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Generalizations of nonlinear and supersymmetric classical electrodynamics

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Abstract

We first write down a very general description of nonlinear classical electrodynamics, making use of generalized constitutive equations and constitutive tensors. Our approach includes non-Lagrangian as well as Lagrangian theories, which allows for electromagnetic fields in the widest possible variety of media (anisotropic, piroelectric, chiral and ferromagnetic) and accommodates the incorporation of nonlocal effects. We formulate electric–magnetic duality in terms of the constitutive tensors. We then propose a supersymmetric version of the general constitutive equations, in a superfield approach.

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1. Introduction

Maxwell's equations for the vector fields **D** (electric displacement), **E** (electric field), **B** (magnetic induction) and **H** (magnetic field) in the vacuum or in media are well known to be incomplete. The system is completed with *constitutive equations*, which establish functional relations among these vector fields [1, 2]. Then **D** and **H** may be regarded as constructs used to describe (via the constitutive equations) how the directly observable fields **E** and **B** are produced by charge and current densities. The explicit form of the constitutive equations depends on the physical properties one assumes for the vacuum or for the medium in which the fields occur; in particular, they are constrained by the symmetry of the medium.

It has also long been known [3], though not always widely appreciated, that Maxwell's equations alone (without constitutive equations) are consistent with both Lorentz symmetry (for any value of the light speed c) and Galilei symmetry. Thus, one can specify a particular Lorentz- or Galilei-covariant theory through the choice of constitutive equations. Linear constitutive equations, taken together with Maxwell's equations, are inconsistent with Galilei-covariant electrodynamics, while a certain class of nonlinear equations are compatible with it [4]. These observations were subsequently generalized to a class of equations for non-Abelian gauge fields, written using nonlinear constitutive equations [5]. Interest in nonlinearity and Maxwell's equations is heightened by experimental results in nonlinear optics, such as optical squeezing and slow light speed [6], and by theoretical ideas such as Born–Infeld theories of superstrings [7–10]. It has recently been suggested that a modified Born–Infeld Lagrangian, proposed originally for the purpose of introducing a Galilean limit in nonlinear electromagnetism, provides a way to introduce a null string (i.e., zero tension) limit in a relativistic theory of four-dimensional superstrings [11]. We also remark on the recent use of modifications of Maxwell's equations as 'test theories' in astrophysical observations, where upper bounds to various sorts of possible deviations from known physical laws can be established through measurement [12, 13]. Our approach provides a very general framework for the construction of such test theories, particularly in the direction of allowing for dissipative effects. Our approach also accommodates various interesting limits of physical constants.

This paper continues the systematic exploration of nonlinear constitutive equations. We first write a very general form for such equations and the 'constitutive tensors' that appear in them. As in earlier work, this includes non-Lagrangian (i.e., dissipative) as well as Lagrangian theories, but also allows for the description of electromagnetic fields in the widest possible variety of media, including anisotropic, piroelectric, chiral and ferromagnetic media. Such a description also accommodates the incorporation of nonlocal effects. We then formulate electric–magnetic duality in terms of the constitutive tensors. Finally, we propose a supersymmetric version of the general constitutive equations, so as to obtain a general, nonlinear supersymmetric electrodynamics within a superfield approach. Some of our results were sketched briefly in [14]; here we provide greater detail and additional development.

2. Maxwell's equations and nonlinear constitutive equations

To set the context and specify notation, we review established results in this section. Let us write Maxwell's equations in SI units,

$$\begin{aligned} \operatorname{curl}\mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \qquad & \operatorname{div}\mathbf{B} &= 0, \\ \operatorname{curl}\mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} + \mathbf{j}, \qquad & \operatorname{div}\mathbf{D} &= \rho, \end{aligned} \tag{1}$$

where $\mathbf{j}$ and ρ are the current and charge densities, respectively. We consider only flat spacetime, so that the metric is given by the Minkowski tensor $\eta_{\mu\nu} = (1, -1, -1, -1)$, $x^\mu = (ct, x^i)$, $\mu, \nu, \ldots = 0, 1, 2, 3$, $i, j, \ldots = 1, 2, 3$, with $\partial_\mu = \partial/\partial x^\mu = [c^{-1}\partial/\partial t, \nabla]$. The antisymmetric Levi-Civita tensor is written $\varepsilon^{\mu\nu\rho\sigma}$, with $\varepsilon^{0123} = 1$. We denote six Lorentz invariants constructed from $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$, $\mathbf{H}$ (in terms of which other invariants may be written) by

$$\begin{aligned} &C_1 = \mathbf{B}^2 - \frac{1}{c^2}\mathbf{E}^2, \qquad C_2 = \mathbf{B}\cdot\mathbf{E}, \qquad C_3 = \mathbf{D}^2 - \frac{1}{c^2}\mathbf{H}^2, \qquad C_4 = \mathbf{D}\cdot\mathbf{H}, \\ &C_5 = \mathbf{B}\cdot\mathbf{H} - \mathbf{E}\cdot\mathbf{D}, \qquad C_6 = \mathbf{B}\cdot\mathbf{D} + \frac{1}{c^2}\mathbf{E}\cdot\mathbf{H}. \end{aligned} \tag{2}$$

The constitutive equations relating $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$ can reduce the symmetry of equations (1) to the Lorentz or Galilean groups; some other possibilities are considered in [15].

In covariant notation, we have the standard electromagnetic tensor fields

$$F_{\mu\nu} = \begin{pmatrix} 0 & \frac{1}{c}E_x & \frac{1}{c}E_y & \frac{1}{c}E_z \\ -\frac{1}{c}E_x & 0 & -B_z & B_y \\ -\frac{1}{c}E_y & B_z & 0 & -B_x \\ -\frac{1}{c}E_z & -B_y & B_x & 0 \end{pmatrix}, \qquad G_{\mu\nu} = \begin{pmatrix} 0 & cD_x & cD_y & cD_z \\ -cD_x & 0 & -H_z & H_y \\ -cD_y & H_z & 0 & -H_x \\ -cD_z & -H_y & H_x & 0 \end{pmatrix}; \tag{3}$$

the Hodge dual tensors are $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ and $\tilde{G}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}$; and Maxwell's equations become

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \qquad \partial_\mu G^{\mu\nu} = j^\nu, \tag{4}$$

where $j^\mu = (c\rho, \mathbf{j})$ is the 4-current. The first equations in (4) imply $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is an Abelian gauge field; but in general there is no similar representation for $G_{\mu\nu}$. The field strength tensors $F_{\mu\nu}$ and $\tilde{F}^{\mu\nu}$ are physically observable, in that their components can be inferred from measurement of the force $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ on a test charge q moving with velocity $\mathbf{v}$. The tensors $G_{\mu\nu}$ and $\tilde{G}^{\mu\nu}$ originate with the currents, and their relation to the observable fields is determined by the properties of the medium (or the vacuum) within which they are being described.

If we write constitutive equations for nonlinear electromagnetism in the form $\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B})$, $\mathbf{H} = \mathbf{H}(\mathbf{E}, \mathbf{B})$, then for a Lorentz-covariant theory, they must take the form [15]

$$\mathbf{D} = M(C_1, C_2)\mathbf{B} + \frac{1}{c^2}N(C_1, C_2)\mathbf{E}, \qquad \mathbf{H} = N(C_1, C_2)\mathbf{B} - M(C_1, C_2)\mathbf{E}, \tag{5}$$

where $M(C_1, C_2)$ and $N(C_1, C_2)$ are some smooth scalar functions of the first two invariants in equations (2). For the (linear) vacuum case, $\mathbf{D} = \varepsilon_0\mathbf{E}$ and $\mathbf{B} = \mu_0\mathbf{H}$, where ε_0 and μ_0 are, respectively, the permittivity and permeability of empty space; and where, consistent with equations (5), $\varepsilon_0\mu_0 = c^{-2}$ (so that $M = 0, N = 1/\mu_0 = \varepsilon_0 c^2$). But in general, the dependence of C_1 and C_2 on $\mathbf{B}$ and $\mathbf{E}$ means that equations (5) are nonlinear. When the constitutive equations take the form given in equations (5), the other Lorentz invariants in equations (2) are determined from the first two invariants C_1 and C_2 by the formulae

$$\begin{aligned} C_3 &= \left[M(C_1, C_2)^2 - \frac{1}{c^2}N(C_1, C_2)^2\right]C_1 + \frac{4}{c^2}M(C_1, C_2)N(C_1, C_2)C_2, \\ C_4 &= M(C_1, C_2)N(C_1, C_2)C_1 - \left[M(C_1, C_2)^2 - \frac{1}{c^2}N(C_1, C_2)^2\right]C_2, \\ C_5 &= N(C_1, C_2)C_1 - 2M(C_1, C_2)C_2, \qquad C_6 = M(C_1, C_2)C_1 + \frac{2}{c^2}N(C_1, C_2). \end{aligned} \tag{6}$$

If the theory is conformally invariant, then the 'constitutive functions' M and N that appear in equations (5) depend only on the quotient C_1/C_2, so that $M(C_1, C_2) = M_{\text{conf}}(C_1/C_2)$ and $N(C_1, C_2) = N_{\text{conf}}(C_1/C_2)$ [16]. The well-known Born–Infeld electrodynamics is specified by choosing M and N in equations (5) according to the formulae

$$M(C_1, C_2) = \frac{C_2}{\mu_0 b^2\sqrt{1 + \frac{c^2}{b^2}C_1 - \frac{c^2}{b^4}C_2^2}}, \tag{7}$$

$$N(C_1, C_2) = \frac{1}{\mu_0\sqrt{1 + \frac{c^2}{b^2}C_1 - \frac{c^2}{b^4}C_2^2}}, \tag{8}$$

where the real parameter b is the maximum permitted value of the electric field strength when the magnetic field is zero—any stronger electric field causes the argument of the square root in equations (7) and (8) to become negative. It follows from the form of (7) and (8) that Born–Infeld nonlinear electrodynamics has no conformal symmetry.

In covariant notation

$$C_1 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \equiv 2X, \qquad C_2 = -\frac{c}{4} F_{\mu\nu} \tilde{F}^{\mu\nu} \equiv -cY, \tag{9}$$

where X and Y are introduced here (for later use) for compatibility with the standard invariant notation. Then we also have

$$G_{\mu\nu} = N(C_1, C_2) F_{\mu\nu} + cM(C_1, C_2) \tilde{F}_{\mu\nu}. \tag{10}$$

Taking the Hodge conjugate of equation (10), we represent these equations in the form

$$\begin{pmatrix} G_{\mu\nu} \\ \tilde{G}_{\mu\nu} \end{pmatrix} = \begin{pmatrix} N(C_1, C_2) & cM(C_1, C_2) \\ -cM(C_1, C_2) & N(C_1, C_2) \end{pmatrix} \begin{pmatrix} F_{\mu\nu} \\ \tilde{F}_{\mu\nu} \end{pmatrix}. \tag{11}$$

In the space of 'spinors'

$$\Pi^F = \begin{pmatrix} F_{\mu\nu} \\ \tilde{F}_{\mu\nu} \end{pmatrix}, \qquad \Pi^G = \begin{pmatrix} G_{\mu\nu} \\ \tilde{G}_{\mu\nu} \end{pmatrix}, \tag{12}$$

we then have a kind of quaternionic structure as discussed in [17, 18],

$$\Pi^G = \mathbb{Q} \cdot \Pi^F, \tag{13}$$

where $\mathbb{Q}$ is defined by

$$\mathbb{Q} = N(C_1, C_2)\sigma_0 + \mathrm{i}cM(C_1, C_2)\sigma_2, \tag{14}$$

and where $\sigma_0 = I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\sigma_2 = \left(\begin{smallmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)$ are the Pauli matrices.

3. Generalized nonlinear constitutive equations

Although the constitutive equations (5) are fairly general, they do not take account of a variety of possibilities, such as anisotropic media [19], chiral materials where derivative terms enter [20], piroelectric and ferromagnetic materials and so forth. Therefore, we propose to generalize equations (5) and (10) by introducing the *constitutive tensors* $S_{\mu\nu}$, $R^{\rho\sigma}_{\mu\nu}$ and $Q^{\rho\sigma\lambda_1\ldots\lambda_n}_{\mu\nu}$, $n = 1, 2, 3, \ldots$. We then write a general nonlinear constitutive equation as follows:

$$G_{\mu\nu} = S_{\mu\nu} + R^{\rho\sigma}_{\mu\nu} F_{\rho\sigma} + Q^{\rho\sigma\lambda_1}_{\mu\nu} \frac{\partial F_{\rho\sigma}}{\partial x^{\lambda_1}} + Q^{\rho\sigma\lambda_1\lambda_2}_{\mu\nu} \frac{\partial F_{\rho\sigma}}{\partial x^{\lambda_1} \partial x^{\lambda_2}} + \cdots + Q^{\rho\sigma\lambda_1\ldots\lambda_n}_{\mu\nu} \frac{\partial F_{\rho\sigma}}{\partial x^{\lambda_1} \partial x^{\lambda_2} \cdots \partial x^{\lambda_n}}. \tag{15}$$

This equation and its superfield analogue introduced in section 6 are the central focus of the approach we are advocating here. Evidently, formula (15), taken together with Maxwell's equations, includes all the possibilities discussed up to now, as well as new ones, defining the *general nonlinear electromagnetic theory*.

Let us discuss the arguments of the constitutive tensors which are the coefficients in equation (15). Evidently, we must distinguish $S_{\mu\nu}$ by the absence of any dependence on $F_{\rho\sigma}$; for if $S_{\mu\nu}$ were to depend on any of the entries in $F_{\rho\sigma}$, we could then just incorporate it into the second term of equation (15) by rewriting it in the form $R^{\rho\sigma}_{\mu\nu} F_{\rho\sigma}$ (where $R^{\rho\sigma}_{\mu\nu} = S_{\mu\nu}/F_{\rho\sigma}$ for each particular value of μ, ν, ρ, σ). Hence the arguments of $S_{\mu\nu}$ should be (for maximum generality) just the spacetime coordinates x^λ. Likewise $R^{\rho\sigma}_{\mu\nu}$ can depend on x^λ and on $F_{\kappa\lambda}$, but not on any derivatives of $F_{\kappa\lambda}$. In general, $Q^{\rho\sigma\lambda_1\ldots\lambda_n}_{\mu\nu}$ depends on x^λ together with derivatives of $F_{\kappa\lambda}$ of nth order or less.

Now one can require $R^{\rho\sigma}_{\mu\nu}$ and $Q^{\rho\sigma\lambda_1\ldots\lambda_n}_{\mu\nu}$ to depend directly on x, $F_{\kappa\lambda}$ and derivatives of appropriate orders, as *local* functions that take into account only their values at x. But one may also consider constitutive tensors that depend on the field strengths and their derivatives via functionals that take account of their values at different spacetime points, or that involve integrals of functions of field strengths and their derivatives over regions of spacetime—for example, over regions that could be defined by the worldsheets of strings, branes, defects or other extended configurations in the spacetime manifold. Thus, we also have the capability of describing a variety of nonlocal effects with this approach.

As we impose Lorentz covariance on the constitutive equations, the constitutive tensors will depend on the fields through the invariants X and Y defined in equations (9), as follows: $S_{\mu\nu}$ is a constant independent of X and Y, while

$$R^{\rho\sigma}_{\mu\nu} = R^{\rho\sigma}_{\mu\nu}(X, Y), \qquad Q^{\rho\sigma\lambda_1\ldots\lambda_n}_{\mu\nu} = Q^{\rho\sigma\lambda_1\ldots\lambda_n}_{\mu\nu}(X, Y, \ldots), \tag{16}$$

where "$\ldots$" denotes invariant derivatives of the invariants X, Y up to nth order. Obviously $S_{\mu\nu}$ is antisymmetric, $R^{\rho\sigma}_{\mu\nu}$ is antisymmetric in its upper and lower indices separately and $Q^{\rho\sigma\lambda_1\ldots\lambda_n}_{\mu\nu}$ are antisymmetric in their upper and first two lower indices; with respect to the λ_i, they are symmetric Lorentz tensors.

Let us write some familiar examples in this form. For the simplest vacuum case, we have

$$S_{\mu\nu} = 0, \qquad R^{\rho\sigma}_{\mu\nu} = \mu_0^{-1}\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]}, \qquad Q^{\rho\sigma\lambda_1\ldots\lambda_n}_{\mu\nu} = 0, \tag{17}$$

where the square brackets about the indices denote antisymmetrization with a factor of $1/2$, i.e., $x_{[\mu\nu]} \equiv (x_{\mu\nu} - x_{\nu\mu})/2$. The only nonvanishing constitutive tensor $R^{\rho\sigma}_{\mu\nu}$ is 'diagonal'. For Born–Infeld nonlinear electrodynamics, we have

$$S_{\mu\nu} = 0, \qquad R^{\rho\sigma}_{\mu\nu} = \frac{\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]} - \frac{c^2}{b^2}Y\varepsilon_{[\mu\nu]\lambda\delta}\eta^{\lambda\rho}\eta^{\delta\sigma}}{\mu_0\sqrt{1 + 2\frac{c^2}{b^2}X - \frac{c^4}{b^4}Y^2}}, \qquad Q^{\rho\sigma\lambda_1\ldots\lambda_n}_{\mu\nu} = 0, \tag{18}$$

where b is again the maximum electric field strength, ε is the Levi-Civita tensor and η is the Minkowski tensor. For an anisotropic medium with tensorial permeability ε_{ij} and permittivity μ_{ij}, the constitutive equations are $\mathbf{D}_i = \varepsilon_{ij}\mathbf{E}_j$ and $\mathbf{B} = \mu_{ij}\mathbf{H}_j$. This case is not described by equation (10), but the corresponding constitutive tensor $R^{\rho\sigma}_{\mu\nu}$ is easily calculated; while again, $S_{\mu\nu}$ and $Q^{\rho\sigma\lambda_1\ldots\lambda_n}_{\mu\nu}$ vanish. The case $S_{\mu\nu} \neq 0$ describes piroelectric and ferromagnetic materials.

When the equations of motion for the nonlinear theory are derived from a Lagrangian $L(X, Y)$, which is a scalar function of the invariants X and Y but does not depend on their derivatives, then from the usual definitions together with equation (10), we have

$$G_{\mu\nu} = -\frac{\partial L(X, Y)}{\partial X}F_{\mu\nu} - \frac{\partial L(X, Y)}{\partial Y}\tilde{F}_{\mu\nu}. \tag{19}$$

Comparing (15) and (19) gives us the constitutive tensors,

$$\begin{aligned} &S_{\mu\nu} = 0, \\ &R^{\rho\sigma}_{\mu\nu} = -\frac{\partial L(X, Y)}{\partial X}\delta^{\rho}_{[\mu}\delta^{\sigma}_{\nu]} - \frac{\partial L(X, Y)}{\partial Y}\varepsilon_{[\mu\nu]\lambda\delta}\eta^{\lambda\rho}\eta^{\delta\sigma}, \\ &Q^{\rho\sigma\lambda_1\ldots\lambda_n}_{\mu\nu} = 0. \end{aligned} \tag{20}$$

In this case, the functions M and N in equations (5) are

$$N_L(X, Y) = -\frac{\partial L(X, Y)}{\partial X}, \qquad M_L(X, Y) = -\frac{1}{c}\frac{\partial L(X, Y)}{\partial Y}. \tag{21}$$

So for the 'constitutive functions' M and N to describe a Lagrangian theory of nonlinear electrodynamics, we need

$$\frac{\partial N_L(X,Y)}{\partial Y} = c\frac{\partial M_L(X,Y)}{\partial X}. \tag{22}$$

We see that dissipative, non-Lagrangian theories are naturally included in the current framework.

4. Duality transformations

Next consider the duality transformation δ transforming $F_{\mu\nu}$ to $\tilde{G}_{\mu\nu}$ and $G_{\mu\nu}$ to $\tilde{F}_{\mu\nu}$ [21]:

$$\delta F_{\mu\nu} = \tilde{G}_{\mu\nu}, \qquad \delta G_{\mu\nu} = \tilde{F}_{\mu\nu}. \tag{23}$$

The self-duality (+) or antiself-duality (−) condition is defined by

$$F_{\mu\nu} = \epsilon\tilde{G}_{\mu\nu}, \qquad \epsilon = \pm 1, \tag{24}$$

which establishes the main relation of a self-dual theory,

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = G_{\mu\nu}\tilde{G}^{\mu\nu} \tag{25}$$

(which is equivalent to $\mathbf{D}\cdot\mathbf{H} = \mathbf{B}\cdot\mathbf{E}$). Making reference to equation (15), we have the corresponding (anti)self-duality conditions for the constitutive tensors $R_{\mu\nu}^{\rho\sigma}$ (with $S_{\mu\nu} = 0$, $Q_{\mu\nu}^{\rho\sigma\lambda_1\ldots\lambda_n} = 0$),

$$R_{\mu\nu}^{\rho\sigma}\varepsilon_{\rho\sigma\lambda\delta}\eta^{\lambda[\mu}\eta^{\delta\nu]} = 2\epsilon. \tag{26}$$

The further requirement that a solution of (24) also obey the (anti)self-duality condition $F_{\mu\nu} = \epsilon\tilde{F}^{\mu\nu}$ implies that $X = Y$, but we see these two conditions as fundamentally distinct. We can obtain equations of motion in this framework by the method of [21].

More generally, the finite duality transformations are given by

$$F'_{\mu\nu} = aF_{\mu\nu} + b\tilde{G}_{\mu\nu}, \qquad G'_{\mu\nu} = eG_{\mu\nu} + f\tilde{F}_{\mu\nu}, \tag{27}$$

where the determinant $af - be = 1$. If we use the constitutive equations (10) and their Hodge conjugates, we can write

$$F'_{\mu\nu} = U_{\mu\nu}^{\rho\sigma}F_{\rho\sigma}, \qquad G'_{\mu\nu} = V_{\mu\nu}^{\rho\sigma}G_{\rho\sigma}, \tag{28}$$

where the 'dual tensors' $U_{\mu\nu}^{\rho\sigma}$ and $V_{\mu\nu}^{\rho\sigma}$ take the form

$$\begin{aligned} U_{\mu\nu}^{\rho\sigma} &= [a - bcM(C_1,C_2)]\delta_\mu^\rho\delta_\nu^\sigma + \frac{1}{2}bN(C_1,C_2)\varepsilon_{\mu\nu\lambda\delta}\eta^{\lambda\rho}\eta^{\delta\sigma}, \\ V_{\mu\nu}^{\rho\sigma} &= \left[e + \frac{fcM(C_1,C_2)}{N\left(C_1,C_2^2\right) + c^2M\left(C_1,C_2^2\right)}\right]\delta_\mu^\rho\delta_\nu^\sigma + \frac{1}{2}\frac{N(C_1,C_2)}{N(C_1,C_2)^2 + c^2M(C_1,C_2)^2}\varepsilon_{\mu\nu\lambda\delta}\eta^{\lambda\rho}\eta^{\delta\sigma}. \end{aligned} \tag{29}$$

5. Supersymmetric electrodynamics

Now we turn to a geometric superfield formulation of classical electrodynamics and the constitutive equations in superspace. We start by fixing some further notation, following mostly [22]. The $N = 1$ four-dimensional superspace is described by coordinates $z^M = \{x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}\}$, where we introduce the unifying index $M = \{\mu, \alpha, \dot{\alpha}\}$ and $\theta^\alpha, \bar{\theta}^{\dot{\alpha}}$ $(\alpha, \dot{\alpha} = 1, 2)$ are additional complex Grassmann coordinates (two-components Majorana spinors). The transformations of $(N = 1, D = 4)$ supersymmetry are given by

$$\tilde{x}^\mu = x^\mu + \mathrm{i}\lambda^\alpha\sigma_{\alpha\dot{\alpha}}^\mu\bar{\theta}^{\dot{\alpha}} - \mathrm{i}\theta^\alpha\sigma_{\alpha\dot{\alpha}}^\mu\lambda^{\dot{\alpha}}, \qquad \tilde{\theta}^\alpha = \theta^\alpha + \lambda^\alpha, \qquad \tilde{\bar{\theta}}^{\dot{\alpha}} = \bar{\theta}^{\dot{\alpha}} + \lambda^{\dot{\alpha}}, \tag{30}$$

where λ^{α} and $\lambda^{\dot{\alpha}}$ are constant Grassmann spinors. The transformations in equations (30) are generated by supercharges

$$\mathrm{Q}_{\alpha} = -\mathrm{i}\frac{\partial}{\partial\theta^{\alpha}} + \sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^{\mu}}, \qquad \bar{\mathrm{Q}}_{\dot{\alpha}} = \mathrm{i}\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^{\mu}}, \tag{31}$$

with

$$\{\mathrm{Q}_{\alpha}, \bar{\mathrm{Q}}_{\dot{\alpha}}\} = 2\mathrm{i}\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^{\mu}}, \tag{32}$$

where $\sigma^{\mu}_{\alpha\dot{\alpha}} = (I, \vec{\sigma})_{\alpha\dot{\alpha}}$ are the Pauli matrices. Then, $\tilde{z}^{M} = \exp[\mathrm{i}(\lambda^{\alpha}\mathrm{Q}_{\alpha} + \bar{\mathrm{Q}}_{\dot{\alpha}}\lambda^{\dot{\alpha}})]z^{M}$. Defining

$$\mathrm{D}_{\mu} = \frac{\partial}{\partial x^{\mu}}, \qquad \mathrm{D}_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} - \mathrm{i}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^{\mu}}, \qquad \bar{\mathrm{D}}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \mathrm{i}\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^{\mu}}, \tag{33}$$

we have

$$\{\mathrm{D}_{\alpha}, \bar{\mathrm{D}}_{\dot{\alpha}}\} = 2\mathrm{i}\sigma^{\mu}_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^{\mu}}. \tag{34}$$

The (anti)commutators other than those of equations (32) and (34) vanish.

Now a general superfield $\Phi(x, \theta, \bar{\theta})$ written as a function of nilpotent Grassmann variables θ^{α} and $\bar{\theta}^{\dot{\alpha}}$ can be expanded as a finite series with respect to them. Its components are ordinary and spinorial functions, members of the corresponding supermultiplet, that are mixed by the (infinitesimal) supersymmetry transformations

$$\delta\Phi(x, \theta, \bar{\theta}) = \mathrm{i}(\lambda^{\alpha}\mathrm{Q}_{\alpha} + \bar{\mathrm{Q}}_{\dot{\alpha}}\lambda^{\dot{\alpha}})\Phi(x, \theta, \bar{\theta}). \tag{35}$$

For further details, see [22].

The Abelian gauge field $A_{\mu}(x)$ is a component of a gauge superfield vector multiplet $\mathsf{V}(x, \theta, \bar{\theta}) = \mathsf{V}^{+}(x, \theta, \bar{\theta})$, where + denotes super-Hermitian conjugation. The supergauge transformations are given by

$$\tilde{\mathsf{V}}(x, \theta, \bar{\theta}) = \mathsf{V}(x, \theta, \bar{\theta}) + \frac{\mathrm{i}}{2}(\Lambda(x, \theta, \bar{\theta}) - \Lambda^{+}(x, \theta, \bar{\theta})), \tag{36}$$

where $\Lambda(x, \theta, \bar{\theta})$ is a chiral superfield parameter satisfying $\mathrm{D}_{\alpha}\Lambda(x, \theta, \bar{\theta}) = 0$ and $\bar{\mathrm{D}}_{\dot{\alpha}}\Lambda^{+}(x, \theta, \bar{\theta}) = 0$. That is, defining $x^{\mu}_{L,R} = x^{\mu} \pm \mathrm{i}\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}$, Λ and Λ^{+} is each actually a function of two variables, $\Lambda(x, \theta, \bar{\theta}) = \Upsilon(x_{L}, \theta)$, and $\Lambda^{+}(x, \theta, \bar{\theta}) = \Upsilon^{+}(x_{R}, \bar{\theta})$. In the Wess–Zumino gauge, half of the component fields are gauged away using supergauge transformations (36); so that $\mathsf{V}(x, \theta, \bar{\theta})$ takes the form

$$\mathsf{V}_{WZ}(x, \theta, \bar{\theta}) = -\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}A_{\mu}(x) - \mathrm{i}\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\theta^{\alpha}\psi_{\alpha}(x) + \mathrm{i}\theta^{\alpha}\theta_{\alpha}\bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}(x) + \tfrac{1}{2}\theta^{\alpha}\theta_{\alpha}\bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}D(x), \tag{37}$$

where $\psi_{\alpha}(x)$ is a Majorana fermion field (photino) and $D(x)$ is an auxiliary field which vanishes on-shell.

If $\Psi(x, \theta, \bar{\theta})$ is a matter superfield, then the supergauge transformations of Ψ are given by

$$\tilde{\Psi}(x, \theta, \bar{\theta}) = \exp\left[-\frac{ie}{\hbar c}(\Lambda(x, \theta, \bar{\theta}) + \Lambda^{+}(x, \theta, \bar{\theta}))\right]\Psi(x, \theta, \bar{\theta}). \tag{38}$$

Together, equations (36) and (38) provide the full set of supergauge transformations of the Abelian $N = 1$ gauge theory.

Next let us introduce the gauge superpotential superfield $\mathsf{A}_{M}(x, \theta, \bar{\theta})$, a superconnection on the supermanifold, corresponding to the superderivatives D_{M}. Then the covariant superderivatives (covariant with respect to the supergauge transformations (36)–(38)) are

$$\nabla_{M} = \mathrm{D}_{M} + \frac{ie}{\hbar c}\mathsf{A}_{M}(x, \theta, \bar{\theta}). \tag{39}$$

The requirement that the covariant superderivatives acting on the matter superfield transform as superfields themselves via equation (38) leads to

$$\begin{aligned}\tilde{\mathsf{A}}_\mu(x,\theta,\bar\theta) &= \mathsf{A}_\mu(x,\theta,\bar\theta) + \tfrac{1}{2}\mathrm{D}_\mu(\Lambda(x,\theta,\bar\theta) + \Lambda^+(x,\theta,\bar\theta)),\\ \tilde{\mathsf{A}}_\alpha(x,\theta,\bar\theta) &= \mathsf{A}_\alpha(x,\theta,\bar\theta) + \tfrac{1}{2}\mathrm{D}_\alpha\Lambda^+(x,\theta,\bar\theta),\\ \tilde{\bar{\mathsf{A}}}_{\dot\alpha}(x,\theta,\bar\theta) &= \bar{\mathsf{A}}_{\dot\alpha}(x,\theta,\bar\theta) + \tfrac{1}{2}\bar{\mathrm{D}}_{\dot\alpha}\Lambda(x,\theta,\bar\theta).\end{aligned} \tag{40}$$

These relations can be satisfied identically by choosing

$$\mathsf{A}_\mu(x,\theta,\bar\theta) = \left(\mathrm{i}\mathrm{D}_\mu - \tfrac{1}{2}\mathrm{D}_\alpha\sigma_\mu^{\alpha\dot\alpha}\bar{\mathrm{D}}_{\dot\alpha}\right)\mathsf{V}(x,\theta,\bar\theta) \tag{41}$$

and

$$\mathsf{A}_\alpha(x,\theta,\bar\theta) = \mathrm{i}\mathrm{D}_\alpha\mathsf{V}(x,\theta,\bar\theta), \qquad \bar{\mathsf{A}}_{\dot\alpha}(x,\theta,\bar\theta) = \mathrm{i}\bar{\mathrm{D}}_{\dot\alpha}\mathsf{V}(x,\theta,\bar\theta), \tag{42}$$

where $\mathsf{V}(x,\theta,\bar\theta)$ is a prepotential of the $N = 1$ Abelian gauge theory. From equations (41) and (42), the relation between the vector and spinor covariant superderivatives follows

$$\nabla_\mu = \frac{\mathrm{i}}{4}\sigma_\mu^{\alpha\dot\alpha}\{\nabla_\alpha, \bar\nabla_{\dot\alpha}\}, \tag{43}$$

using the relation between covariant superderivatives and superderivatives,

$$\nabla_\alpha = \mathrm{e}^{\frac{e}{\hbar c}\mathsf{V}(x,\theta,\bar\theta)}\mathrm{D}_\alpha\mathrm{e}^{-\frac{e}{\hbar c}\mathsf{V}(x,\theta,\bar\theta)}, \qquad \bar\nabla_{\dot\alpha} = \mathrm{e}^{-\frac{e}{\hbar c}\mathsf{V}(x,\theta,\bar\theta)}\bar{\mathrm{D}}_{\dot\alpha}\mathrm{e}^{\frac{e}{\hbar c}\mathsf{V}(x,\theta,\bar\theta)}. \tag{44}$$

Now one constructs the corresponding (anti)commutators of the covariant superderivatives, introducing the gauge-invariant torsion T_{MN}^{K} and the superfield strength $\mathsf{F}_{MN}(x,\theta,\bar\theta)$, as follows. Let {] denote the 'mixed commutator'—the anticommutator when both entries are odd, the commutator for other combinations. Then, we have

$$\{\nabla_M, \nabla_N] = \mathrm{i}\mathsf{T}_{MN}^{K}\nabla_K + \mathrm{i}\mathsf{F}_{MN}(x,\theta,\bar\theta). \tag{45}$$

From equation (43), it follows that the only nonvanishing components of the torsion are

$$\mathsf{T}_{\alpha\dot\alpha}^{\mu} = 2\sigma_{\alpha\dot\alpha}^{\mu}. \tag{46}$$

Thus, in the $N = 1$ Abelian theory, the torsion constraints are the same as in flat $N = 1$ superspace.

For the superfield strength, it follows from equation (43) that all the components for which both indices are fermionic vanish, i.e.,

$$\mathsf{F}_{\alpha\beta}(x,\theta,\bar\theta) = \mathsf{F}_{\alpha\dot\beta}(x,\theta,\bar\theta) = \mathsf{F}_{\dot\alpha\dot\beta}(x,\theta,\bar\theta) = 0. \tag{47}$$

These constraints are called 'representation preserving', because they allow one to introduce the chiral and anti-chiral superfields which survive in the presence of nonzero gauge coupling. Then the lowest dimensional surviving superfield strengths are mixed spin-vector (odd-valued) superfields

$$\begin{aligned}\mathsf{F}_{\alpha\mu}(x,\theta,\bar\theta) &= -\mathrm{i}[\nabla_\alpha, \nabla_\mu] = \mathrm{D}_\alpha\mathsf{A}_\mu(x,\theta,\bar\theta) - \mathrm{D}_\mu\mathsf{A}_\alpha(x,\theta,\bar\theta),\\ \bar{\mathsf{F}}_{\dot\alpha\mu}(x,\theta,\bar\theta) &= -\mathrm{i}[\bar\nabla_{\dot\alpha}, \nabla_\mu] = \bar{\mathrm{D}}_{\dot\alpha}\mathsf{A}_\mu(x,\theta,\bar\theta) - \mathrm{D}_\mu\bar{\mathsf{A}}_{\dot\alpha}(x,\theta,\bar\theta).\end{aligned} \tag{48}$$

These are the actual super analogues of the field strength $F_{\mu\nu}$ in ordinary electromagnetism.

From equations (41) and (42), the manifest form of the superfield strengths in terms of the prepotential is

$$\begin{aligned}\mathsf{F}_{\alpha\mu}(x,\theta,\bar\theta) &= -\tfrac{1}{2}\mathrm{D}_\alpha\mathrm{D}_\beta\sigma_\mu^{\beta\dot\beta}\bar{\mathrm{D}}_{\dot\beta}\mathsf{V}(x,\theta,\bar\theta),\\ \bar{\mathsf{F}}_{\dot\alpha\mu}(x,\theta,\bar\theta) &= -\tfrac{1}{2}\bar{\mathrm{D}}_{\dot\alpha}\bar{\mathrm{D}}_{\dot\beta}\sigma_\mu^{\dot\beta\beta}\mathrm{D}_\beta\mathsf{V}(x,\theta,\bar\theta).\end{aligned} \tag{49}$$

The superfield strengths can be expressed in terms of a chiral spinor superfield that depends on only one spinorial coordinate, as follows:

$$\mathsf{F}_{\alpha\mu}(x,\bar\theta) = -\mathrm{i}\varepsilon_{\alpha\beta}\sigma_\mu^{\beta\dot\beta}\bar{\mathsf{W}}_{\dot\beta}(x,\bar\theta), \qquad \bar{\mathsf{F}}_{\dot\alpha\mu}(x,\theta) = -\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}\bar\sigma_\mu^{\dot\beta\beta}\mathsf{W}_\beta(x,\theta), \tag{50}$$

where

$$\mathsf{W}_\beta(x,\theta) = \tfrac{1}{2}\bar{\mathrm{D}}_{\dot\alpha}\bar{\mathrm{D}}^{\dot\beta}\mathrm{D}_\beta \mathsf{V}(x,\theta,\bar\theta), \qquad \bar{\mathrm{D}}_{\dot\alpha}\mathsf{W}_\beta(x,\theta) = 0,$$
$$\bar{\mathsf{W}}_{\dot\beta}(x,\bar\theta) = \tfrac{1}{2}\mathrm{D}^\alpha \mathrm{D}_\beta \bar{\mathrm{D}}_{\dot\beta}\mathsf{V}(x,\theta,\bar\theta), \qquad \mathrm{D}_\alpha \bar{\mathsf{W}}_{\dot\beta}(x,\bar\theta) = 0. \tag{51}$$

These chiral superfields satisfy an additional constraint, $\mathrm{D}^\alpha \mathsf{W}_\alpha(x,\theta) = \bar{\mathrm{D}}_{\dot\alpha}\bar{\mathsf{W}}^{\dot\alpha}(x,\bar\theta)$.

Using the component expansion of $\mathsf{V}(x,\theta,\bar\theta)$ in the Wess–Zumino gauge (37), we obtain

$$\mathsf{W}_\alpha(x,\theta) = -\mathrm{i}\psi_\alpha(x) + \left(\varepsilon_{\alpha\gamma} D(x) - \frac{\mathrm{i}}{2}\sigma^\mu_{\alpha\dot\alpha}\varepsilon^{\dot\alpha\dot\beta}\bar\sigma^\nu_{\dot\beta\gamma}F_{\mu\nu}(x)\right)\theta^\gamma - \theta^\beta\theta_\beta\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\psi^{\dot\alpha}(x),$$
$$\bar{\mathsf{W}}_{\dot\alpha}(x,\bar\theta) = \mathrm{i}\bar\psi_{\dot\alpha}(x) + \left(\varepsilon_{\dot\alpha\dot\gamma} D(x) + \frac{\mathrm{i}}{2}\bar\sigma^\mu_{\dot\alpha\alpha}\varepsilon^{\alpha\beta}\sigma^\nu_{\beta\dot\gamma}F_{\mu\nu}(x)\right)\bar\theta^{\dot\gamma} + \bar\theta_{\dot\beta}\bar\theta^{\dot\beta}\bar\sigma^\mu_{\dot\alpha\alpha}\partial_\mu\psi^\alpha(x). \tag{52}$$

From equations (50), it follows that the role the gauge invariants X and Y in equations (9) played for nonlinear electromagnetism is now played by the superfield invariants,

$$\mathsf{X}(x,\theta) = \tfrac{1}{4}\bar{\mathsf{F}}_{\dot\alpha\mu}(x,\theta)\bar{\mathsf{F}}^{\dot\alpha\mu}(x,\theta) = \mathsf{W}^\alpha(x,\theta)\mathsf{W}_\alpha(x,\theta),$$
$$\mathsf{Y}(x,\bar\theta) = \tfrac{1}{4}\mathsf{F}^{\alpha\mu}(x,\bar\theta)\mathsf{F}_{\alpha\mu}(x,\bar\theta) = \bar{\mathsf{W}}_{\dot\alpha}(x,\bar\theta)\bar{\mathsf{W}}^{\dot\alpha}(x,\bar\theta). \tag{53}$$

Applying the component expansions (52), we observe that

$$\mathsf{X}(x,\theta) = \cdots + \theta^\alpha\theta_\alpha(X - \mathrm{i}Y), \qquad \mathsf{Y}(x,\bar\theta) = \cdots + \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha}(X + \mathrm{i}Y), \tag{54}$$

which, after integration over the Grassmann coordinates, will give the correct contributions of X and Y to the Lagrangian.

6. Supersymmetric constitutive equations

Finally, we are ready to consider the $N = 1$ supersymmetric theory in 'supermedia', with the goal of obtaining the 'super' version of the constitutive equations (15).

By analogy with the case of nonlinear electromagnetism, we introduce the superfield strengths in media $\mathsf{G}^{\alpha\mu}(x,\bar\theta)$ and $\bar{\mathsf{G}}_{\dot\alpha\mu}(x,\theta)$, without expressing G or $\bar{\mathsf{G}}$ via any prepotential. Rather we assume them to have a representation similar to that of equations (50),

$$\mathsf{G}_{\alpha\mu}(x,\bar\theta) = -\mathrm{i}\varepsilon_{\alpha\beta}\sigma^{\beta\dot\beta}_\mu\bar{\mathsf{W}}^G_{\dot\beta}(x,\bar\theta), \qquad \bar{\mathsf{G}}_{\dot\alpha\mu}(x,\theta) = -\mathrm{i}\varepsilon_{\dot\alpha\dot\beta}\bar\sigma^{\dot\beta\beta}_\mu\mathsf{W}^G_\beta(x,\theta). \tag{55}$$

Here the chiral superfields in media $\mathsf{W}^G_\beta(x,\theta)$ and $\bar{\mathsf{W}}^G_{\dot\beta}(x,\bar\theta)$ likewise are not expressed through a prepotential superfield, but nevertheless have a component expansion similar to that of equations (52),

$$\mathsf{W}^G_\alpha(x,\theta) = -\mathrm{i}\psi^G_\alpha(x) + \left(\varepsilon_{\alpha\gamma} D^G(x) - \frac{\mathrm{i}}{2}\sigma^\mu_{\alpha\dot\alpha}\varepsilon^{\dot\alpha\dot\beta}\bar\sigma^\nu_{\dot\beta\gamma}G_{\mu\nu}(x)\right)\theta^\gamma - \theta^\beta\theta_\beta\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\bar\psi^{G\dot\alpha}(x),$$
$$\bar{\mathsf{W}}^G_{\dot\alpha}(x,\bar\theta) = \mathrm{i}\bar\psi^G_{\dot\alpha}(x) + \left(\varepsilon_{\dot\alpha\dot\gamma} D^G(x) + \frac{\mathrm{i}}{2}\bar\sigma^\mu_{\dot\alpha\alpha}\varepsilon^{\alpha\beta}\sigma^\nu_{\beta\dot\gamma}G_{\mu\nu}(x)\right)\bar\theta^{\dot\gamma} + \bar\theta_{\dot\beta}\bar\theta^{\dot\beta}\bar\sigma^\mu_{\dot\alpha\alpha}\partial_\mu\psi^{G\alpha}(x), \tag{56}$$

where $G_{\mu\nu}(x)$ satisfies equation (4).

In writing the analogue of the constitutive equations (15), it is obviously much more convenient to deal with variables having only one spinor index (i.e., the chiral superfields) than with those having spin-vector indices (i.e., the superfield strengths). Furthermore, these are related to each other by constant matrices, through equations (50) and (55). Therefore, we formulate the superconstitutive equations in terms of the chiral superfields, as follows:

$$\mathsf{W}^G_\alpha = \mathsf{S}_\alpha + \mathsf{R}^\beta_\alpha\mathsf{W}_\beta + \mathsf{Q}^{\beta_1\gamma}_\alpha\mathrm{D}_{\beta_1}\mathsf{W}_\gamma + \mathsf{Q}^{\beta_1\beta_2\gamma}_\alpha\mathrm{D}_{\beta_1}\mathrm{D}_{\beta_2}\mathsf{W}_\gamma + \cdots + \mathsf{Q}^{\beta_1\cdots\beta_n\gamma}_\alpha\mathrm{D}_{\beta_1}\cdots\mathrm{D}_{\beta_n}\mathsf{W}_\gamma,$$
$$\bar{\mathsf{W}}^G_{\dot\alpha} = \bar{\mathsf{S}}_{\dot\alpha} + \bar{\mathsf{R}}^{\dot\beta}_{\dot\alpha}\bar{\mathsf{W}}_{\dot\beta} + \bar{\mathsf{Q}}^{\dot\beta_1\dot\gamma}_{\dot\alpha}\bar{\mathrm{D}}_{\dot\beta_1}\bar{\mathsf{W}}_{\dot\gamma} + \bar{\mathsf{Q}}^{\dot\beta_1\dot\beta_2\dot\gamma}_{\dot\alpha}\bar{\mathrm{D}}_{\dot\beta_1}\bar{D}_{\dot\beta_2}\bar{\mathsf{W}}_{\dot\gamma} + \cdots + \bar{\mathsf{Q}}^{\dot\beta_1\cdots\dot\beta_n\dot\gamma}_{\dot\alpha}\mathrm{D}_{\dot\beta_1}\cdots\mathrm{D}_{\dot\beta_n}\mathsf{W}_{\dot\gamma}. \tag{57}$$

Imposing supergauge invariance requires that the constitutive supertensors S_α, R^β_α, $\mathsf{Q}^{\beta_1\gamma}_\alpha$, etc, depend only on (x,θ), the gauge superinvariant $\mathsf{X}(x,\theta)$ that was given by the first of equations

(53) and the superderivatives of $\mathsf{X}(x,\theta)$ with respect to (x,θ). In turn, the constitutive supertensors $\bar{\mathsf{S}}_{\dot\alpha}$, $\bar{\mathsf{R}}_{\dot\alpha}^{\dot\beta}$, $\bar{\mathsf{Q}}_{\dot\alpha}^{\dot\beta_1\dot\gamma}$, etc, depend only on $(x,\bar\theta)$, the gauge superinvariant $\mathsf{Y}(x,\bar\theta)$ given by the second of equations (53) and the superderivatives of $\mathsf{Y}(x,\bar\theta)$ with respect to $(x,\bar\theta)$. In analogy with the case of nonlinear electromagnetism, we have S depending only on (x,θ) and $\bar{\mathsf{S}}$ depending only on $(x,\bar\theta)$, while R can depend on $\mathsf{X}(x,\theta)$ as well as (x,θ) and $\bar{\mathsf{R}}$ can depend on $\mathsf{Y}(x,\bar\theta)$ as well as $(x,\bar\theta)$. Finally, the arguments of $\mathsf{Q}_\alpha^{\beta_1\cdots\beta_n\gamma}$ and $\bar{\mathsf{Q}}_{\dot\alpha}^{\dot\beta_1\cdots\dot\beta_n\dot\gamma}$ can include up to nth-order superderivatives of the respective supergauge invariants.

As before, one can also incorporate nonlocal effects into the formalism. Instead of demanding that R, $\bar{\mathsf{R}}$, Q and $\bar{\mathsf{Q}}$ depend on the superfield strengths of equations (48) only as local functions of their values at $(x,\theta,\bar\theta)$, one can admit constitutive tensors that depend on the superfield strengths and their superderivatives at distinct points in the supermanifold, or evaluated over extended regions of the supermanifold. This is important in the context of potential applications to superstrings and to branes.

Using expansions (52) and (56), it is straightforward to write down the superfield constitutive equations (57) in components.

7. Conclusion

We have proposed a way to take an extremely general approach to nonlinear classical electrodynamics and supersymmetric electrodynamics, for the purpose either of describing fields and superfields in general kinds of media, or of exploring their behavior *in vacua* with general properties. The framework formally takes into account media of various types, includes non-Lagrangian as well as Lagrangian theories, and accommodates the description of nonlocal effects. This is accomplished through generalized constitutive equations (and constitutive tensors), and their further generalization to include superfields. We expect future research directions to include the detailed development of new examples within this framework.

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