Math 454 Lecture 10: 7/12/2017

Cole Franks

July 23, 2017

Reading these notes is worse in almost every way than reading sections 5.1 and 5.2 of Keller and Trotter, which you can find here:

http://www.rellek.net/book/s_graphs_intro.html.

I tried to write down things that aren't emphasized in that book.

Contents

1 Basic graph definitions

Reading these notes worse in almost every way than reading sections 5.1 and 5.2 of Keller and Trotter. However, I will put a few things in that we covered.

Definition 1.1. A graph $G = (V, E)$ is a pair of sets V, called the *vertex set*, and E, called the *edge set*. E is a set of 2-subsets of V. In other word, it is a collection of unordered pairs in V. The elements of the edge set (the unordered pairs) are called the *edges* of G. We say an edge e is *incident* to a vertex v if $v \in e$, and that v and w are *adjacent* if $\{v, w\} \in E$ (there is an edge containing v and w).

A graph with n vertices and m edges is said to be of *order* n and *size* m. That is, the order of a graph is the number of vertices, and the size is the number of edges.

Example 1. $G = (\{a, b, c, d\}, \{\{a, b\}, \{c, d\}, \{a, b\}, \{c, d\}\})$ has vertex set

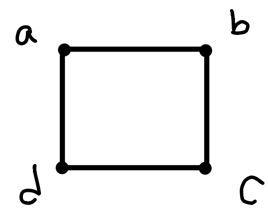
$$V = \{a, b, c, d\}$$

and edge set

$$E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}.$$

$\{ab\}$ is an edge of G. Commonly, this edge is denoted ab, and ba denotes the same edge. c is a vertex of G. da is incident to d, and the vertices a and d are adjacent.

Often we draw a graph where vertices are represented by points, and edges by lines going through the points corresponding to vertices they contain. The graph G just described could be drawn as follows:



Definition 1.2. Two graphs $G_1 = (V_1, E_2)$ and $G_2 = (V_2, E_2)$ are isomorphic if, informally, $G_1's$ vertex set can be relabeled with vertices of G_2 so that the relabeled graph is equal to G_2. More precisely, there is a one-to-one and onto function $f : V_1 \to V_2$ such that

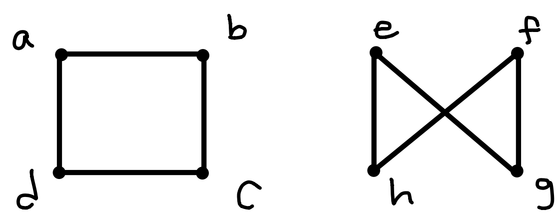
$$E_2 = \{f(v)f(w) : vw \in E_1\}.$$

Alternately, $f(v)$ and $f(w)$ are adjacent if and only if v and w are.

The a function f as above is called an *isomorphism.*

Example 2. The following graphs are isomorphic with an isomorphism f given by

$$f(a) = e, f(b) = g, f(c) = f, \text{ and } f(d) = h.$$

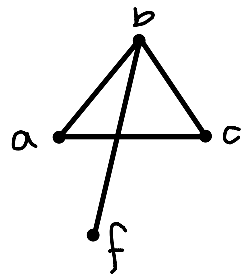


Definition 1.3. The *degree* of a vertex v in a graph G, which we will denote $d(v)$, is the number of edges incident to v, which is also the number of vertices adjacent to v.

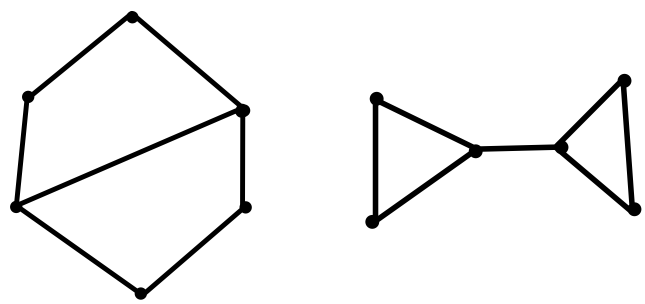
Definition 1.4. The *degree sequence* of a graph is a sequence of the degrees of the vertices of G in non-increasing order.

Example 3. The following graph has degrees $d(a) = 2, d(b) = 3, d(c) = 2, d(f) = 1$, and has degree sequence

$$(3, 2, 2, 1) :$$



Example 4. As we saw in class, two non-isomorphic graphs can have the same degree sequence, such as the following two graphs.



Theorem 1.1 (Handshake Lemma/First Theorem of Graph Theory)**.** *If $G = (V, E)$ is a graph then*

$$\sum_{v \in V} d(v) = 2|E|.$$

In particular, the sum of the degree sequence is even.

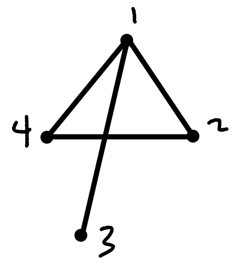
2 Adjacency Matrix

Definition 2.1. The *adjacency matrix* of a graph $G = (\{1, \dots, n\}, E)$ is a $n \times n$ matrix A with entries in $\{0, 1\}$ satisfying

$$A_{ij} = \begin{cases} 1 : & ij \in E \\ 0 : & ij \notin E. \end{cases}$$

There is a distinction between *the* adjacency matrix an *an* adjacency matrix: *an* adjacency matrix of G is *the* adjacency matrix of some graph isomorphic to G. This amounts to picking an ordering of the vertices of G.

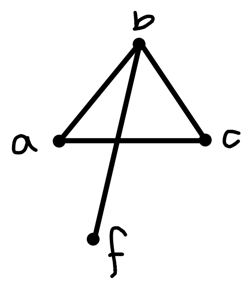
Example 5. *The* adjacency matrix of



is

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}.$$

An adjacency matrix of



is

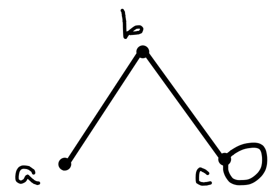
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

(this just has rows 2 and 3 swapped and columns 2 and 3 swapped from the previous matrix).

3 Variants of graphs

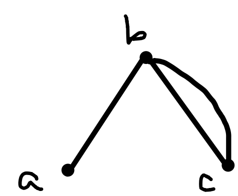
Definition 3.1. A *looped graph*, or a graph with self loops, is a graph where E is also allowed to contain sets of size 1 indicating that a vertex has a "loop", or is adjacent to itself.

Example 6 (A looped graph)**.** The following represents the looped graph
$G = (\{a, b, c\}, \{\{a, b\}, \{b, c\}, \{c\})$:



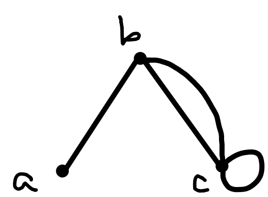
Definition 3.2. A *multigraph* $G = (V, E)$ consists of a set V, called the *vertex set*, and a **multiset** E, called the *edge set*. E is a multiset set of 2-subsets of V. In other word, it is a collection of unordered pairs in V where each unordered pair can occur any number of times.

Example 7 (A multigraph)**.** The following represents the multigraph
$G = (\{a, b, c\}, \{\{a, b\}, 2 \cdot \{b, c\})$:



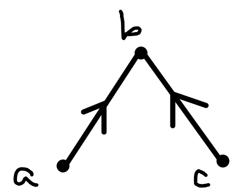
You can probably guess what a looped multigraph is.

Example 8 (A looped multigraph)**.** The following represents the looped multigraph
$G = (\{a, b, c\}, \{\{a, b\}, 2 \cdot \{b, c\}, \{c\}\})$:



Definition 3.3. A *directed graph* $G = (V, A)$, or *digraph*, is a pair of sets V, called the *vertex set*, and A, called the *arc set*. A is a set of **ordered pairs** of distinct elements of V. The ordered pairs in A are called *arcs*.

Example 9 (A digraph)**.** Digraphs are often drawn with arrows on the edges. The arc (a, b) is drawn as an arrow from a to b. The following represents the digraph
$G = (\{a, b, c\}, \{(a, b), (b, c)\})$:



I won't define *looped digraph*, *multidigraph*, or *looped multidigraph*, but you can guess what they are.