwise* $A = \max(\mathcal{P})$.
3. $A_+ \cap A_- = A$, *because if it had any element* $x \notin A$, *then there would be* $y \preceq x \preceq z$ *with* $y, z \in A$, *a contradiction.*
4. $A_+ \cup A_- = X$, *because otherwise we could increase* $A$ *to size* $\text{width}(\mathcal{P}) + 1$, *a contradiction.*

Since $|A_+| < k$ and $|A_-| < k$, the posets $\mathcal{P}_+ = (A_+, \preceq)$ and $\mathcal{P}_- = (A_-, \preceq)$ have decompositions into $\text{width}(\mathcal{P})$ chains ($A$ is still the largest chain in either!). The least elements of the chains of $\mathcal{P}_+$ are exactly $A$, and the top elements of the chains in $\mathcal{P}_-$ are exactly $A$, and this is the only place these chains overlap, so they can be "glued" to get a chain decomposition of $\mathcal{P}$.

**Case 2. The only antichains of size** $\text{width}(\mathcal{P})$ **are one or both of** $\min(\mathcal{P}), \max(\mathcal{P})$**:** Let $x$ be a minimal element and let $y$ be a maximal element with $x \preceq y$ ($x$ may equal $Y$). Then $\{x, y\}$ is a chain, and the poset $\mathcal{P}' = (X \setminus \{x, y\}, \preceq)$ has width $\text{width}(\mathcal{P}) - 1$ and smaller ground set, so by indcution it has a decomposition into $\text{width}(\mathcal{P}) - 1$ chains. Add the chain $\{x, y\}$ back to get a decomposition of $\mathcal{P}$ into $\text{width}(\mathcal{P})$ chains.

□

# 3 The Boolean Lattice

## 3.1 Sperner's Theorem

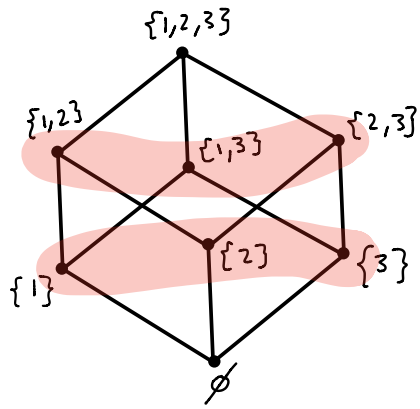
**Question 1.** Suppose the UN had 10 official languages (which would be true if the four proposed new languages became official). Say two interpreters are redundant if one can speak all the languages the other speaks. How many interpreters can the UN have if no two interpreters are redundant?

We can state this language in terms of families of sets. Let $A_i$ be the set of languages spoken by interpreter $i$; where $i \in [m]$ if there are $m$ interpreters. Then a pair of redundant interpreters is a pair $A_i \subset A_j$. So the question becomes:

**Question 2.** What is the largest family of subsets of [10] such that no set in the family is contained within another? Equivalently, what is the size of the largest antichain in the Boolean lattice?

One idea is just to let each interpreter speak some 5-subset of the languages. These certainly don't overlap, and this way we can have $\binom{10}{5} = 252$ interpreters. Is this the largest number? It turns out to be so. The "middle level" of $\mathcal{B}_n$ is the largest antichain.

**Example 14.** Two antichains of size 3 in $\mathcal{B}_3$:




**Theorem 3.1** (Sperner's Theorem)**.** *If $\mathcal{F}$ is a family of subsets of $[n]$ such that there are no distinct subsets $A, B \in \mathcal{F}$ satisfying $A \subset B$, then*

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

*In other words, the largest antichain of $\mathcal{B}_n$ is of size*

$$\binom{n}{\lfloor n/2 \rfloor},$$

*formed by all subsets of $[n]$ of size $\lfloor n/2 \rfloor$.*

The proof of this theorem is in the next lecture.