Matchings, Ramsey Theory: Math 454 Lecture 15 (7/24/2017)

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Keller and Trotter Graph Theory chapter:

http://www.rellek.net/book/ch_graphs.html.

Keller and Trotter Network Flows chapter:

http://www.rellek.net/book/ch_networkflow.html.

Keller and Trotter Intro to Ramsey Theory:

http://www.rellek.net/book/s_probmeth_graph-ramsey.html.

More can be found about matchings in Chapter 9, 11.4, and 12.5 of Brualdi. Ramsey Theory shows up in Chapter 3.3 of Brualdi.

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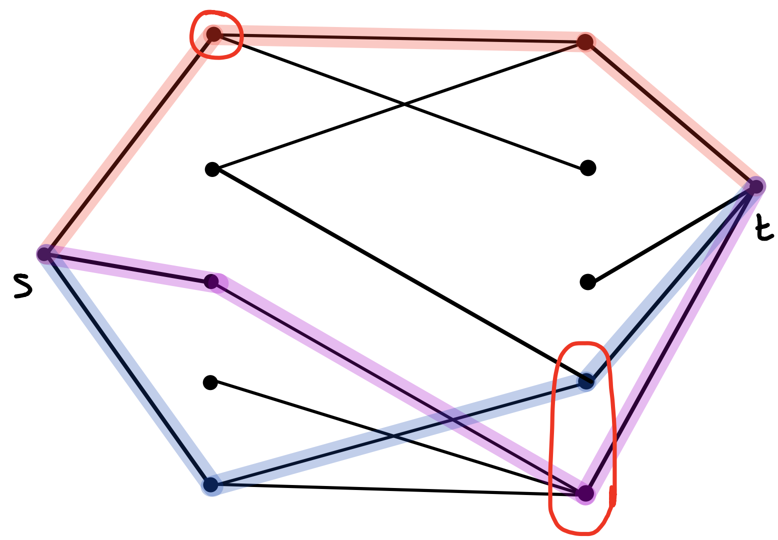
1 Menger's Theorem

1.1 Vertex version of Menger

Definition 1.1. Paths P_1 and P_2 from x to y are *independent* if the only vertices they share are x and y.

Theorem 1.1. *The number of vertices required to separate s from t in a graph G is exactly the number of independent, directed paths from s to t.*

Example 1. The highlighted paths are independent, and circled in red are vertices that can be removed to separate s and t.



Proof. Clearly if there are k independent paths, you need to delete at least one vertex from each to disconnect s and t, so the number of vertices required to separate s from t in a graph G is at least the number of independent, directed paths from s to t. Next we need to show that the number of independent paths from s to t is *at least* the number of vertices required to separate s from t in G. That is, we need to find the independent paths.

Form a network $N = (D, c, s_+, t_-)$ from the graph G. For each vertex v of G, add vertices v_+ and v_- to D and an arc (v_-, v_+) to D with capacity 1. For each arc (v, w) of G, add an arc (v_+, w_-) to D with infinite capacity. Delete s_- and t_+, and name the source of this network to be s_+ and the sink t_-. See Figure 1.

Suppose some cut S in N has capacity $k < \infty$. The arcs of D leaving S must be of the form (v_-, v_+), because these are the only arcs with finite capacities. A directed path from s to t in G corresponds to exactly one directed path s_+ to t_- in D, and this path must

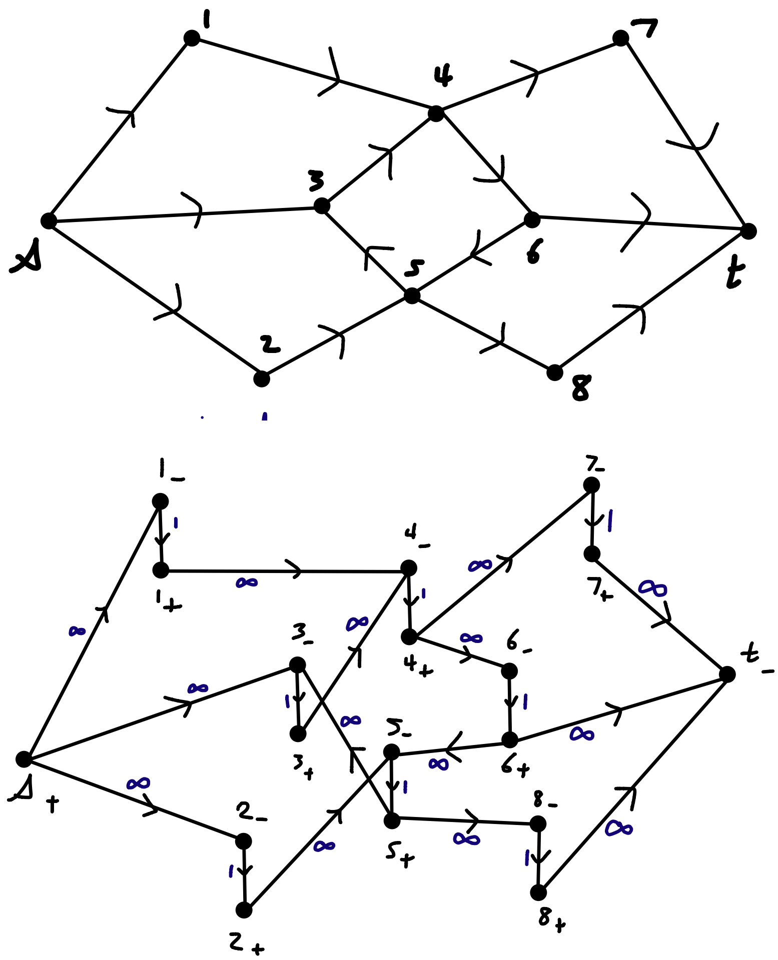


Figure 1: The bottom network is obtained from the digraph at top by the procedure in the proof.

leave S at some point, so it must visit a vertex x_+ such that x_- is not in S. Thus, the set X of vertices x of G such that $x_- \in S$ but $x_+ \notin S$ separates s from t in G. Because $\text{cap}(S) = k$, we must have $|X| = k$.

The preceding argument shows that a cut of capacity k in N gives rise to a set of k vertices separating s and t. If at least k vertices are required to separate s and t, then

$$\min \text{cap}(S) \geq k.$$

Because of this, the Max-Flow Min-Cut theorem tells us there is an integral flow in N with value at least k.

An integral flow, however, has flow value zero or 1 on every edge, because flow entering v_- through an arc (w_+, v_-) must leave v_- via the arc (v_-, v_+) which has capacity only 1. Form a subgraph of G by adding the arcs (v, w) such that the flow on $(v_+, w_-) = 1$. This subgraph has in-degree 1 and out-degree 1 on all vertices except for s and t, and s will have outdegree c and t will have in-degree c, so the subgraph is a union of c independent paths. $\square$

Remark 1.1. This works for directed or undirected graphs. If you want to do it for undirected graphs, just apply the theorem to the directed graph formed by replacing each edge $\{u, v\}$ by the pair of arcs $(u, v), (v, u)$.

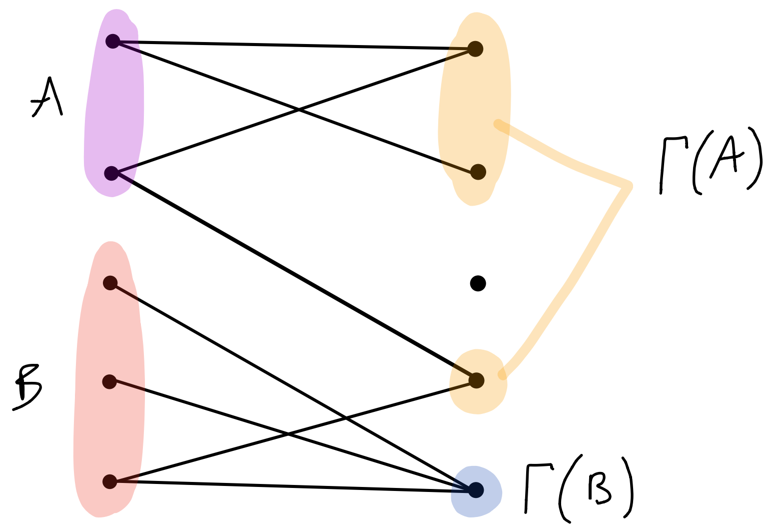
2 Hall's Marriage theorem

In this section we will consider undirected graphs, though it doesn't matter much, because usually you can think of edges as going from one side of the bipartite graph to the other.

Definition 2.1. If $G = (X \cup Y, E)$ is a bipartite graph and $R \subset X$, then

$$\Gamma(R) = \{y : y \text{ is adjacent to some vertex in } R\}.$$

In other words, $\Gamma(R)$ is all vertices adjacent to some vertex in R. It's like the "neighborhood" of the set R.



We saw in Lecture 13 that if $|\Gamma(R)| < |R|$ for some R subset X then there is no hope of a complete matching from X to Y. This is because R must be matched to $\Gamma(R)$, and there are not enough vertices in $\Gamma(R)$ to accomplish this. It turns out that this is the *only* issue we must avoid.

Definition 2.2. $G = (X \cup Y, E)$ is said to satisfy *Hall's marriage condition* if

$$|\Gamma(R)| \geq |R|$$

for all subsets $R \subset X$.

Theorem 2.1. *G contains a complete matching of X to Y if and only if G satisfies Hall's marriage condition, or equivalently*

$$|\Gamma(R)| \geq |R|$$

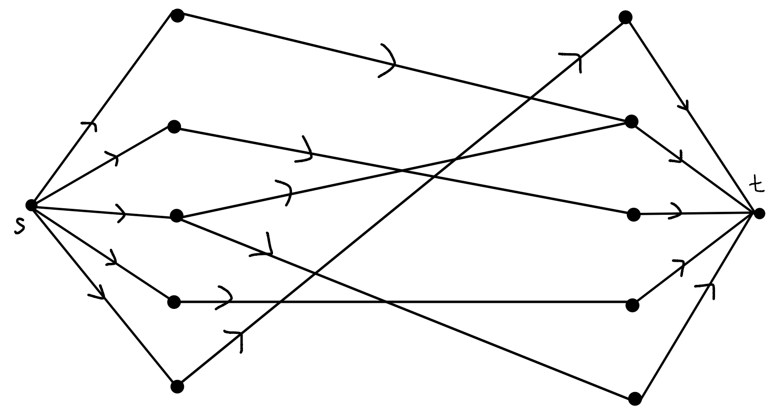
for all subsets $R \subset X$.

Proof. We already noticed that if there is a perfect matching, then Hall's marriage condition must be satisfied.

Next, we need to show if G satisfies Hall's marriage condition then there is a perfect matching. Let $G = (X \cup Y, E)$ be a bipartite graph satisfying Hall's marriage condition. Form a directed graph D with

- vertices $X \cup Y \cup \{s, t\}$, that is, the same vertex set but with two additional vertices.

- a directed edge (s, x) for every vertex $x \in X$.
- a directed edge (x, y) for every edge $E = \{x, y\}$ for $x \in X$ and $y \in Y$.
- a directed edge (y, t) for every vertex $y \in Y$.



Now a complete matching of X to Y exactly corresponds to $|X|$ independent paths from s to t in D. By Menger's Theorem, we know that the maximum number of independent paths is exactly the size of the smallest set of vertices that separate s from t.

Suppose there is no matching of size $|X|$, then there must be a set S of vertices of size $|S| < |X|$ that separates s from t. Assume $S = S_1 \cup S_2$, where $S_1 \subset X$ and $S_2 \subset Y$. This means there are no edges from $X \setminus S_1$ to $Y \setminus S_2$ in D. Thus, $\Gamma(X - S_1) \subset S_2$, or

$$\Gamma(X - S_1) \leq S_2 < |X| - |S_1|$$

so that

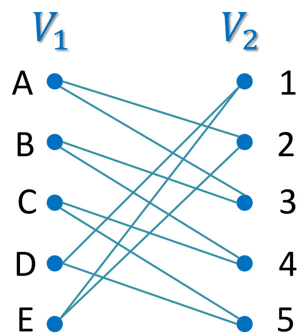
$$|\Gamma(X - S_1)| < |X - S_1|,$$

a contradiction. $\square$

Example 2. A *latin square* is an $n \times n$ square filled with the numbers $\{1, \ldots, n\}$ in which each row and each column contain each number exactly once. How to make these? It turns out that no matter how we fill out the first $m < n$ columns, we can fill out the next column. We do this by making a bipartite graph $G = (R \cup [n], E)$, where $\{r, j\} \in E$ if the number j has not yet appeared in row r.

- Each vertex $r \in R$ has degree $(m - n)$ because m symbols have appeared so far, and
- each vertex $j \in [n]$ has degree $(m - n)$ because every number has already appeared in m rows (because every column contains each number once!).

This means the bipartite graph is *regular*, that is, all the degrees are the same.



A	B	C	D	E
B	C	D	E	A
C	D	E	A	B

Theorem 2.2. *If $G = (X \cup Y, E)$ is a regular bipartite graph, then G contains a complete matching of X to Y.*

Proof. We just have to check Hall's marriage condition for regular bipartite graphs. Let $S \subset X$, and consider $\Gamma(S)$. For $y \in \Gamma(S)$, let $d_S(y)$ be the number of neighbors y has among S. Then we can count the edges between S and $\Gamma(S)$ in two ways; namely, the number of edges there is

$$d|S| = \sum_{y \in \Gamma(s)} d_S(y) \leq d|\Gamma(S)|.$$

Dividing by d, $|S| \leq |\Gamma(S)|$, meaning Hall's condition holds. $\square$

Once we get this perfect matching $\{1, j_1\}, \{2, j_2\}, \ldots \{n, j_n\}$, just put j_i in row i to make the $m + 1^{st}$ column.

Remark 2.1. Another way to think about Latin squares is as decompositions of $K_{n,n}$ into n edge-disjoint, perfect matchings.

3 Ramsey Theory

Ramsey Theory is a theory built upon the idea that often when you partition objects into several parts, at least one of the parts must contain some nice substructure. An easy example of this is the pigeonhole principle, which dictates simply that if you partition a set of objects, you will get a large part.

3.1 More Pigeonhole Principle

Theorem 3.1 ("Strong" Pigeonhole Principle)**.** *If a set of size n is partitioned into m parts, there is a part of size $\lceil n/m \rceil$.*

3.2 Ramsey numbers

People at a party.

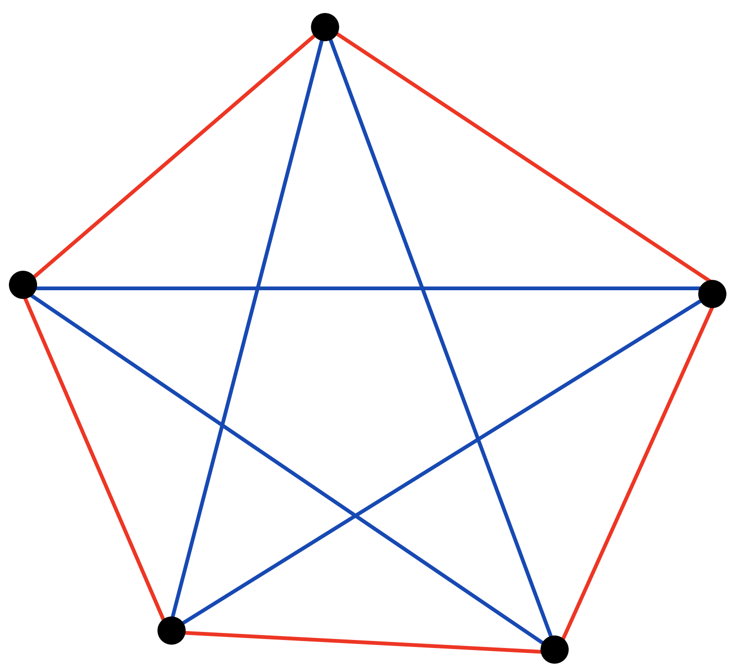
Fact 3.2.1. If the edges of K_6 are colored red and blue, then there is either a red copy of K_3 or a blue copy of K_3.

Proof. Look at a vertex v. It has five incident edges in K_6, so some 3 of them are the same color. Assume, without loss of generality, that this color is blue. Thus, v is adjacent to at least 3 vertices by blue edges. These 3 vertices have 3 edges among them. If any one of these edges is blue, then its endpoints and v form a blue triangle. If all the edges are red, then they form a red triangle. Hence, either way, K_6 contains triangle of red or blue. $\square$

We can try to come up with theorems of the same form for red K_m's and blue $K_n's$.

Definition 3.1. $R(m, n)$ is the smallest number r such that every coloring of the edges of K_r contains a red copy of K_m or a blue copy of K_n.

Example 3. Since the following contains no red or blue K_3, $R(3, 3) > 5$.



However, the fact above shows $R(3, 3) \leq 6$, so in fact $R(3, 3) = 6$.

3.3 Ramsey's Theorem

Theorem 3.2. *For all $m \geq 1, n \geq 1$, $R(m,n)$ actually exists, and*

$$R(m,n) \leq \binom{m+n-2}{m-1}.$$

Proof. We need to show there is a number r such that if the edges of K_r are colored with red and blue, then K_r contains a red copy of K_m or a blue copy of K_n. First we show

$$R(m,n) \leq R_{m-1,n} + R_{m,n-1}.$$

Suppose $s = R_{m-1,n}$ and $t = R_{m-1,n}$. Consider a coloring of the edges of K_{s+t} with red and blue. Any vertex v has either at least s neighbors through red edges or at least t neighbors through blue edges; otherwise there are at most $1 + (s-1) + (t-1) = s+t-1$ vertices. Suppose v has s neighbors through red edges. Since $s = R(m-1,n)$, there is either a red K_{m-1} among these neighbors, in which case we can add v to get a red K_m, or a blue K_n. The argument is similar if v instead has at least t neighbors through blue edges. Either way, K_{s+t} has either a red K_m or a blue K_n.

We can use the above fact to show

$$R(m,n) \leq \binom{m+n-2}{m-1}.$$

Let the above proposition be denoted $P(n,m)$. We prove it for all $n \geq 1, m \geq 1$ by double induction:
First off, note if $m \leq 2$ or $n \leq 2$, the claim is very easy. In fact,

$$R(m,2) = m = \binom{m+2-2}{m-1}$$

and

$$R(2,n) = n = \binom{n+2-2}{2-1}.$$

Thus, $P(m,2)$ for all $m \geq 0$, $P_{2,n}$ for all $n \geq 0$ are our base cases. We assume $n, m \geq 3$ to do the induction step. We use $P_{m-1,n}$ and $P_{m,n-1}$ to prove $P_{m,n}$. The idea of double induction is illustrated in Figure 2.
Thus, we assume $P_{m-1,n}$ holds and $P_{m,n-1}$ holds, that is

$$R(m-1,n) \leq \binom{m+n-3}{m-2}$$

and

$$R(m,n-1) \leq \binom{m+n-3}{m-1}.$$

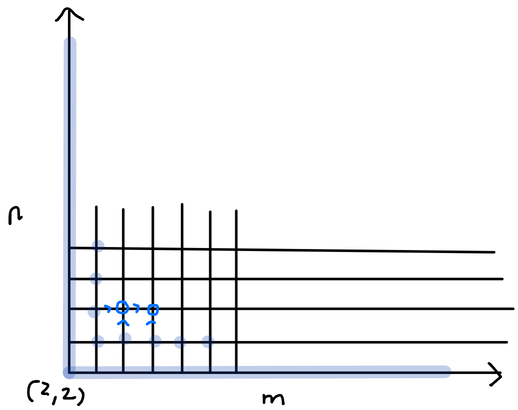


Figure 2: The idea of double induction. One could think of the axes as all the propositions $P(m,2)$ and $P(2,n)$, highlighted in blue to indicate they are true. Then, once you know $P_{m-1,n}$ and $P_{m,n-1}$ imply $P_{m,n}$, the truth of the propositions "spreads" up and to the right via the blue arrows to fill out the whole quadrant.

We already showed

$$R(m,n) \leq R\left(m-1,n\right) + R\left(m,n-1\right).$$

However, by the induction hypothesis,

$$R\left(m-1,n\right) + R\left(m,n-1\right) \leq \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} = \binom{m+n-2}{m-1}.$$

The last equality is Pascal's Formula. $\square$

3.4 Best known bounds for diagonal Ramsey numbers

$$\begin{aligned} R(2,2) &= 2 \\ R(3,3) &= 6 \\ R(4,4) &= 5 \\ R(5,5) &\in [43,48] \\ R(6,6) &\in [102,165] \end{aligned}$$

> Erdős asks us to imagine an alien force, vastly more powerful than us, landing on Earth and demanding the value of R(5, 5) or they will destroy our planet.

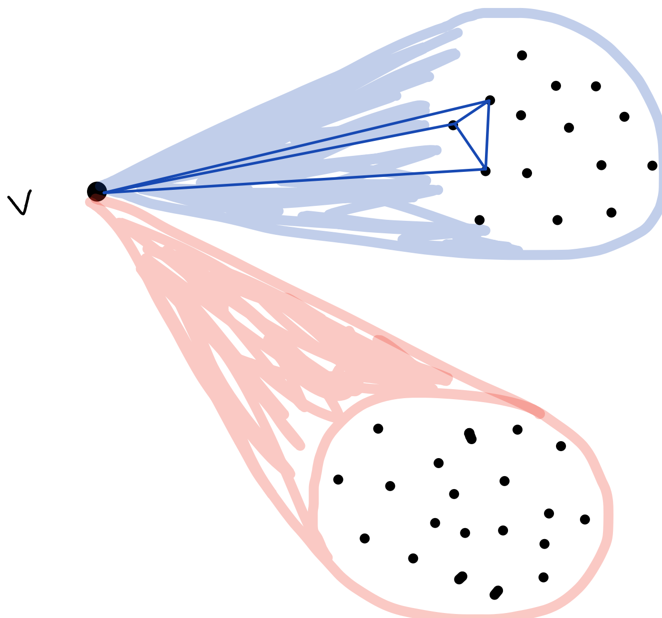


Figure 3: Finding a blue K_4 by finding a blue K_3 among neighbors attached with blue edges.

In that case, he claims, we should marshal all our computers and all our mathematicians and attempt to find the value. But suppose, instead, that they ask for R(6, 6). In that case, he believes, we should attempt to destroy the aliens.
- Joel Spencer