UNIFORM ASYMPTOTIC EXPANSION OF THE VOLTAGE POTENTIAL IN THE PRESENCE OF THIN INHOMOGENEITIES WITH ARBITRARY CONDUCTIVITY

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Contents

1. Introduction

Asymptotic expansions of the voltage potential in terms of the "radius" ε of a diametrically small (or several diametrically small) material inhomogeneity(ies) are by now quite well known [4, 11]. Let ω_ε denote the inhomogeneity and let $0 < a_\varepsilon < \infty$ denote the conductivity inside the inhomogeneity. The potential u_ε converges (in the far field) to a limit "background" potential u_0, which is independent of the conductivity a_ε; this convergence (and for that matter the approximation rate of any finite number of terms in the asymtotic expansion) is uniform with respect to a_ε [21].

As was shown in [10], the existence of the first two terms of the asymptotic expansion carries over to a situation much more general than that of a finite collection of diametrically small inhomogeneities, *namely that of an arbitrary set* ω_ϵ whose Lebesgue measure converges to zero. The convergence statement there is modulo the extraction of a subsequence, and so it is really a compactness result. Furthermore the convergence is not generally uniform with respect to the inhomogeneity conductivity a_ε.

Thin inhomogeneities, whose limit set is a smooth, codimension 1 manifold, are indeed examples of inhomogeneities for which the convergence to the background potential u_0 or the standard expansion cannot be valid uniformly in a_ε. Indeed, by taking a_ε close to 0 or to ∞ one obtains either a nearly homogeneous Neumann condition or nearly constant Dirichlet condition at the boundary $\partial\omega_\varepsilon$ of the inhomogeneity. This boundary, however, does not shrink to a single point as $\varepsilon \to 0$, as is the case when the inhomogeneity is of small radius, but rather it "converges" to a codimension 1 manifold, σ, which has positive capacity. Neither the problem with homogeneous Neumann boundary condition nor the one with constant Dirichlet condition on σ has u_0 as its solution; consequently, the convergence of u_ε towards u_0 cannot take place uniformly in a_ε.

The purpose of this paper is to find a "simple" replacement for u_0, say u_ε^0, with the properties that:

(1) u_ε^0 may be (simply) calculated from the limiting domain $\Omega \setminus \sigma$, the boundary data on $\partial\Omega$, and the right hand side.
(2) u_ε^0 depends on ε and a_ε through its boundary conditions on σ,
(3) $u_\varepsilon - u_\varepsilon^0$ converges to 0 uniformly in a_ε, as ε tends to 0.

Such a convergence result is useful for theoretical as well as for practical purposes:

- For theoretical purposes, it easily allows one to identify the (ε independent) limit of the potential u_ε, when the behavior of a_ε is more precisely known.
- For numerical purposes, it allows to trade a problem posed on a very thin domain, which may be difficult to simulate due to the requirements of a very small mesh size, for a problem posed on a fixed domain with a single additional interphase boundary condition; see the numerical experiments in [22].

We also briefly discuss the derivation of the next term in a "uniform" asymptotic expansion of u_ε. From a practical point of view, knowledge of the first two terms would give a very effective tool for the determination of ω_ϵ from the knowledge of far field data of u_ε, in a fashion that would work independently of the conductivity a_ε; see [3] for the description of such a reconstruction algorithm in the context where the conductivity inside the inhomogeneity is constant and does not depend on ε: $a_\varepsilon = a$, where $0 < a < \infty$.

There are other studies of asymptotic expansions, specifically related to thin inhomogeneities. In [7], the authors establish a first-order asymptotic expansion of u_ε when the conductivity coefficient a_ε is independent of ε; they consider both the case of a *closed*, and an *open* curve σ as far as the limiting set of the inhomogeneity is concerned. They rely on very sharp regularity estimates for u_ε near the boundary of the inhomogeneity; this analysis is carried over to the Helmholtz equation in [6]. In [5], a (closed) thin conductivity inhomogeneity is considered and analyzed in the case where the coefficient a_ε degenerates to 0 as $\varepsilon \to 0$, using Γ-convergence techniques. This situation is also investigated in [1] in the context of the minimization of non linear energy functionals, and in [9] in a situation where the boundary of the inhomogeneity is oscillating. In [22], the resistive limit $a_\varepsilon/\varepsilon \to 0$ is considered, a case of particular relevance as an approximation to the behavior of the membrane of a biological cell. In this very particular situation, the authors establish the existence of a limiting potential. The analysis is very different from the one presented here and relies on matched asymptotic expansions in all three subdomains: the interior region, the membrane, and the exterior region. It seems difficult to extend such an analysis to the general case studied here.

The technique we use here to verify the uniform approximation property of u_ε^0 estimates the norm distance between u_ε and u_ε^0 in terms of the gap between the corresponding energies, using both the primal and dual formulation. This technique goes back to at least [20]. It has the additional nice feature that it only relies on uniform regularity estimates for the approximate solution u_ε^0, *not* for u_ε.

2. Preliminaries and main notations

2.1. Setting of the problem.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and σ be a *closed* $\mathcal{C}^{2,\alpha}$ curve, included in Ω and lying at positive distance from $\partial\Omega$. The closed curve σ divides Ω into two subdomains Ω^- and Ω^+. Ω^- (resp. Ω^+) denotes the subdomain interior (resp. exterior) to the curve σ, and unless otherwise specified, n stands for the normal vector to σ, pointing outward from Ω^-. For any subset $V \subset \Omega$ we denote $V^\pm := V \cap \Omega^\pm$ (remark that, with this notation, $\partial V^\pm \neq \partial(V^\pm)$). If u is any function defined on Ω, we denote by $u^\pm$ its restriction to $\Omega^\pm$. If u^+ and u^- have traces $u^+|_\sigma$ and $u^-|_\sigma$ on σ, we denote by $[u] := u^+|_\sigma - u^-|_\sigma$ the *jump* of u across σ. Moreover, when u is sufficiently regular, we denote by

$$\frac{\partial u^\pm}{\partial n}(x) = \lim_{t\to 0} \nabla u(x \pm tn(x)) \cdot n(x)$$

the *exterior* and *interior normal components* of ∇u at $x \in \sigma$. The associated *normal jump* across σ is denoted by $\left[\frac{\partial u}{\partial n}\right]$.

Except for the thin inhomogeneity the domain Ω is occupied by a conductive material, with conductivity 1. The thin inhomogeneity (with mid-surface σ, and width 2ε; see Figure 1) is

$$\omega_\varepsilon := \{x + tn(x), x \in \sigma, t \in (-\varepsilon, \varepsilon)\} \ ;$$

and it has conductivity a_ε. The conductivity γ_ε in the entire domain is therefore given by

$$\gamma_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega \setminus \overline{\omega_\varepsilon}, \\ a_\varepsilon & \text{if } x \in \omega_\varepsilon. \end{cases} \tag{2.1}$$

We assume that $a_\varepsilon \in (0, \infty)$ is a scalar constant, but this constant may change with ε. In particular, a_ε may go to 0 or ∞ as $\varepsilon \to 0$.

A potential $\varphi \in H^{1/2}(\partial\Omega)$ is applied to $\partial\Omega$, and Ω has a charge distribution $f \in L^2(\Omega)$. The electric potential u_ε in Ω is the solution to

$$\begin{cases} -\mathrm{div}(\gamma_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega \ , \\ u_\varepsilon = \varphi & \text{on } \partial\Omega \ . \end{cases} \tag{2.2}$$

It is well-known that under the above hypotheses, the system (2.2) has a unique solution $u_\varepsilon \in H^1(\Omega)$. The following notations will prove useful

- For any open subset $U \subset \mathbb{R}^2$, $L^2_0(U)$ denotes the subspace of $L^2(U)$ composed of functions u such that $\int_U u\,dx = 0$. There is a natural mapping $L^2(U) \ni u \mapsto \left(u - \frac{1}{|U|}\int_U u\,dx\right) \in L^2_0(U)$. By a small abuse of notation, for any function $u \in L^2(U)$, we shall write:

$$||u||_{L^2_0(U)} = \left|\left| u - \frac{1}{|U|}\int_U u\,dx \right|\right|_{L^2(U)} .$$

- For sufficiently small $\delta > 0$, $\mathcal{F}_\delta$ denotes the following closed subspace of $L^2(\Omega)$:

$$\mathcal{F}_\delta = \left\{ f \in L^2(\Omega), \ \mathrm{supp}(f) \subset \Omega \setminus \omega_\delta \ , \ \int_{\Omega^-} f\,dx = 0 \right\} \ .$$

  This Hilbert space may also be identified as $\mathcal{F}_\delta = L^2(\Omega^+ \setminus \overline{\omega_\delta}) \times L^2_0(\Omega^- \setminus \overline{\omega_\delta})$.

The goal of this paper is to understand the uniform asymptotic behavior of the potential u_ε, as the width 2ε of the thin inhomogeneity goes to 0, – uniform, that is, with respect to the conductivity a_ε inside the inclusion. More precisely, we will derive an approximate problem posed on the fixed domain $\Omega \setminus \sigma$ (with boundary conditions on σ, depending on ε and a_ε), whose solution u^0_ε is uniformly close to u_ε as $\varepsilon \to 0$, *independently of the behavior of the sequence* a_ε.

Remark 1. Let us briefly comment on the hypotheses of the above model and the possible generalizations of our results.

- We assume that the background conductivity γ_0, that is, the conductivity outside the inhomogeneity, is equal to 1. This is only a matter of convenience, and it would be straightforward to replace it by a smooth, variable conductivity distribution $\gamma_0(x)$, with $0 < c_0 < \gamma_0(x) < c_1$.
- We consider the case of only one internal inhomogeneity, but our analysis immediately carries over to the case of finitely many well separated, internal inhomogeneities.

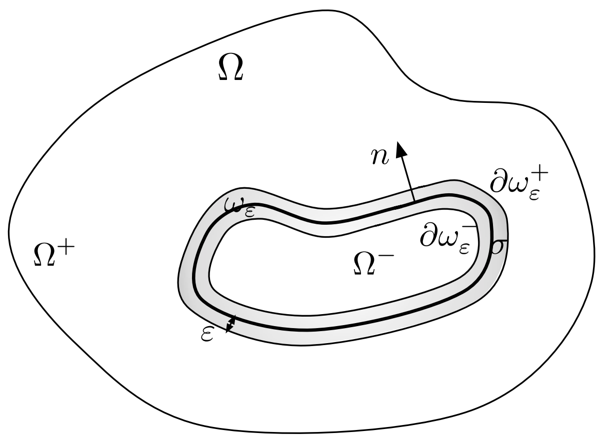

FIGURE 1. Setting of the thin inhomogeneity problem.

- We have chosen for simplicity to restrict our analysis to the case of two space dimensions, but with some additional work it carries over to thin inhomogeneities in higher dimension as well; the curve σ then gets replaced by a closed, smooth (codimension 1) hypersurface.
- We also assume that a_ε is constant inside ω_ε. As we will show, the limit behavior of u_ε is completely different depending on whether a_ε degenerates to 0 or to ∞ as $\varepsilon \to 0$ (and at what rate). We do not currently know how to (rigorously) generalize the analysis presented here to the situation where a_ε is variable inside ω_ε and degenerates to 0 on some parts of ω_ε and to ∞ on other parts. A somewhat related problem would be to consider the case of a simple *open* curve σ.
- Our present results pertain to the conductivity problem (zero frequency). It should be interesting to study the the same geometric setting in the context of the Helmholtz problem. We expect the generalization to a single fixed frequency to be rather straightforward, a more challenging problem would be to obtain results that are also uniform over a broad range of frequencies.

2.2. **Some facts about distances and projections.**

In this subsection, we present some material about distances and projections, as well as a version of the coarea formula that will prove very useful when calculating integrals on a set of the form ω_ε. The context is the same as in Section 2.1: σ is a closed curve of class $\mathcal{C}^{2,\alpha}$ defining two subdomains Ω^-, Ω^+ of a larger (smooth) bounded domain $\Omega \subset \mathbb{R}^2$. For any $x \in \Omega$, let $d(x,\sigma) := \min_{y\in\sigma} d(x,y)$ be the *Euclidean distance* from x to σ. The *signed distance function* d_{Ω^-} to the interior subdomain Ω^- is defined as:

$$\forall x \in \Omega,\ \ d_{\Omega^-}(x) = \left\{ \begin{array}{cl} -d(x,\sigma) & \text{if } x \in \Omega^- \\ 0 & \text{if } x \in \sigma \\ d(x,\sigma) & \text{if } x \in \Omega^+ \end{array} \right. .$$

It is well-known that the *projection mapping*

$$p_\sigma : x \mapsto \text{ the unique } y \in \sigma \text{ s.t. } d(x,y) = d(x,\sigma)$$

is well-defined on a sufficiently small tubular neighborhood ω_δ of σ; see e.g. [18], prop 5.4.14; the maximum thickness of such a neighborhood depends on the curvature of σ. In the remainder of this note, we shall assume that

$$\omega_1 \subset \Omega, \text{ and } p_\sigma \text{ is well-defined on } \omega_1 \ . \tag{2.3}$$

This hypothesis is only a matter of scaling, and all the analysis adapts mutatis mutandis to the general case. Properties (2.3) allow us to define an extension of the normal vector field $n : \sigma \to \mathbb{S}^1$ to the whole ω_1 as: $n(x) := n(p_\sigma(x))$; other quantities which are intrinsically defined on σ can be extended likewise. Thus, for any point $x \in \omega_1$, we shall denote by $\kappa(x)$ the curvature of σ at the point $p_\sigma(x)$.

The derivatives of d_{Ω^-} and p_σ are (see e.g. [2]):

$$
\nabla d_{\Omega^-}(x) = n(x),\ \ \nabla^2(d_{\Omega^-})(x) = \begin{pmatrix} \frac{\kappa(x)}{1+\kappa(x)d_{\Omega^-}(x)} & 0 \\ 0 & 0 \end{pmatrix},
$$

(2.4)

$$
\nabla p_\sigma(x) = \begin{pmatrix} \frac{1}{1+\kappa(x)d_{\Omega^-}(x)} & 0 \\ 0 & 0 \end{pmatrix},
$$

where the above matrix identities are expressed in the orthonormal basis $(\tau(x), n(x))$ of $\mathbb{R}^2$. Here τ denotes the 90 degree clockwise rotate of $n(x)$, in other words the extension of a smooth tangent field on σ, and $\nabla^2 u$ stands for the Hessian matrix of a function u.

These observations, together with the coarea formula [12] yield:

Proposition 1. *Let $g \in L^1(\Omega)$. Then,*

$$
\int_{\omega_\varepsilon} g\, dx = \int_\sigma \int_{p_\sigma^{-1}(y)\cap\omega_\varepsilon} g(z)(1+\kappa(y)d_{\Omega^-}(z))\, d\mu^1(z)\, ds(y)\ ,\ \varepsilon \le 1\ ,
$$

where $d\mu^1$ is the one-dimensional Hausdorff measure on the pre-images $p_\sigma^{-1}(y)\cap\omega_\varepsilon$, and $ds(y)$ is the Hausdorff measure on the codimension 1 subset σ.

Remark 2. This formula may seem ill-defined at first glance, since g being only integrable over Ω, it is a priori not defined on all the one-dimensional sets $p_\sigma^{-1}(y)$, $y \in \sigma$. However, it turns out to be defined on almost every such set (see [15], §3.4.3, Theorem 2), and that is sufficient.

As explained above, the normal vector field n and the tangent vector field τ on σ can be extended as orthonormal vector fields to a tubular neighborhood of σ. The coordinates $(\xi\cdot\tau, \xi\cdot n)$ of a vector ξ in this basis will be denoted (ξ_τ, ξ_n).

It is convenient to express the two-dimensional divergence operator in the local basis (τ, n).

Lemma 2. *Let ξ be a vector field of class $\mathcal{C}^1$ defined on a tubular neighborhood of σ. Then,*

$$
\operatorname{div}(\xi) = \frac{\partial}{\partial\tau}(\xi_\tau) + \frac{\partial}{\partial n}(\xi_n) + \frac{\kappa}{1+\kappa d_{\Omega^-}}\xi_n\ .
$$

Proof. We calculate

$$
\begin{array}{rcl}
\frac{\partial}{\partial\tau}(\xi_\tau) & = & \nabla(\xi\cdot\tau)\cdot\tau \\
 & = & \left(\nabla\xi^T\tau + \nabla\tau^T\xi\right)\cdot\tau \\
 & = & (\nabla\xi\,\tau)\cdot\tau + (\nabla\tau\,\tau)\cdot\xi\ ,
\end{array}
$$

and similarly, $\frac{\partial}{\partial n}(\xi_n) = (\nabla\xi\, n)\cdot n$. For the latter identity, we relied on the fact that $\nabla n\, n = \nabla n^T n = 0$ (which follows, e.g. from (2.4)). Since $\operatorname{div}(\xi) = \operatorname{tr}(\nabla\xi)$ can be evaluated in any orthonormal basis,

$$
\begin{array}{rcl}
\operatorname{div}(\xi) & = & (\nabla\xi\,\tau)\cdot\tau + (\nabla\xi\, n)\cdot n \\
 & = & \frac{\partial}{\partial\tau}(\xi\cdot\tau) + \frac{\partial}{\partial n}(\xi\cdot n) - (\nabla\tau\,\tau)\cdot\xi\ .
\end{array} \tag{2.5}
$$

By differentiation of $\tau\cdot\tau = 1$, one obtains $(\nabla\tau\,\tau)\cdot\tau = (\nabla\tau^T\,\tau)\cdot\tau = 0$. Similarly, by differentiation of $n\cdot\tau = 0$, and use of (2.4), one obtains

$$
(\nabla\tau\,\tau)\cdot n = (\nabla\tau^T\, n)\cdot\tau = -(\nabla n^T\,\tau)\cdot\tau = -\frac{\kappa}{1+\kappa d_{\Omega^-}}\ .
$$

The desired result follows from a combination of these two observations with (2.5). □

Remark 3. Arguments similar to those of the last proof reveal that

$$
\frac{\partial^2 g}{\partial\tau\partial n} = \frac{\partial^2 g}{\partial n\partial\tau} + \frac{\kappa}{1+\kappa d_{\Omega^-}}\frac{\partial g}{\partial\tau}\ ,
$$

for any function g of class $\mathcal{C}^2$ on a neighborhood of σ. Thus, for any such function, Lemma 2 allows us to conclude that the vector field $-\frac{\partial g}{\partial n}\tau + \frac{\partial g}{\partial\tau}n$ is divergence-free.

3. A general argument to estimate the difference between energy minimizers

In this section we introduce our main tool for assessing the convergence of minimizers of variational problems, defined on possibly varying domains. We also present the special considerations required to apply this tool to inhomogeneous Dirichlet problems, which are of most relevance to the present studies.

3.1. **An energy lemma.**

The following lemma may be viewed as a generalization of a rather standard fact about the difference between minimizers of quadratic functionals.

Lemma 3. *Let $V_\varepsilon, W_\varepsilon$ be two families of Hilbert spaces, and let H be another Hilbert space, which continuosly contains all the V_ε and W_ε. Consider also $a_\varepsilon : V_\varepsilon \times V_\varepsilon \to \mathbb{R}$ and $b_\varepsilon : W_\varepsilon \times W_\varepsilon \to \mathbb{R}$ two families of symmetric bilinear forms that are continuous and coercive. For any $\ell \in H'$, define the energy functionals E_ε and F_ε (whose dependence on ℓ is omitted) by*

$$\forall v \in V_\varepsilon,\;\; E_\varepsilon(v) = \frac{1}{2} a_\varepsilon(v,v) - \ell(v)\ ,$$

$$\forall w \in W_\varepsilon,\;\; F_\varepsilon(w) = \frac{1}{2} b_\varepsilon(w,w) - \ell(w)\ .$$

E_ε and F_ε admit unique minimizers $v_\varepsilon^\ell \in V_\varepsilon$, $w_\varepsilon^\ell \in W_\varepsilon$, due to the usual Lax-Milgram theory. The gap between v_ε^ℓ and w_ε^ℓ can be controlled in terms of the gap between the corresponding energies as follows

$$\sup_{||\ell||_{H'} \le 1} ||v_\varepsilon^\ell - w_\varepsilon^\ell||_H \le 4 \sup_{||\ell||_{H'} \le 1} |E_\varepsilon(v_\varepsilon^\ell) - F_\varepsilon(w_\varepsilon^\ell)|\ . \tag{3.1}$$

Proof. Let ℓ be an arbitrary linear form in H'. By the standard Lax-Milgram theory, we know that v_ε^ℓ and w_ε^ℓ are characterized by the fact that

$$\forall v \in V_\varepsilon,\;\; a_\varepsilon(v_\varepsilon^\ell, v) = \ell(v),\;\; \forall w \in W_\varepsilon,\;\; b_\varepsilon(w_\varepsilon^\ell, w) = \ell(w)\ . \tag{3.2}$$

This in particular implies that

$$E_\varepsilon(v_\varepsilon^\ell) = -\frac{1}{2}\ell(v_\varepsilon^\ell),\;\; F_\varepsilon(w_\varepsilon^\ell) = -\frac{1}{2}\ell(w_\varepsilon^\ell)\ . \tag{3.3}$$

Consequently, for any $\ell \in H'$, one has

$$|\ell(v_\varepsilon^\ell - w_\varepsilon^\ell)| = 2|E_\varepsilon(v_\varepsilon^\ell) - F_\varepsilon(w_\varepsilon^\ell)|\ . \tag{3.4}$$

Now, define the bilinear form $q : H' \times H' \to \mathbb{R}$ by

$$\forall \ell_1, \ell_2 \in H',\;\; q(\ell_1, \ell_2) = \ell_1(v_\varepsilon^{\ell_2} - w_\varepsilon^{\ell_2})\ .$$

Using (3.2) we obtain that

$$q(\ell_1, \ell_2) = a_\varepsilon(v_\varepsilon^{\ell_1}, v_\varepsilon^{\ell_2}) - b_\varepsilon(w_\varepsilon^{\ell_1}, w_\varepsilon^{\ell_2})\ ,$$

from which it is clear that q is symmetric. We are thus in position to use the polarization identity for q:

$$q(\ell_1, \ell_2) = \frac{1}{4}\left(q(\ell_1 + \ell_2, \ell_1 + \ell_2) - q(\ell_1 - \ell_2, \ell_1 - \ell_2)\right)\ ,$$

to conclude that

$$\sup_{\substack{||\ell_1||_{H'} \le 1, \\ ||\ell_2||_{H'} \le 1}} |q(\ell_1, \ell_2)| \le 2 \sup_{||\ell||_{H'} \le 1} |q(\ell, \ell)|\ .$$

In combination with (3.4) this last inequality yields

$$\sup_{||\ell_2||_{H'} \le 1} \sup_{||\ell_1||_{H'} \le 1} |\ell_1(v_\varepsilon^{\ell_2} - w_\varepsilon^{\ell_2})| \le 4 \sup_{||\ell||_{H'} \le 1} |E_\varepsilon(v_\varepsilon^\ell) - F_\varepsilon(w_\varepsilon^\ell)|\ ,$$

which immediately gives

$$\sup_{||\ell||_{H'} \le 1} ||v_\varepsilon^\ell - w_\varepsilon^\ell||_H \le 4 \sup_{||\ell||_{H'} \le 1} |E_\varepsilon(v_\varepsilon^\ell) - F_\varepsilon(w_\varepsilon^\ell)|\ .$$

This completes the proof of the lemma. □

Remark 4. Suppose the spaces V_ε and W_ε are only "weakly" contained in H, in the sense that there exist linear continuous mappings $P_\varepsilon : V_\varepsilon \to H$, and $Q_\varepsilon : W_\varepsilon \to H$ through which they may be identified with subspaces of H (we might even allow for the possibility that these mappings are not injective). Change the quadratic functionals slightly to accommodate for these mappings:

$$\forall v \in V_\varepsilon, \;\; E_\varepsilon(v) = \frac{1}{2} a_\varepsilon(v,v) - P_\varepsilon^* \ell(v) \;,$$

$$\forall w \in W_\varepsilon, \;\; F_\varepsilon(w) = \frac{1}{2} b_\varepsilon(w,w) - Q_\varepsilon^* \ell(w) \;,$$

with P_ε^* and Q_ε^* being the adjoints of P_ε and Q_ε, respectively. The equivalent of Lemma 3 now asserts that

$$\sup_{||\ell||_{H'} \leq 1} ||P_\varepsilon v_\varepsilon^\ell - Q_\varepsilon w_\varepsilon^\ell||_H \leq 4 \sup_{||\ell||_{H'} \leq 1} |E_\varepsilon(v_\varepsilon^\ell) - F_\varepsilon(w_\varepsilon^\ell)| \;. \tag{3.5}$$

Remark 5. Some comments are in order about the meaning of Lemma 3, and the way we intend to use it. Our purpose is to prove an estimate for the difference $(v_\varepsilon - w_\varepsilon)$ between the minimizers $v_\varepsilon \in V_\varepsilon$, and $w_\varepsilon \in W_\varepsilon$ of two energy functionals E_ε, and F_ε. In the applications ahead, v_ε and w_ε are solutions to some elliptic PDEs whose coefficients, or domains of definition, depend on ε. Of course, such an estimate can only be realized in terms of the norm $||\cdot||_H$ of a "larger" space H, which "contains" all the $V_\varepsilon, W_\varepsilon$. Lemma 3 states that such an estimate can be obtained in terms of the difference between the corresponding minimized energies - a quantity which should in principle be simpler to compute. To be more precise such an estimate may be obtained provided we are able to calculate the energy differences in a slightly more general context, namely in the case when a (common) additional, and rather arbitrary linear term $\ell \in H'$ has been added to the energies E_ε, F_ε. Somehow, this additional linear term plays the role of a "sentinel", and is meant to "observe" functions in V_ε and W_ε, or at least the features of these that are expressed in the space H through which they are "seen".

3.2. **Extension of Lemma 3 to the case of inhomogeneous Dirichlet boundary conditions.**

The purpose of this subsection is to describe the adjustments needed to the framework of the previous lemma when dealing with inhomogeneous Dirichlet boundary conditions.

3.2.1. *A short remark about minimization of functionals over sets of functions satisfying an inhomogeneous Dirichlet boundary condition.*

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and V be a Hilbert space of functions over Ω, such that the trace mapping

$$V \ni u \;\mapsto\; u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$$

is well-defined, continuous, and has a continuous right inverse (e.g. $V = H^1(\Omega)$). Let $V_0 = \{u \in V,\; v = 0 \text{ on } \partial\Omega\}$ be the associated homogeneous space. Let $a : V \times V \to \mathbb{R}$ be a continuous and coercive bilinear form over V, and $\ell : V \to \mathbb{R}$ be a continuous linear form over V. We are interested in the following minimization problem:

$$\min_{\substack{v \in V \\ v = \varphi \text{ on } \partial\Omega}} E(v), \;\; E(v) := \frac{1}{2} a(v,v) - \ell(v) \;, \tag{3.6}$$

the solution, u, of which solves the variational problem

$$\left\{ \begin{array}{ll} a(u,v) = \ell(v) & \text{for all } v \in V_0 \\ u = \varphi & \text{on } \partial\Omega \end{array} \right. . \tag{3.7}$$

As is well known, (3.7) (and thus the minimization problem (3.6)) has a unique solution $u = \widehat{u} + u_\varphi \in V$, where $u_\varphi \in V$ is a right inverse of φ for the trace operator (i.e. $u_\varphi = \varphi$ on $\partial\Omega$), and $\widehat{u} \in V_0$ is defined by:

$$\forall v \in V_0, \;\; a(\widehat{u}, v) = \ell(v) - a(u_\varphi, v) \;. \tag{3.8}$$

The existence and uniqueness of $\widehat{u}$ are straightforward consequences of the Lax-Milgram Theorem. From a slightly different point of view, $\widehat{u}$ can also be regarded as the unique solution to the following minimization problem:

$$F(\widehat{u}) = \min_{v \in V_0} F(v), \;\; F(v) := \frac{1}{2} a(v,v) - \ell(v) + a(u_\varphi, v) \;.$$

By using (3.8), we actually have

$$F(\widehat{u}) = -\frac{1}{2}a(\widehat{u},\widehat{u}) = -\frac{1}{2}\ell(\widehat{u}) + \frac{1}{2}a(u_\varphi,\widehat{u})\ . \tag{3.9}$$

We return to (3.6). As a straightforward consequence of the definition of u_φ,

$$\min_{\substack{v\in V\\ v=\varphi \text{ on } \partial\Omega}} E(v) = \min_{v\in V_0} E_0(v),\ \text{ where } E_0(v) := \frac{1}{2}a(v,v) - \ell(v) + a(u_\varphi,v) + \frac{1}{2}a(u_\varphi,u_\varphi) - \ell(u_\varphi).$$

Note that the quantity $E_0(v)$ differs from $F(v)$ by a term which is independent of v. Owing to the previous considerations, E_0 has a unique minimum point $v=\widehat{u}$, and

$$\begin{aligned}\min_{\substack{v\in V\\ v=\varphi \text{ on } \partial\Omega}} E(v) &= \tfrac{1}{2}a(\widehat{u},\widehat{u}) + a(u_\varphi,\widehat{u}) - \ell(\widehat{u}) + \tfrac{1}{2}a(u_\varphi,u_\varphi) - \ell(u_\varphi)\\ &= \tfrac{1}{2}a(u,u) - \ell(u)\ ,\end{aligned}$$

or, by use of (3.9),

$$\begin{aligned}E(u) = \min_{\substack{v\in V\\ v=\varphi \text{ on } \partial\Omega}} E(v) &= -\tfrac{1}{2}\ell(\widehat{u}) + \tfrac{1}{2}a(u_\varphi,\widehat{u}) + \tfrac{1}{2}a(u_\varphi,u_\varphi) - \ell(u_\varphi)\\ &= -\tfrac{1}{2}\ell(u) + \tfrac{1}{2}a(u_\varphi,u) - \tfrac{1}{2}\ell(u_\varphi)\ .\end{aligned} \tag{3.10}$$

This last formula is particularly convenient since it is an affine expression of $E(u)$ in terms of u, depending on the data ℓ and φ of the problem (3.7). It is the equivalent of (3.3) in the context of variational problems of the form (3.7), posed on *affine* function spaces.

3.2.2. *The energy lemma, the Dirichlet version.*

The following result adapts Lemma 3 to the case when inhomogeneous Dirichlet boundary conditions are considered.

Lemma 4. *Let Ω be a bounded domain in $\mathbb{R}^2$, and let $V_\varepsilon, W_\varepsilon$ be two families of Hilbert spaces of functions defined on Ω, such that, for any $\varepsilon>0$, the trace operator*

$$V_\varepsilon \ni v\ \mapsto\ v|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega)$$

is well-defined, continuous, and has a linear continuous right inverse $\varphi\mapsto v_\varphi$ (similarly for W_ε with a mapping $\varphi\mapsto w_\varphi$). Let H be another Hilbert space, which continuously contains all the V_ε and W_ε. Denote also by $a_\varepsilon : V_\varepsilon\times V_\varepsilon\to\mathbb{R}$ and $b_\varepsilon : W_\varepsilon\times W_\varepsilon\to\mathbb{R}$ two families of symmetric bilinear forms that are continuous and coercive. For any $\varphi\in H^{\frac{1}{2}}(\partial\Omega)$, $\ell\in H'$, consider the minimization problems:

$$\min_{\substack{v\in V_\varepsilon\\ v=\varphi \text{ on } \partial\Omega}} E_\varepsilon(v),\ \ E_\varepsilon(v) = \frac{1}{2}a_\varepsilon(v,v) - \ell(v),$$

$$\min_{\substack{w\in W_\varepsilon\\ w=\varphi \text{ on } \partial\Omega}} F_\varepsilon(w),\ \ F_\varepsilon(w) = \frac{1}{2}b_\varepsilon(w,w) - \ell(w),$$

which admit unique minimizers $v_\varepsilon^{\ell,\varphi}\in V_\varepsilon$, $w_\varepsilon^{\ell,\varphi}\in W_\varepsilon$ (again, the dependence of E_ε, F_ε on ℓ is omitted). Then, for any $s\geq 1/2$, the following estimate holds

$$\sup_{\substack{||\ell||_{H'}\leq 1\\ ||\varphi||_{H^s(\partial\Omega)}\leq 1}} ||v_\varepsilon^{\ell,\varphi} - w_\varepsilon^{\ell,\varphi}||_H \leq 4 \sup_{\substack{||\ell||_{H'}\leq 1\\ ||\varphi||_{H^s(\partial\Omega)}\leq 1}} |E_\varepsilon(v_\varepsilon^{\ell,\varphi}) - F_\varepsilon(w_\varepsilon^{\ell,\varphi})|. \tag{3.11}$$

Proof. For any elements $\varphi\in H^s(\partial\Omega)$ and $\ell\in H'$, (3.10) implies that

$$|E_\varepsilon(v_\varepsilon^{\ell,\varphi}) - F_\varepsilon(w_\varepsilon^{\ell,\varphi})| = \frac{1}{2}|-\ell(v_\varepsilon^{\ell,\varphi} - w_\varepsilon^{\ell,\varphi}) + a_\varepsilon(v_\varphi, v_\varepsilon^{\ell,\varphi}) - b_\varepsilon(w_\varphi, w_\varepsilon^{\ell,\varphi}) - \ell(v_\varphi - w_\varphi)|\ .$$

Consider the space $\mathcal{H} := H'\times H^s(\partial\Omega)$ equipped with the norm

$$|||(\ell,\varphi)||| = \max\left(||\ell||_{H'}, ||\varphi||_{H^s(\partial\Omega)}\right)\ ,$$

and introduce the bilinear form $q:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$, defined for $(\ell_1,\varphi_1),(\ell_2,\varphi_2)\in\mathcal{H}$ by the expression:

$$q((\ell_1,\varphi_1),(\ell_2,\varphi_2)) = -\ell_1(v_\varepsilon^{\ell_2,\varphi_2} - w_\varepsilon^{\ell_2,\varphi_2}) + a_\varepsilon(v_{\varphi_1}, v_\varepsilon^{\ell_2,\varphi_2}) - b_\varepsilon(w_{\varphi_1}, w_\varepsilon^{\ell_2,\varphi_2}) - \ell_2(v_{\varphi_1} - w_{\varphi_1})\ .$$

The form q is symmetric. Indeed, introducing $\widehat{v_\varepsilon^{\ell_i}} := v_\varepsilon^{\ell_i,\varphi_i} - v_{\varphi_i}$ and $\widehat{w_\varepsilon^{\ell_i}} := w_\varepsilon^{\ell_i,\varphi_i} - w_{\varphi_i}$, one obtains

$$\begin{aligned}
q((\ell_1,\varphi_1),(\ell_2,\varphi_2)) &= -\ell_1(\widehat{v_\varepsilon^{\ell_2}} - \widehat{w_\varepsilon^{\ell_2}}) + a_\varepsilon(v_{\varphi_1}, v_\varepsilon^{\ell_2,\varphi_2}) - b_\varepsilon(w_{\varphi_1}, w_\varepsilon^{\ell_2,\varphi_2}) - \ell_1(v_{\varphi_2} - w_{\varphi_2}) - \ell_2(v_{\varphi_1} - w_{\varphi_1}) \\
&= -a_\varepsilon(v_\varepsilon^{\ell_1,\varphi_1}, \widehat{v_\varepsilon^{\ell_2}}) + b_\varepsilon(w_\varepsilon^{\ell_1,\varphi_1}, \widehat{w_\varepsilon^{\ell_2}}) + a_\varepsilon(v_{\varphi_1}, v_\varepsilon^{\ell_2,\varphi_2}) - b_\varepsilon(w_{\varphi_1}, w_\varepsilon^{\ell_2,\varphi_2}) \\
&\quad -\ell_1(v_{\varphi_2} - w_{\varphi_2}) - \ell_2(v_{\varphi_1} - w_{\varphi_1}) \\
&= a_\varepsilon(v_{\varphi_1}, v_{\varphi_2}) - a_\varepsilon(\widehat{v_\varepsilon^{\ell_1}}, \widehat{v_\varepsilon^{\ell_2}}) - b_\varepsilon(w_{\varphi_1}, w_{\varphi_2}) + b_\varepsilon(\widehat{w_\varepsilon^{\ell_1}}, \widehat{w_\varepsilon^{\ell_2}}) \\
&\quad -\ell_1(v_{\varphi_2} - w_{\varphi_2}) - \ell_2(v_{\varphi_1} - w_{\varphi_1}) \ .
\end{aligned}$$

The polarization identity now yields

$$\sup_{\substack{|||(\ell_1,\varphi_1)|||\le 1,\\ |||(\ell_2,\varphi_2)|||\le 1}} |q((\ell_1,\varphi_1),(\ell_2,\varphi_2))| \le 2 \sup_{|||(\ell,\varphi)|||\le 1} |q((\ell,\varphi),(\ell,\varphi))| \ ,$$

and therefore by the same technique as in the proof of Lemma 3

$$\begin{aligned}
\sup_{\substack{||\ell||_{H'}\le 1\\ ||\varphi||_{H^s(\partial\Omega)}\le 1}} ||v_\varepsilon^{\ell,\varphi} - w_\varepsilon^{\ell,\varphi}||_H &= \sup_{|||(\ell_2,\varphi_2)|||\le 1} \sup_{||\ell_1||_{H'}\le 1} |q((\ell_1,0),(\ell_2,\varphi_2))| \\
&\le \sup_{\substack{|||(\ell_1,\varphi_1)|||\le 1,\\ |||(\ell_2,\varphi_2)|||\le 1}} |q((\ell_1,\varphi_1),(\ell_2,\varphi_2))| \\
&\le 2 \sup_{|||(\ell,\varphi)|||\le 1} |q((\ell,\varphi)(\ell,\varphi))| \\
&= 4 \sup_{\substack{||\ell||_{H'}\le 1\\ ||\varphi||_{H^s(\partial\Omega)}\le 1}} |E_\varepsilon(v_\varepsilon^{\ell,\varphi}) - F_\varepsilon(w_\varepsilon^{\ell,\varphi})| \ .
\end{aligned}$$

This is the desired estimate. □

Remark 6.

(1) For the estimates (3.1) and (3.11) of Lemma 3 and Lemma 4, it is sufficient (on the right hand side) to envoke the supremum for ℓ belonging to a dense subset of H', due to the continuity of the mappings $\ell \mapsto v_\varepsilon^\ell$, $\ell \mapsto w_\varepsilon^\ell$.

(2) Lemmas 3 and 4 do not generally hold when the energies E_ε and F_ε contain additional linear terms $c_\varepsilon \in V_\varepsilon'$ and $d_\varepsilon \in W_\varepsilon'$ (i.e. contain linear terms from a larger class than H')

$$E_\varepsilon(v) = \frac{1}{2}a_\varepsilon(v,v) - c_\varepsilon(v) - \ell(v), \ \ F_\varepsilon(w) = \frac{1}{2}b_\varepsilon(w,w) - d_\varepsilon(w) - \ell(w) \ .$$

In this case it may still be possible to control the difference $||v_\varepsilon^\ell - w_\varepsilon^\ell||$ in terms of the difference $|E_\varepsilon(v_\varepsilon^\ell) - F_\varepsilon(w_\varepsilon^\ell)|$ between the corresponding energies; however, this control will in general be 'weaker', and may require assumptions that are not so naturally formulated in an abstract framework.

4. Derivation of the 0^{th} order approximation of u_ε

In this section we formally construct a *uniform* 0^{th}-order approximation to the solution u_ε to (2.2). This approximation u_ε^0 is, as explained earlier, the solution to a "simpler" problem with the same data f, φ, but posed on a fixed domain. Some of the coefficients of this "simpler" problem depend on ε and a_ε, and as we have explained in the introduction this is inevitable. Later, in Section 6, we shall rigorously prove a uniform approximation estimate for u_ε^0. To be more precise at that point we shall prove that there exists a constant C which only depends on the data Ω, σ, f and φ, and not on ε and a_ε, such that:

$$||u_\varepsilon - u_\varepsilon^0|| \le C\varepsilon \ .$$

The norm $||.||$, and the dependence of C on f and φ will be specified later.

To construct the approximation u_ε^0, we rely on the fact that u_ε is the minimizer of an energy functional E_ε, and that the flux $(\gamma_\varepsilon \nabla u_\varepsilon)$ is the maximizer of a dual energy E_ε^c. We begin with the construction of an approximate energy E_ε^0 to E_ε, and then we shall search the desired approximation u_ε^0 as the minimizer of E_ε^0. We also analyze the dual energy E_ε^c to obtain additional information about the behavior of the flux $(\gamma_\varepsilon \nabla u_\varepsilon)$, which we shall need for the proof of the estimate of $(u_\varepsilon - u_\varepsilon^0)$.

4.1. Asymptotic expansions of the energy functionals associated with u_ε.

4.1.1. *Asymptotic expansion of the primal Dirichlet energy.*

As is well-known, the solution u_ε to (2.2) is the unique solution of the minimization problem

$$\min_{\substack{u\in H^1(\Omega)\\ u=\varphi \text{ on } \partial\Omega}} E_\varepsilon(u),\ \ E_\varepsilon(u) = \frac{1}{2}\int_\Omega \gamma_\varepsilon |\nabla u|^2\,dx - \int_\Omega fu\,dx\ . \tag{4.1}$$

First, we transform part of this energy expression by means of the mapping $H_\varepsilon : \omega_1 \to \omega_\varepsilon$, defined by

$$H_\varepsilon(x) = p_\sigma(x) + \varepsilon d_{\Omega^-}(x) n(x)\ . \tag{4.2}$$

A straightforward calculation based on (2.4) yields

$$\nabla H_\varepsilon = \begin{pmatrix} \frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}} & 0 \\ 0 & \varepsilon \end{pmatrix}\ , \tag{4.3}$$

where the above matrix is expressed in the local basis (τ, n) of the plane. For any function $u \in H^1(\omega_\varepsilon)$ we denote by $\widehat{u} := u \circ H_\varepsilon$; a change of variables now leads to

$$\begin{aligned} \int_{\omega_\varepsilon} |\nabla u|^2\,dx &= \int_{\omega_1} \left((\det\nabla H_\varepsilon)\,\nabla H_\varepsilon^{-1}(\nabla H_\varepsilon^{-1})^T\right)\nabla\widehat{u}\cdot\nabla\widehat{u}\,dx \\ &= \varepsilon\int_{\omega_1} \frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\left(\frac{\partial\widehat{u}}{\partial\tau}\right)^2 dx + \frac{1}{\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\left(\frac{\partial\widehat{u}}{\partial n}\right)^2 dx\ . \end{aligned}$$

Using this change of variables, we may now equivalently restate Problem (4.1) as

$$\min_{\substack{(u,v)\in\overline{V_\varepsilon^0}\\ u=\varphi \text{ on } \partial\Omega}} \overline{F_\varepsilon^0}(u,v)\ , \tag{4.4}$$

where the set $\overline{V_\varepsilon^0}$ is defined as

$$\overline{V_\varepsilon^0} = \left\{(u,v)\in H^1(\Omega\setminus\overline{\omega_\varepsilon})\times H^1(\omega_1),\ \forall x\in\sigma,\ \ v(x\pm n(x)) = u(x\pm\varepsilon n(x))\right\}\ ,$$

and the rescaled energy $\overline{F_\varepsilon^0}$ is given by

$$\overline{F_\varepsilon^0}(u,v) = \frac{1}{2}\int_{\Omega\setminus\overline{\omega_\varepsilon}}|\nabla u|^2\,dx + \frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1}\frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\left(\frac{\partial v}{\partial\tau}\right)^2 dx + \frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\left(\frac{\partial v}{\partial n}\right)^2 dx - \int_\Omega fu\,dx\ .$$

Obviously, the equalities featured in the above definition of the space $\overline{V_\varepsilon^0}$ are understood in the sense of traces. We now proceed to formally simplify this problem. Retaining only the leading order contribution in the definition of the energy functional $\overline{F_\varepsilon^0}$ (and of the space $\overline{V_\varepsilon^0}$) we are led to the approximate problem

$$\min_{\substack{(u,v)\in V^0\\ u=\varphi \text{ on } \partial\Omega}} F_\varepsilon^0(u,v)\ , \tag{4.5}$$

where we have introduced the function space

$$V^0 = \left\{(u,v)\in H^1(\Omega\setminus\sigma)\times H^1(\omega_1),\ \text{s.t. } \forall x\in\sigma,\ v(x\pm n(x)) = u^\pm(x)\right\}\ , \tag{4.6}$$

and the approximate energy

$$F_\varepsilon^0(u,v) = \frac{1}{2}\int_{\Omega\setminus\sigma}|\nabla u|^2\,dx + \frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1}(1+\kappa d_{\Omega^-})\left(\frac{\partial v}{\partial\tau}\right)^2 dx + \frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1}\frac{1}{1+\kappa d_{\Omega^-}}\left(\frac{\partial v}{\partial n}\right)^2 dx - \int_\Omega fu\,dx\ .$$

This problem can be further simplified, by performing the "inner" minimization in v and expressing the result in terms of u. The Problem (4.5) can thus be rewritten

$$\min_{\substack{u\in H^1(\Omega\setminus\sigma)\\ u=\varphi \text{ on } \partial\Omega}} \left\{\frac{1}{2}\int_\Omega|\nabla u|^2\,dx - \int_\Omega fu\,dx + G_\varepsilon^0(u)\right\}\ , \tag{4.7}$$

where

$$G_\varepsilon^0(u) = \min_{\substack{v\in H^1(\omega_1)\\ v(x+n(x))=u^+(x),\ x\in\sigma\\ v(x-n(x))=u^-(x),\ x\in\sigma}} \left\{\frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1}(1+\kappa d_{\Omega^-})\left(\frac{\partial v}{\partial\tau}\right)^2 dx + \frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1}\frac{1}{1+\kappa d_{\Omega^-}}\left(\frac{\partial v}{\partial n}\right)^2 dx\right\}\ . \tag{4.8}$$

This problem can be solved in terms of u which would give rise to an explicit expression for $G^0_\varepsilon(u)$. Before doing so, we note that the two terms of the energy are of different orders when $\varepsilon \to 0$; one might therefore naturally expect that the behavior of the minimizer v of the previous expression to leading order should be dictated by the term $\frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1}\frac{1}{1+\kappa d_{\Omega^-}}\left(\frac{\partial v}{\partial n}\right)^2 dx$. From the Euler-Lagrange equation associated with this minimization, it follows that v should satisfy

$$\forall w \in H^1_0(\omega_1),\ \int_{\omega_1}\frac{1}{1+\kappa d_{\Omega^-}}\frac{\partial v}{\partial n}\frac{\partial w}{\partial n}\,dx = 0\ .$$

If we introduce the coarea formula of Proposition 1, this simplifies to

$$\forall w \in H^1_0(\omega_1),\ \int_\sigma\int_{-1}^1 \frac{\partial v}{\partial n}(x+tn(x))\frac{\partial w}{\partial n}(x+tn(x))\,dt\,ds(x) = 0\ .$$

Choosing a test function w of the form $w(x+tn(x)) = \phi(x)\psi(t)$, with arbitrary $\phi \in \mathcal{C}^\infty(\sigma)$ and $\psi \in \mathcal{C}^\infty_c(-1,1)$, we now arrive at

$$\int_\sigma \phi(x)\int_{-1}^1 \frac{d}{dt}(v(x+tn(x)))\psi'(t)\,dt\,ds(x) = 0\ ,$$

from which we conclude that for any $x\in\sigma$, and any function $\psi \in \mathcal{C}^\infty_c(-1,1)$,

$$\int_{-1}^1 \frac{d}{dt}(v(x+tn(x)))\psi'(t)\,dt = 0\ .$$

As a consequence, for any $x\in\sigma$, the function $t\mapsto v(x+tn(x))$ is affine. Introducing the boundary conditions for v (*cf.* 4.8), we now arrive at

$$\forall x \in \sigma,\ t\in(-1,1),\ v(x+tn(x)) = \frac{t}{2}[u](x) + \frac{1}{2}(u^+(x)+u^-(x))\ .$$

Substituting this expression for the minimizer in (4.8) we obtain

$$\begin{aligned}
G^0_\varepsilon(u) &\approx \frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1}(1+d_{\Omega^-}\kappa)\left(\frac{\partial v}{\partial\tau}\right)^2 dx + \frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1}\frac{1}{1+d_{\Omega^-}\kappa}\left(\frac{\partial v}{\partial n}\right)^2 dx\\
&= \frac{\varepsilon a_\varepsilon}{2}\int_\sigma\int_{-1}^1 (1+t\kappa)^2\left(\frac{\partial v}{\partial\tau}(x+tn(x))\right)^2 dt\,ds(x) + \frac{a_\varepsilon}{2\varepsilon}\int_\sigma\int_{-1}^1\left(\frac{\partial v}{\partial n}(x+tn(x))\right)^2 dt\,ds(x)\\
&= \frac{\varepsilon a_\varepsilon}{2}\int_\sigma\int_{-1}^1\left(\frac{\partial}{\partial\tau}(v(x+tn(x)))\right)^2 dt\,ds(x) + \frac{a_\varepsilon}{4\varepsilon}\int_\sigma\left(u^+-u^-\right)^2 ds\\
&= \frac{\varepsilon a_\varepsilon}{8}\int_\sigma\int_{-1}^1\left(\frac{\partial u^+}{\partial\tau}(x)+\frac{\partial u^-}{\partial\tau}(x)+t\left(\frac{\partial u^+}{\partial\tau}(x)-\frac{\partial u^-}{\partial\tau}(x)\right)\right)^2 dt\,ds(x) + \frac{a_\varepsilon}{4\varepsilon}\int_\sigma\left(u^+-u^-\right)^2 ds\ ,
\end{aligned}$$

where Proposition 1 was used for the first identity. Finally, after integration in t

$$G^0_\varepsilon(u) \approx \frac{\varepsilon a_\varepsilon}{3}\int_\sigma\left(\left(\frac{\partial u^+}{\partial\tau}\right)^2+\left(\frac{\partial u^-}{\partial\tau}\right)^2+\frac{\partial u^+}{\partial\tau}\frac{\partial u^-}{\partial\tau}\right)ds + \frac{a_\varepsilon}{4\varepsilon}\int_\sigma\left(u^+-u^-\right)^2 ds\ . \tag{4.9}$$

Let us draw some conclusions of these formal calculations: (4.7) and (4.9) suggest to search for an approximation u^0_ε to u_ε by solving

$$\min_{\substack{u\in V_\sigma\\ u=\varphi \text{ on } \partial\Omega}} E^0_\varepsilon(u)\ , \tag{4.10}$$

where V_σ denotes the space

$$V_\sigma = \left\{v\in H^1(\Omega\setminus\sigma),\ v^+|_\sigma, v^-|_\sigma \in H^1(\sigma)\right\}\ , \tag{4.11}$$

and the approximate energy E^0_ε reads:

(4.12)

$$E^0_\varepsilon(u) = \frac{1}{2}\int_{\Omega\setminus\sigma}|\nabla u|^2\,dx + \frac{\varepsilon a_\varepsilon}{3}\int_\sigma\left(\left(\frac{\partial u^+}{\partial\tau}\right)^2+\left(\frac{\partial u^-}{\partial\tau}\right)^2+\frac{\partial u^+}{\partial\tau}\frac{\partial u^-}{\partial\tau}\right)ds + \frac{a_\varepsilon}{4\varepsilon}\int_\sigma\left(u^+-u^-\right)^2 ds - \int_\Omega fu\,dx\ .$$

We also note that according to these calculations the (rescaled) potential $(u_\varepsilon \circ H_\varepsilon)$, inside the inhomogeneity ω_1, should be approximated by the function $v_\varepsilon^0 \in H^1(\omega_1)$, given by

$$\forall x \in \sigma,\ t \in (-1,1),\ \ v_\varepsilon^0(x+tn(x)) = \frac{t}{2}[u_\varepsilon^0](x) + \frac{1}{2}\left(u_\varepsilon^{0+}(x) + u_\varepsilon^{0-}(x)\right) \ . \tag{4.13}$$

4.1.2. *Asymptotic expansion of the dual energy and its maximizer.*

Before turning to a rigorous study of the function u_ε^0 and its distance to u_ε, we perform in this section a formal study of the *dual energy* E_ε^c corresponding to E_ε in the spirit of [19].

The dual energy principle associated with E_ε asserts that

$$\min_{\substack{u\in H^1(\Omega)\\ u=\varphi \text{ on } \partial\Omega}} E_\varepsilon(u) = \max_{\substack{\xi\in L^2(\Omega)^2\\ -\mathrm{div}(\xi)=f}} E_\varepsilon^c(\xi)\ ,$$

with

$$E_\varepsilon^c(\xi) = \int_{\partial\Omega} \xi\cdot n\varphi\, ds - \frac{1}{2}\int_\Omega \gamma_\varepsilon^{-1}|\xi|^2\, dx\ . \tag{4.14}$$

The last extremal problem admits $(\gamma_\varepsilon\nabla u_\varepsilon)$ as the unique maximal argument. We shall now apply the same strategy as in the previous subsection, namely, to split the integral $\frac{1}{2}\int_\Omega \gamma_\varepsilon^{-1}|\xi|^2\, dx$ into two, one over $\Omega\setminus\overline{\omega_\varepsilon}$, the other over ω_ε, and rescale the second one by using a change of variables. The following lemma provides a hint of what is the relevant rescaling when the objects in question are vector fields:

Lemma 5. *Let U,V be two smooth subdomains of $\mathbb{R}^2$, $\psi: U\to V$ be a diffeomorphism of class $\mathcal{C}^1$; let $\xi\in L^2(V)^2$ be a vector field, and $f\in L^2(V)$. Then the (weak) divergence of ξ equals f if and only if the vector field $|\det(\nabla\psi)|(\nabla\psi)^{-1}(\xi\circ\psi)\in L^2(U)^2$ has divergence $|\det(\nabla\psi)|f\circ\psi$. In particular, ξ is (weakly) divergence-free if and only if $|\det(\nabla\psi)|(\nabla\psi)^{-1}(\xi\circ\psi)$ is.*

Proof. We have, successively,

$$\begin{aligned}
\mathrm{div}(\xi)=f \quad &\Leftrightarrow \quad \forall p\in\mathcal{C}_c^\infty(V),\ \int_V \xi\cdot\nabla p\, dx = -\int_V fp\, dx\\
&\Leftrightarrow \quad \forall p\in\mathcal{C}_c^\infty(V),\ \int_U |\det(\nabla\psi)|(\xi\circ\psi)\cdot(\nabla p)\circ\psi\, dx = -\int_U |\det(\nabla\psi)|(f\circ\psi)(p\circ\psi)\, dx\\
&\Leftrightarrow \quad \forall p\in\mathcal{C}_c^\infty(V),\ \int_U |\det(\nabla\psi)|(\xi\circ\psi)\cdot\left((\nabla\psi)^{-1}\right)^T\nabla(p\circ\psi)\, dx = -\int_U |\det(\nabla\psi)|(f\circ\psi)(p\circ\psi)\, dx\\
&\Leftrightarrow \quad \forall \widehat{p}\in\mathcal{C}_c^\infty(U),\ \int_U |\det(\nabla\psi)|(\nabla\psi)^{-1}(\xi\circ\psi)\cdot\nabla\widehat{p}\, dx = -\int_U |\det(\nabla\psi)|(f\circ\psi)\widehat{p}\, dx\\
&\Leftrightarrow \quad |\det(\nabla\psi)|(\nabla\psi)^{-1}(\xi\circ\psi) \text{ has divergence } |\det(\nabla\psi)|f\circ\psi\ ,
\end{aligned}$$

which proves the desired result. □

Remark 7. In the same way we established Lemma 5 we may establish that if $\xi\in H_{\mathrm{div}}(V)$ with $\xi\cdot n = g$ on ∂V in a weak sense, then $|\det(\nabla\psi)|(\nabla\psi)^{-1}(\xi\circ\psi)\cdot n = g\circ\psi|\frac{\partial}{\partial\tau}\psi|$ on ∂U.

For any $\xi\in L^2(\Omega)^2$

$$\begin{aligned}
\int_{\omega_\varepsilon}|\xi|^2\, dx &= \int_{\omega_1}\det(\nabla H_\varepsilon)(\xi\circ H_\varepsilon)\cdot(\xi\circ H_\varepsilon)\, dx\\
&= \int_{\omega_1}\left(\frac{1}{\det(\nabla H_\varepsilon)}\nabla H_\varepsilon^T\nabla H_\varepsilon\right)\widehat{\xi}\cdot\widehat{\xi}\, dx\ ,
\end{aligned}$$

where we have denoted $\widehat{\xi} = \det(\nabla H_\varepsilon)\,(\nabla H_\varepsilon)^{-1}(\xi\circ H_\varepsilon)$. We also calculate that

$$\left|\frac{\partial}{\partial\tau}H_\varepsilon\right| = |\nabla H_\varepsilon\tau\cdot\tau| = \frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\ .$$

Performing a change of variables on ω_ε, and using these two identities in combination with (4.3), Lemma 5 and Remark 7 we are led to rewrite the maximization problem for E^c_ε in the form

$$\max_{\substack{(\xi,\eta)\in\overline{V^{c0}_\varepsilon}\\ -\mathrm{div}(\xi)=f\\ -\mathrm{div}(\eta)=0}} \overline{F^{c0}_\varepsilon}(\xi,\eta)\ , \tag{4.15}$$

where

$$\overline{V^{c0}_\varepsilon} = \left\{ (\xi,\eta)\in H_{\mathrm{div}}(\Omega\setminus\overline{\omega_\varepsilon})\times H_{\mathrm{div}}(\omega_1),\ \ \forall x\in\sigma,\quad \begin{array}{l} \frac{1+\kappa}{1+\varepsilon\kappa}\eta_n(x+n(x)) = \xi_n(x+\varepsilon n(x)) \\ \frac{1-\kappa}{1-\varepsilon\kappa}\eta_n(x-n(x)) = \xi_n(x-\varepsilon n(x)) \end{array} \right\}\ , \tag{4.16}$$

and the functional $\overline{F^{c0}_\varepsilon}$ is given by

$$\overline{F^{c0}_\varepsilon}(\xi,\eta) = \int_{\partial\Omega}\xi\cdot n\varphi\, ds - \frac{1}{2}\int_{\Omega\setminus\overline{\omega_\varepsilon}}|\xi|^2\,dx - \frac{1}{2\varepsilon a_\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\eta_\tau^2\,dx - \frac{\varepsilon}{2a_\varepsilon}\int_{\omega_1}\frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\eta_n^2\,dx\ .$$

Here we have used that the support of f is away from ω_ε (since $f\in\mathcal{F}_\delta$ for some fixed $\delta>0$).
As before, only the leading order terms in the definitions of $\overline{V^{c0}_\varepsilon}$ and $\overline{F^{c0}_\varepsilon}$ are now retained in the construction of the approximate extremal problem

$$\max_{\substack{(\xi,\eta)\in V^{0c}\\ -\mathrm{div}(\xi)=f\\ -\mathrm{div}(\eta)=0}} F^{c0}_\varepsilon(\xi,\eta)\ . \tag{4.17}$$

The approximate set V^{c0} is

$$V^{c0} = \left\{ (\xi,\eta)\in H_{\mathrm{div}}(\Omega\setminus\sigma)\times H_{\mathrm{div}}(\omega_1),\ \int_\sigma[\xi_n]=0\ ,\ \text{and } \forall x\in\sigma\ \ \begin{array}{l} (1+\kappa)\eta_n(x+n(x)) = \xi_n^+(x) \\ (1-\kappa)\eta_n(x-n(x)) = \xi_n^-(x) \end{array} \right\}\ , \tag{4.18}$$

and the approximate energy F^{c0}_ε is

$$F^{c0}_\varepsilon(\xi,\eta) = \int_{\partial\Omega}\xi\cdot n\varphi\,ds - \frac{1}{2}\int_{\Omega\setminus\sigma}|\xi|^2\,dx - \frac{1}{2\varepsilon a_\varepsilon}\int_{\omega_1}\frac{1}{1+\kappa d_{\Omega^-}}\eta_\tau^2\,dx - \frac{\varepsilon}{2a_\varepsilon}\int_{\omega_1}(1+\kappa d_{\Omega^-})\eta_n^2\,dx\ . \tag{4.19}$$

Note that we have included the integral constraint $\int_\sigma[\xi_n]=0$ as part of the description of the set V^{c0}; this additional constraint is a consequence of the interface conditions imposed on ξ and η, and the constraint $\mathrm{div}(\eta)=0$, and so it leaves the maximization unchanged. To simplify (4.17) further, we remark as in Section 4.1.1 that the extremal problem in η can be solved explicitely (at least approximately) in terms of ξ. Indeed, we rewrite (4.17) as

$$\max_{\substack{\xi\in H_{\mathrm{div}}(\Omega\setminus\sigma)\\ -\mathrm{div}(\xi)=f\\ \int_\sigma[\xi_n]=0}} \left\{ \int_{\partial\Omega}\xi\cdot n\varphi\,ds - \frac{1}{2}\int_{\Omega\setminus\sigma}|\xi|^2\,dx - G^{c0}_\varepsilon(\xi) \right\}\ ,$$

where

$$G^{c0}_\varepsilon(\xi) := \min_{\substack{\eta\in W^{0c}\\ -\mathrm{div}(\eta)=0}} \left\{ \frac{1}{2\varepsilon a_\varepsilon}\int_{\omega_1}\frac{1}{1+\kappa d_{\Omega^-}}\eta_\tau^2\,dx + \frac{\varepsilon}{2a_\varepsilon}\int_{\omega_1}(1+\kappa d_{\Omega^-})\eta_n^2\,dx \right\}\ . \tag{4.20}$$

Here the set W^{c0} is given by

$$W^{c0} = \left\{ \eta\in H_{\mathrm{div}}(\omega_1),\ \ \forall x\in\sigma\ \ \begin{array}{l} (1+\kappa)\eta_n(x+n(x)) = \xi_n^+(x) \\ (1-\kappa)\eta_n(x-n(x)) = \xi_n^-(x) \end{array} \right\}\ .$$

We then proceed to calculate explicitely the expression (4.20). Intuitively, the minimizer η should be characterized to leading order by the minimization of the term $\frac{1}{2\varepsilon a_\varepsilon}\int_{\omega_1}\frac{1}{1+\kappa d_{\Omega^-}}\eta_\tau^2\,dx$. The associated Euler-Lagrange equation reads:

$$\int_{\omega_1}\frac{1}{1+\kappa d_{\Omega^-}}\eta_\tau\zeta_\tau\,dx = 0\ ,$$

for any $\zeta\in H_{\mathrm{div}}(\omega_1)$ s.t. $-\mathrm{div}(\zeta)=0$, and $(1\pm\kappa(x))\zeta_n(x\pm n(x))=0$. Since for any $\psi\in\mathcal{C}^\infty_c(\omega_1)$, the field $(-\frac{\partial\psi}{\partial n},\frac{\partial\psi}{\partial\tau})$, is divergence-free (see Remark 3), and has a vanishing normal component ($\frac{\partial\psi}{\partial\tau}$) on $\partial\omega_1$, we obtain

$$\int_{\omega_1}\frac{1}{1+\kappa d_{\Omega^-}}\eta_\tau\frac{\partial\psi}{\partial n}\,dx = 0\ ;$$

and now using Proposition 1,

$$\int_\sigma \int_{-1}^{1} \eta_\tau(x+tn(x)) \frac{\partial \psi}{\partial n}(x+tn(x))\, dt\, ds(x) = 0 \ .$$

Due to the same argument as in Section 4.1.1, we conclude that the quantity $\eta_\tau(x+tn(x))$ is independent of $t \in (-1,1)$, that is, there exists a function $a : \sigma \to \mathbb{R}$ such that

$$\forall x \in \sigma,\ t \in (-1,1),\ \ \eta_\tau(x+tn(x)) = a(x) \ .$$

We now rely on the divergence-free property of η to complete the calculation. Using Lemma 2, one has, for any fixed $x \in \sigma$ and $t \in (-1,1)$,

$$\frac{\partial \eta_\tau}{\partial \tau}(x+tn(x)) + \frac{\partial \eta_n}{\partial n}(x+tn(x)) + \frac{\kappa(x)}{1+t\kappa(x)}\eta_n(x+tn(x)) = \mathrm{div}(\eta)(x+tn(x)) = 0 \ ,$$

that is, letting $z(t) = \eta_n(x+tn(x))$,

$$z'(t) + \frac{\kappa(x)}{1+t\kappa(x)} z(t) = -\frac{1}{1+t\kappa(x)} \frac{\partial}{\partial \tau}(\eta_\tau(x+tn(x))) = -\frac{1}{1+t\kappa(x)} \frac{\partial a}{\partial \tau}(x) \ ,$$

which is nothing but an ODE for z. A simple calculation now gives that there exists a function $b : \sigma \to \mathbb{R}$ such that

$$\eta_n(x+tn(x)) = -\frac{t}{1+t\kappa(x)} \frac{\partial a}{\partial \tau}(x) + \frac{b(x)}{1+t\kappa(x)} \ .$$

Owing to the boundary conditions for η_n in the definition of the set W^{c0}, the functions a and b must satisfy

$$\forall x \in \sigma,\ \left\{ \begin{array}{rcl} -\frac{\partial a}{\partial \tau}(x) + b(x) & = & \xi_n^+(x), \\ \frac{\partial a}{\partial \tau}(x) + b(x) & = & \xi_n^-(x) \ , \end{array} \right.$$

which after straightforward manipulations leads to

$$\begin{aligned} \frac{\partial}{\partial \tau}(\eta_\tau(x+tn(x))) &= -\frac{1}{2}[\xi_n](x), \text{ and} \\ \eta_n(x+tn(x)) &= \frac{1}{2}\left(\frac{t}{1+t\kappa(x)}[\xi_n](x) + \frac{1}{1+t\kappa(x)}(\xi_n^+(x) + \xi_n^-(x)) \right) \ . \end{aligned} \tag{4.21}$$

These expressions are unfortunately not as explicit as those obtained in Section 4.1.1, and in particular they do not lead to a similarly simple variational problem for ξ. However, they do (approximately) connect the exterior and interior components, ξ and η, of the maximizer of F_ε^{c0}, which hopefully is close to that of $\overline{F_\varepsilon^{c0}}$.

5. Study of the approximate function u_ε^0: uniform energy and regularity estimates

In this section, we study properties of the solution u_ε^0 to (4.10), which is our candidate for the 0^{th} order term of the asymptotic expansion of u_ε.

We assume the data to be such that $f \in L^2(\Omega)$ with support away from σ, and with $\int_{\Omega^-} f\ dx = 0$ – this is expressed by requiring that $f \in \mathcal{F}_\delta$ for some fixed $\delta > 0$ (see the definitions in Section 2.1); we also assume that $\varphi \in H^{1/2}(\partial\Omega)$. After first proving existence and uniqueness of the solution u_ε^0, our main purpose is to establish energy and regularity estimates for u_ε^0 (and its derivatives) which are uniform with respect to ε and the sequence a_ε (see Subsections 5.3 and 5.4).

5.1. **Existence, uniqueness, and a classical formulation of (4.10).**

Let $V_{\sigma,0}$ be the subspace of V_σ - the latter being defined by (4.11) - composed of functions with vanishing trace on $\partial\Omega$. We define the following semi-norm and norm on V_σ:

$$|u|_{V_\sigma}^2 = \int_{\Omega\setminus\sigma} |\nabla u|^2\, dx + \int_\sigma \left(\left(\frac{\partial u^+}{\partial \tau}\right)^2 + \left(\frac{\partial u^-}{\partial \tau}\right)^2 \right) ds + \int_\sigma (u^+ - u^-)^2\, ds, \quad ||u||_{V_\sigma}^2 = ||u||_{L^2(\Omega)}^2 + |u|_{V_\sigma}^2 \ .$$

We note that due to a standard Poincaré inequality the seminorm $|\cdot|_{V_\sigma}$ is actually a norm on $V_{\sigma,0}$, equivalent to $||u||^2_{V_\sigma}$. The variational formulation associated to (4.10) is

(5.1) Find $u^0_\varepsilon \in V_\sigma$ with $u^0_\varepsilon|_{\partial\Omega}= \varphi$, such that

$$\forall v \in V_{\sigma,0},\ \int_{\Omega\setminus\sigma} \nabla u^0_\varepsilon \cdot \nabla v\, dx + \frac{2\varepsilon a_\varepsilon}{3} \int_\sigma \left(\frac{\partial u^{0+}_\varepsilon}{\partial \tau}\frac{\partial v^+}{\partial \tau} + \frac{\partial u^{0-}_\varepsilon}{\partial \tau}\frac{\partial v^-}{\partial \tau} + \frac{1}{2}\left(\frac{\partial u^{0+}_\varepsilon}{\partial \tau}\frac{\partial v^-}{\partial \tau} + \frac{\partial u^{0-}_\varepsilon}{\partial \tau}\frac{\partial v^+}{\partial \tau}\right)\right) ds$$
$$+ \frac{a_\varepsilon}{2\varepsilon}\int_\sigma (u^{0+}_\varepsilon - u^{0-}_\varepsilon)(v^+ - v^-)\, ds = \int_\Omega f v\, dx.$$

Proposition 6. *The minimization problem (4.10), or equivalently the variational problem (5.1), has a unique solution* $u^0_\varepsilon \in V_\sigma$.

Proof. The existence and uniqueness of u^0_ε follow from the standard Lax-Milgram theory – the only point which deserves comment is the (non uniform in ε and a_ε) coercivity of the bilinear form involved in (5.1) on the space $V_{\sigma,0}$. This coercivity follows from the inequality

$$\forall a, b \in \mathbb{R},\quad \frac{1}{3}(a^2 + b^2 + ab) = \frac{1}{6}(a^2 + b^2) + \frac{1}{6}(a + b)^2 \geq \frac{1}{6}(a^2 + b^2)\ , \tag{5.2}$$

and the fact (noted above) that the seminorm $|\cdot|_{V_\sigma}$ is a norm on $V_{\sigma,0}$, equivalent to $||\cdot||^2_{V_\sigma}$. □

Problem (4.10) can be stated in a "classical" form. Indeed, using smooth test functions $v \in C^\infty_c(\Omega \setminus \sigma)$ in (5.1), we first see that u^0_ε satisfies

$$-\Delta u^0_\varepsilon = f \text{ in } \Omega \setminus \sigma\ ,$$

in the sense of distributions. If f and φ are smooth then it is fairly easy to prove that u^0_ε is actually $C^{2,\alpha}$ up to the boundary $\partial\Omega$ and up to the curve σ, and it solves the equation $-\Delta u^0_\varepsilon = f$ in a classical sense. The proof of regularity is a very standard elliptic regularity argument, that we leave to the reader, however, in Sections 5.3 and 5.4 (and the appendix) we shall show exactly what a priori estimates hold uniformly in ε and a_ε. Now using again (5.1), and an integration by parts, we obtain that

$$\int_\sigma \left(-\frac{\partial u^{0+}_\varepsilon}{\partial n} v^+ + \frac{\partial u^{0-}_\varepsilon}{\partial n} v^- \right) ds + \frac{2\varepsilon a_\varepsilon}{3} \int_\sigma \left(\frac{\partial u^{0+}_\varepsilon}{\partial \tau}\frac{\partial v^+}{\partial \tau} + \frac{\partial u^{0-}_\varepsilon}{\partial \tau}\frac{\partial v^-}{\partial \tau} + \frac{1}{2}\left(\frac{\partial u^{0+}_\varepsilon}{\partial \tau}\frac{\partial v^-}{\partial \tau} + \frac{\partial u^{0-}_\varepsilon}{\partial \tau}\frac{\partial v^+}{\partial \tau}\right)\right) ds \tag{5.3}$$
$$+ \frac{a_\varepsilon}{2\varepsilon}\int_\sigma (u^{0+}_\varepsilon - u^{0-}_\varepsilon)(v^+ - v^-)\, ds = 0\ ,$$

for all functions $v \in V_{\sigma,0}$. Using this last equality with test functions $v \in H^1(\Omega \setminus \sigma)$ such that $v = 0$ on $\partial\Omega$, v^+ is smooth on σ, and $v^- = 0$ on σ, we obtain that

$$\frac{\partial u^{0+}_\varepsilon}{\partial n} + \frac{\varepsilon a_\varepsilon}{3}\left(2\frac{\partial^2 u^{0+}_\varepsilon}{\partial \tau^2} + \frac{\partial^2 u^{0-}_\varepsilon}{\partial \tau^2}\right) - \frac{a_\varepsilon}{2\varepsilon}(u^{0+}_\varepsilon - u^{0-}_\varepsilon) = 0 \text{ on } \sigma\ .$$

Symmetrically, by exchanging the roles of v^- and v^+, one obtains

$$\frac{\partial u^{0-}_\varepsilon}{\partial n} - \frac{\varepsilon a_\varepsilon}{3}\left(\frac{\partial^2 u^{0+}_\varepsilon}{\partial \tau^2} + 2\frac{\partial^2 u^{0-}_\varepsilon}{\partial \tau^2}\right) - \frac{a_\varepsilon}{2\varepsilon}(u^{0+}_\varepsilon - u^{0-}_\varepsilon) = 0 \text{ on } \sigma\ .$$

In summary, u^0_ε is a solution to the following problem on $\Omega \setminus \sigma$

$$\begin{cases} -\Delta u^0_\varepsilon = f & \text{in } \Omega \setminus \sigma \\ u^0_\varepsilon = \varphi & \text{on } \partial\Omega \\ \frac{\partial u^{0+}_\varepsilon}{\partial n} + \frac{\varepsilon a_\varepsilon}{3}\left(2\frac{\partial^2 u^{0+}_\varepsilon}{\partial \tau^2} + \frac{\partial^2 u^{0-}_\varepsilon}{\partial \tau^2}\right) - \frac{a_\varepsilon}{2\varepsilon}(u^{0+}_\varepsilon - u^{0-}_\varepsilon) = 0 & \text{on } \sigma \\ \frac{\partial u^{0-}_\varepsilon}{\partial n} - \frac{\varepsilon a_\varepsilon}{3}\left(\frac{\partial^2 u^{0+}_\varepsilon}{\partial \tau^2} + 2\frac{\partial^2 u^{0-}_\varepsilon}{\partial \tau^2}\right) - \frac{a_\varepsilon}{2\varepsilon}(u^{0+}_\varepsilon - u^{0-}_\varepsilon) = 0 & \text{on } \sigma \end{cases} \tag{5.4}$$

Let us also notice that insertion of $v \in C^\infty_c(\Omega)$, $v \equiv 1$ in a neighborhood of σ, into (5.3) yields

$$\int_\sigma \left[\frac{\partial u^0_\varepsilon}{\partial n} \right] ds = 0\ .$$

This identity, in combination with the fact that $\int_{\Omega^-} f\, ds = 0$, gives

$$\int_\sigma \frac{\partial u_\varepsilon^{0+}}{\partial n}\, ds = \int_\sigma \frac{\partial u_\varepsilon^{0-}}{\partial n}\, ds = 0\ . \tag{5.5}$$

5.2. **The dual energy maximization problem for u_ε^0.**

In this paper, it will prove convenient on several occasions to use the dual energy maximization principle for u_ε^0. We remind the reader that the hypotheses for f and φ are:

$$f \in \mathcal{F}_\delta = \left\{ f \in L^2(\Omega),\ \operatorname{supp}(f) \subset \Omega \setminus \omega_\delta\ ,\ \int_{\Omega^-} f\, dx = 0 \right\},\ \text{and } \varphi \in H^{\frac{1}{2}}(\partial\Omega)\ .$$

We write

$$\begin{aligned} &E_\varepsilon^0(u_\varepsilon^0) \\ &= \min_{\substack{u \in V_\sigma \\ u=\varphi \text{ on } \partial\Omega}} \left\{ \begin{array}{c} \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u|^2\, dx + \frac{\varepsilon a_\varepsilon}{3}\int_\sigma \left(\left(\frac{\partial u^+}{\partial \tau}\right)^2 + \left(\frac{\partial u^-}{\partial \tau}\right)^2 + \frac{\partial u^+}{\partial \tau}\frac{\partial u^-}{\partial \tau} \right) ds \\ +\frac{a_\varepsilon}{4\varepsilon}\int_\sigma \left(u^+ - u^-\right)^2\, ds - \int_\Omega f u\, dx \end{array} \right\} \\ &= \min_{\substack{u \in V_\sigma \\ u=\varphi \text{ on } \partial\Omega}} \max_{\substack{\xi \in L^2(\Omega\setminus\sigma)^2 \\ w^+, w^-, z \in L^2(\sigma)}} \left\{ \begin{array}{c} \int_{\Omega\setminus\sigma} \xi \cdot \nabla u\, dx - \frac{1}{2}\int_{\Omega\setminus\sigma} |\xi|^2\, dx \\ +\frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \left(\frac{\partial u^+}{\partial \tau} w^+ + \frac{\partial u^-}{\partial \tau} w^- + \frac{1}{2}\left(\frac{\partial u^+}{\partial \tau} w^- + \frac{\partial u^-}{\partial \tau} w^+ \right)\right) ds \\ -\frac{\varepsilon a_\varepsilon}{3}\int_\sigma \left(w^{+2} + w^{-2} + w^+ w^- \right)\, ds \\ +\frac{a_\varepsilon}{2\varepsilon}\int_\sigma \left(u^+ - u^-\right) z\, ds - \frac{a_\varepsilon}{4\varepsilon}\int_\sigma z^2\, ds - \int_\Omega f u\, dx \end{array} \right\}, \end{aligned}$$

where the maximum in the last expression is achieved uniquely at $\xi = \nabla u$, $w^+ = \frac{\partial u^+}{\partial \tau}$, $w^- = \frac{\partial u^-}{\partial \tau}$ and $z = (u^+ - u^-)$. We can now exchange the min and max in the above formula (see [14]) to rewrite

$$E_\varepsilon^0(u_\varepsilon^0) = \max \left\{ \int_{\partial\Omega} \xi \cdot n \varphi\, ds - \frac{1}{2}\int_{\Omega\setminus\sigma} |\xi|^2\, dx - \frac{\varepsilon a_\varepsilon}{3}\int_\sigma \left(w^{+2} + w^{-2} + w^+ w^- \right)\, ds - \frac{a_\varepsilon}{4\varepsilon}\int_\sigma z^2\, ds \right\}.$$

In this last expression, the maximum is taken over all functions $\xi \in L^2(\Omega \setminus \sigma)^2$, $w^+, w^-, z \in L^2(\sigma)$ such that

$$\begin{array}{rl} -\operatorname{div}(\xi) = f & \text{in } \Omega^+ \cup \Omega^-, \\ \xi^+ \cdot n + \frac{\varepsilon a_\varepsilon}{3}\left(2\frac{\partial w^+}{\partial \tau} + \frac{\partial w^-}{\partial \tau} \right) - \frac{a_\varepsilon}{2\varepsilon} z = 0 & \text{on } \sigma\ , \\ \xi^- \cdot n - \frac{\varepsilon a_\varepsilon}{3}\left(\frac{\partial w^+}{\partial \tau} + 2\frac{\partial w^-}{\partial \tau} \right) - \frac{a_\varepsilon}{2\varepsilon} z = 0 & \text{on } \sigma\ . \end{array} \tag{5.6}$$

We note that, in this particular context, the above exchange of the minimum and maximum can be justified very simply, since the functionals at stake are quadratic and we know explicitly the associated minimizer and maximizer.

This last maximum is achieved uniquely at $\xi = \nabla u_\varepsilon^0$, $w^+ = \frac{\partial u_\varepsilon^{0+}}{\partial \tau}$, $w^- = \frac{\partial u_\varepsilon^{0-}}{\partial \tau}$ and $z = (u_\varepsilon^{0+} - u_\varepsilon^{0-})$. We thus end up with the following convenient alternative expression for the minimum energy $E_\varepsilon^0(u_\varepsilon^0)$

$$\begin{aligned} E_\varepsilon^0(u_\varepsilon^0) = \int_{\partial\Omega} \frac{\partial u_\varepsilon^0}{\partial n}\varphi\, ds - \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u_\varepsilon^0|^2\, dx - \frac{\varepsilon a_\varepsilon}{3}\int_\sigma \left(\left(\frac{\partial u_\varepsilon^{0+}}{\partial \tau}\right)^2 + \left(\frac{\partial u_\varepsilon^{0-}}{\partial \tau}\right)^2 + \frac{\partial u_\varepsilon^{0+}}{\partial \tau}\frac{\partial u_\varepsilon^{0-}}{\partial \tau} \right) ds \\ - \frac{a_\varepsilon}{4\varepsilon}\int_\sigma \left(u_\varepsilon^{0+} - u_\varepsilon^{0-}\right)^2\, ds\ . \end{aligned} \tag{5.7}$$

5.3. **Uniform energy estimates for u_ε^0.**

The following lemma provides preliminary energy estimates for the function u_ε^0.

Lemma 7. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, and σ be a closed $\mathcal{C}^{2,\alpha}$ curve in Ω, lying at positive distance from $\partial\Omega$. Let $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ and $f \in \mathcal{F}_\delta$, for some $\delta > 0$. Then,*

(1) There exists a constant $C > 0$, independent of ε and a_ε (but dependent on Ω and σ) such that

$$||\nabla u_\varepsilon^0||_{L^2(\Omega\setminus\sigma)}+(\varepsilon a_\varepsilon)^{\frac{1}{2}}\left(\left|\left|\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right|\right|_{L^2(\sigma)}+\left|\left|\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right|\right|_{L^2(\sigma)}\right)+\left(\frac{a_\varepsilon}{\varepsilon}\right)^{\frac{1}{2}}\left|\left|u_\varepsilon^{0+}-u_\varepsilon^{0-}\right|\right|_{L^2(\sigma)}$$
$$\leq C\left(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}\right) .$$

(2) There exists a constant $C > 0$ independent of ε and a_ε (but dependent on Ω and σ) such that

$$||u_\varepsilon^0||_{L^2(\Omega^+)}\leq C\left(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}\right) , \quad \textit{and} \quad ||u_\varepsilon^0||_{L_0^2(\Omega^-)}\leq C\left(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}\right) .$$

Proof. (1)*:* By definition of $\varphi \in H^{1/2}(\partial\Omega)$, there exists $u_\varphi \in H^1(\Omega)$ which we may assume to have compact support in $\Omega^+ \setminus \overline{\omega_\delta}$ for some $\delta > 0$, such that $u_\varphi = \varphi$ on $\partial\Omega$ and $||u_\varphi||_{H^1(\Omega)}\leq C||\varphi||_{H^{1/2}(\partial\Omega)}$. The variational formulation of problem (4.10) may be expressed in terms of $w_\varepsilon := u_\varepsilon^0 - u_\varphi$

$$\text{(5.8)}\quad \forall v \in V_{\sigma,0},\ \int_{\Omega\setminus\sigma}\nabla w_\varepsilon\cdot\nabla v\,dx+\frac{2\varepsilon a_\varepsilon}{3}\int_\sigma\left(\frac{\partial w_\varepsilon^+}{\partial\tau}\frac{\partial v^+}{\partial\tau}+\frac{\partial w_\varepsilon^-}{\partial\tau}\frac{\partial v^-}{\partial\tau}+\frac{1}{2}\left(\frac{\partial w_\varepsilon^+}{\partial\tau}\frac{\partial v^-}{\partial\tau}+\frac{\partial w_\varepsilon^-}{\partial\tau}\frac{\partial v^+}{\partial\tau}\right)\right)ds$$
$$+\frac{a_\varepsilon}{2\varepsilon}\int_\sigma(w_\varepsilon^+-w_\varepsilon^-)(v^+-v^-)\,ds=\int_\Omega fv\,dx-\int_{\Omega\setminus\sigma}\nabla u_\varphi\cdot\nabla v\,dx .$$

Inserting $v = w_\varepsilon$ as a test function, and relying on the inequality (5.2), we immediately obtain

$$\text{(5.9)}\quad ||\nabla w_\varepsilon||^2_{L^2(\Omega\setminus\sigma)}+\varepsilon a_\varepsilon\left(\left|\left|\frac{\partial w_\varepsilon^+}{\partial\tau}\right|\right|^2_{L^2(\sigma)}+\left|\left|\frac{\partial w_\varepsilon^-}{\partial\tau}\right|\right|^2_{L^2(\sigma)}\right)+\frac{a_\varepsilon}{\varepsilon}\left|\left|w_\varepsilon^+-w_\varepsilon^-\right|\right|^2_{L^2(\sigma)}\leq$$
$$C\left(||\varphi||_{H^{1/2}(\partial\Omega)}+||f||_{L^2(\Omega^+)}\right)||w_\varepsilon||_{H^1(\Omega^+)}+||f||_{L^2(\Omega^-)}||w_\varepsilon-m||_{L^2(\Omega^-)} ,$$

for any value $m \in \mathbb{R}$ (since $\int_{\Omega^-} f = 0$). Due to the Poincaré inequality for functions on Ω^+ which vanish on $\partial\Omega$, we have

$$||w_\varepsilon||_{H^1(\Omega^+)}\leq C||\nabla w_\varepsilon||_{L^2(\Omega^+)} ,$$

and from the Poincaré-Wirtinger inequality on Ω^-

$$\left|\left|w_\varepsilon-\frac{1}{|\Omega^-|}\int_{\Omega^-}w_\varepsilon\right|\right|_{L^2(\Omega^-)}\leq C||\nabla w_\varepsilon||_{L^2(\Omega^-)} .$$

It follows from a combination of these estimates and (5.9) that

$$||\nabla w_\varepsilon||_{L^2(\Omega\setminus\sigma)}+(\varepsilon a_\varepsilon)^{\frac{1}{2}}\left(\left|\left|\frac{\partial w_\varepsilon^+}{\partial\tau}\right|\right|_{L^2(\sigma)}+\left|\left|\frac{\partial w_\varepsilon^-}{\partial\tau}\right|\right|_{L^2(\sigma)}\right)+\left(\frac{a_\varepsilon}{\varepsilon}\right)^{\frac{1}{2}}\left|\left|w_\varepsilon^+-w_\varepsilon^-\right|\right|_{L^2(\sigma)}$$
$$\text{(5.10)}\qquad \leq C\left(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}\right) .$$

The desired result follows from this estimate and the facts that $u_\varepsilon^0 = w_\varepsilon + u_\varphi$, $||u_\varphi||_{H^1(\Omega)}\leq C||\varphi||_{H^{1/2}(\partial\Omega)}$, and u_φ vanishes on σ.

(2)*:* The first inequality is a consequence of (5.10) and the decomposition $u_\varepsilon^0 = w_\varepsilon + u_\varphi$, combined with the Poincaré inequality for functions on Ω^+ which vanish on $\partial\Omega$. The second inequality similarly follows from (5.10) and the Poincaré-Wirtinger inequality on the domain Ω^-. Note that this latter estimate concerns the $L_0^2(\Omega^-)$ semi-norm, not the $L^2(\Omega^-)$ norm. □

5.4. **Uniform regularity estimates for u_ε^0.**

We now proceed to state the uniform regularity estimates for the function u_ε^0, which we shall require for our later analysis. The results needed are stated in the following Theorem, whose proof is postponed to Appendix A.

Theorem 8. *Assume that Ω and σ are of class $\mathcal{C}^{2,\alpha}$, that the source term f belongs to $\mathcal{F}_\delta$ for some $\delta > 0$, and that $\varphi \in H^{\frac{3}{2}}(\partial\Omega)$. Then the unique solution u_ε^0 to the problem (4.10) belongs to $H^2(\Omega\setminus\sigma)\cap H^2(\sigma)$, and the following estimates hold*

$$|u_\varepsilon^0|_{H^2(\Omega\setminus\sigma)} \leq C(||f||_{L^2(\Omega)} + ||\varphi||_{H^{3/2}(\partial\Omega)}) \ , \tag{5.11}$$

$$(\varepsilon a_\varepsilon)^{\frac{1}{2}} \left(\left|\left| \frac{\partial^2 u_\varepsilon^{0+}}{\partial\tau^2} \right|\right|_{L^2(\sigma)} + \left|\left| \frac{\partial^2 u_\varepsilon^{0-}}{\partial\tau^2} \right|\right|_{L^2(\sigma)} \right) + \left(\frac{a_\varepsilon}{\varepsilon}\right)^{\frac{1}{2}} \left|\left| \frac{\partial u_\varepsilon^{0+}}{\partial\tau} - \frac{\partial u_\varepsilon^{0-}}{\partial\tau} \right|\right|_{L^2(\sigma)} \leq C(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}) \ , \tag{5.12}$$

where $|u|_{H^2(V)} := \left(\sum_{\substack{\beta\in\mathbb{N}^2 \\ |\beta|=2}} \left|\left| \frac{\partial^{|\beta|}u}{\partial x^\beta} \right|\right|^2_{L^2(V)} \right)^{1/2}$ stands for the H^2 semi norm of a function $u \in H^2(V)$, and the constant C depends only on Ω and σ (and not on ε and a_ε).

Remark 8.

(1) The proof of Theorem 8 can be iterated, if one assumes higher regularity of Ω, σ, f and φ. More precisely, if Ω and σ are of class $\mathcal{C}^{m,\alpha}$, $f \in \mathcal{F}_\delta \cap H^{m-2}(\Omega)$ and $\varphi \in H^{m-\frac{1}{2}}(\partial\Omega)$ for some $m \geq 2$, then
$$\left|\left| \frac{\partial^{|\beta|} u_\varepsilon^0}{\partial x^\beta} \right|\right|_{L^2(\Omega\setminus\sigma)} \leq C(||f||_{H^{m-2}(\Omega)} + ||\varphi||_{H^{m-\frac{1}{2}}(\partial\Omega)}) \ ,$$
for any multi-index β of length $\leq m$. Note also that these results are local. Thus, even if f only belongs to $\mathcal{F}_\delta$, for some $\delta > 0$, but σ is a $\mathcal{C}^{m,\alpha}$ curve, then u_ε^0 is of class $\mathcal{C}^m(V\setminus\sigma)$ for any open set V such that $\overline{V} \Subset \omega_\delta$, and
$$\left|\left| \frac{\partial^{|\beta|} u_\varepsilon^0}{\partial x^\beta} \right|\right|_{L^2(V\setminus\sigma)} \leq C(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}) \ ,$$
for any multi-index β of length $\leq m$.

(2) The two estimates (5.11) and (5.12) are of a quite different nature; they are complementary in the sense that, depending on the behavior of the sequence a_ε, one may prove more precise than the other. Estimate (5.11) expresses the fact that all the derivatives of u_ε^0 are uniformly bounded with respect to ε and a_ε, provided that the data of the problem have enough regularity. On the other hand, the estimate (5.12) is analogous to the preliminary estimates of Lemma 7: it does not carry much information in the low conductivity regime (i.e., $a_\varepsilon \ll \varepsilon$), but it is in some sense much stronger than (5.11) in the high conductivity regime (i.e., $a_\varepsilon \gg \varepsilon$).

(3) Recall that, due to Lemma 7, $u_\varepsilon^0|_{\Omega^+}$ (and not just its derivatives) also turns out to be uniformly bounded with respect to ε and a_ε. However, in general, this is not the case of $u_\varepsilon^0|_{\Omega^-}$, which is only uniformly bounded *up to a constant.*

6. Proof of the asymptotic exactness of u_ε^0

We are now in position to verify the asymptotic exactness of u_ε^0, in other words to show that the gap $||u_\varepsilon - u_\varepsilon^0||$ tends to zero as ε tends to zero. The precise estimate we establish is the following

Theorem 9. *Assume the "center" curve σ is of class $\mathcal{C}^\infty$, and that $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$. Let $\delta > 0$ be a fixed positive real number, and suppose $f \in \mathcal{F}_\delta$. Let $u_\varepsilon \in H^1(\Omega)$ (resp. $u_\varepsilon^0 \in V_\sigma$) be the unique solution to the minimization problem (4.1) (resp. (4.10)). Then the following estimates hold, for $\varepsilon > 0$ sufficiently small*

$$||u_\varepsilon - u_\varepsilon^0||_{L^2(\Omega^+\setminus\overline{\omega_\delta})} \leq C\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)\varepsilon,$$
$$||u_\varepsilon - u_\varepsilon^0||_{L^2_0(\Omega^-\setminus\overline{\omega_\delta})} \leq C\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)\varepsilon,$$

where the constant C is independent of ε, and of a_ε.

Proof. The technique used here is very close to that used in [21] – a main idea of which is already found in [20] – it relies on two key ingredients:

- The uniform energy and regularity estimates for u^0_ε and ∇u^0_ε presented in Section 5.3 and Section 5.4. Interestingly enough, neither energy nor regularity estimates for the exact solution u_ε are required.
- The general argument of Lemma 4, which controls the discrepancy between u_ε and u^0_ε in terms of the discrepancy between the minimum values of the corresponding energies E_ε and E^0_ε.

Using the notation of Lemma 4, we choose $V_\varepsilon = H^1(\Omega)$, $W_\varepsilon = V_\sigma$, and $H = \mathcal{F}_\delta$ (and we identify H' with $\mathcal{F}_\delta$). The natural mapping $P_\varepsilon \; : \; V_\varepsilon \to H$ is

$$H^1(\Omega) \ni u \mapsto P_\varepsilon u = \begin{cases} u|_{\Omega^+\setminus\overline{\omega_\delta}} & \text{in } \Omega^+ \setminus \overline{\omega_\delta} \\ 0 & \text{in } \omega_\delta \\ u|_{\Omega^-\setminus\overline{\omega_\delta}} - \frac{1}{|\Omega^-\setminus\overline{\omega_\delta}|}\int_{\Omega^-\setminus\overline{\omega_\delta}} u \, dx & \text{in } \Omega^- \setminus \overline{\omega_\delta} \end{cases} \in \mathcal{F}_\delta \; .$$

The operator P_ε (which, like V_ε and W_ε, in this case actually does not depend on ε) also naturally maps W_ε into H. According to Lemma 4 (and Remark 6) the following estimates hold

$$(6.1) \qquad ||u_\varepsilon - u^0_\varepsilon||_{L^2(\Omega^+\setminus\overline{\omega_\delta})} \leq C \left(\sup_{\substack{f\in\mathcal{F}_\delta,\ \varphi\in H^{1/2}(\partial\Omega) \\ f,\varphi \text{ smooth}}} \frac{|E_\varepsilon(u_\varepsilon) - E^0_\varepsilon(u^0_\varepsilon)|}{\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)^2} \right) \left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right) \; ,$$

$$(6.2) \qquad ||u_\varepsilon - u^0_\varepsilon||_{L^2_0(\Omega^-\setminus\overline{\omega_\delta})} \leq C \left(\sup_{\substack{f\in\mathcal{F}_\delta,\ \varphi\in H^{1/2}(\partial\Omega) \\ f,\varphi \text{ smooth}}} \frac{|E_\varepsilon(u_\varepsilon) - E^0_\varepsilon(u^0_\varepsilon)|}{\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)^2} \right) \left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right) \; .$$

The idea is then to estimate the discrepancy $(E_\varepsilon(u_\varepsilon) - E^0_\varepsilon(u^0_\varepsilon))$ between the minimum values of the energies by using particular 'test functions' in place of u_ε (or its gradient) which make E_ε (or its dual) mimic the behavior of the functional E^0_ε near the limiting curve σ. The existence of such test functions is made possible by the regularity estimates for u^0_ε stated the Section 5.3. Sections 6.1 and 6.2 below are devoted to establishing the desired control over this energy discrepancy.

□

In the following, for the sake of brevity, we denote by C a constant, possibly changing from one instance to the other, which only depends on Ω and σ, but is independent of $\varepsilon, a_\varepsilon, f$ and φ. We also use the shorthand

$$C(f,\varphi) \equiv C\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right) \; .$$

6.1. Proof of the upper bound $E_\varepsilon(u_\varepsilon) - E^0_\varepsilon(u^0_\varepsilon) \leq C(f,\varphi)^2\varepsilon$.

As a straightforward consequence of the definition (4.1), one has, for any function $u \in H^1(\Omega)$ such that $u = \varphi$ on $\partial\Omega$,

$$E_\varepsilon(u_\varepsilon) - E^0_\varepsilon(u^0_\varepsilon) \leq E_\varepsilon(u) - E^0_\varepsilon(u^0_\varepsilon) \; .$$

We proceed to construct a "test function" u which makes the right hand side of the above inequality small. To this end, a natural idea is to exploit the equivalent form (4.4) of the problem, and use the pair $(u^0_\varepsilon, v^0_\varepsilon \circ H_\varepsilon^{-1})$ as a test function, where u^0_ε is the unique solution to (4.10), and v^0_ε is given by (4.13). This is unfortunately not possible as is, since the pair $(u^0_\varepsilon, v^0_\varepsilon)$ does not belong to the space $\overline{V^0_\varepsilon}$; indeed, it does not satisfy the boundary conditions

$$\forall x \in \sigma \; \begin{cases} v(x + n(x)) = u(x + \varepsilon n(x)) \\ v(x - n(x)) = u(x - \varepsilon n(x)) \end{cases} \; ,$$

but satisfies instead

$$\forall x \in \sigma \; \begin{cases} v(x + n(x)) = u^+(x) \\ v(x - n(x)) = u^-(x) \end{cases} \; .$$

To remedy this, let us define $z_\varepsilon \in H^1(\Omega \setminus \overline{\omega_\varepsilon})$ as the unique solution to

$$\begin{cases} -\Delta z_\varepsilon = 0 & \text{in } \Omega \setminus \overline{\omega_\varepsilon} \\ z_\varepsilon = 0 & \text{on } \partial\Omega \\ z_\varepsilon = u_\varepsilon^{0+} \circ p_\sigma - u_\varepsilon^0 & \text{on } \partial\omega_\varepsilon^+ \\ z_\varepsilon = u_\varepsilon^{0-} \circ p_\sigma - u_\varepsilon^0 & \text{on } \partial\omega_\varepsilon^- \end{cases} .$$

By construction, the pair $(u_\varepsilon^0 + z_\varepsilon, v_\varepsilon^0)$ belongs to $\overline{V_\varepsilon^0}$. Let us now work toward estimating the function z_ε; as an easy consequence of definitions,

$$\begin{aligned} ||z_\varepsilon|_{\partial\omega_\varepsilon}||_{\mathcal{C}^1(\partial\omega_\varepsilon)} &\leq C\varepsilon \left(||u_\varepsilon^0||_{\mathcal{C}^2(V^+)} + ||u_\varepsilon^0 - m||_{\mathcal{C}^2(V^-)} \right), \\ &\leq C\varepsilon \left(||u_\varepsilon^0||_{H^4(V^+)} + ||u_\varepsilon^0 - m||_{H^4(V^-)} \right) . \end{aligned}$$

Here V is a neighboorhood contained in ω_δ, for a fixed δ with $f \in \mathcal{F}_\delta$ and $m = \frac{1}{|\Omega^-|}\int_{\Omega^-} u_\varepsilon^0$. According to Theorem 8 (and Remark 8), it follows that

$$||z_\varepsilon|_{\partial\omega_\varepsilon}||_{\mathcal{C}^1(\partial\omega_\varepsilon)} \leq C(f,\varphi)\varepsilon .$$

By a very simple construction we may extend the trace $z_\varepsilon|_{\partial\omega_\varepsilon}$ to a function Z_ε defined on the whole domain $\Omega \setminus \overline{\omega_\varepsilon}$ with $Z_\varepsilon = 0$ on $\partial\Omega$ and

$$||Z_\varepsilon||_{\mathcal{C}^1(\Omega\setminus\overline{\omega_\varepsilon})} \leq C||z_\varepsilon||_{\mathcal{C}^1(\partial\omega_\varepsilon)} \leq C(f,\varphi)\varepsilon .$$

A simple calculation gives that

$$\int_{\Omega\setminus\overline{\omega_\varepsilon}} \nabla(z_\varepsilon - Z_\varepsilon)\nabla z_\varepsilon \, dx = 0 ;$$

in other words

$$\int_{\Omega\setminus\overline{\omega_\varepsilon}} |\nabla z_\varepsilon|^2 \, dx = \int_{\Omega\setminus\overline{\omega_\varepsilon}} \nabla Z_\varepsilon \nabla z_\varepsilon \, dx ,$$

and so

$$||\nabla z_\varepsilon||_{L^2(\Omega\setminus\overline{\omega_\varepsilon})} \leq ||\nabla Z_\varepsilon||_{L^2(\Omega\setminus\overline{\omega_\varepsilon})} \leq C||Z_\varepsilon||_{\mathcal{C}^1(\Omega\setminus\overline{\omega_\varepsilon})} \leq C(f,\varphi)\varepsilon . \tag{6.3}$$

Now, using the pair $(u_\varepsilon^0 + z_\varepsilon, v_\varepsilon^0)$ as a "test function" in (4.4), we calculate:

$$\begin{aligned} \overline{F_\varepsilon^0}(u_\varepsilon^0 + z_\varepsilon, v_\varepsilon^0) = \frac{1}{2}\int_{\Omega\setminus\overline{\omega_\varepsilon}} |\nabla u_\varepsilon^0 + \nabla z_\varepsilon|^2 \, dx - \int_\Omega f(u_\varepsilon^0 + z_\varepsilon) \, dx \\ + \frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1} \frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}} \left(\frac{\partial v_\varepsilon^0}{\partial \tau}\right)^2 dx + \frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1} \frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}} \left(\frac{\partial v_\varepsilon^0}{\partial n}\right)^2 dx . \end{aligned}$$

Here

$$\begin{aligned} \int_{\Omega\setminus\overline{\omega_\varepsilon}} |\nabla u_\varepsilon^0 + \nabla z_\varepsilon|^2 \, dx &= \int_{\Omega\setminus\overline{\omega_\varepsilon}} |\nabla u_\varepsilon^0|^2 \, dx + 2\int_{\Omega\setminus\overline{\omega_\varepsilon}} \nabla u_\varepsilon^0 \cdot \nabla z_\varepsilon \, dx + \int_{\Omega\setminus\overline{\omega_\varepsilon}} |\nabla z_\varepsilon|^2 \, dx \\ &\leq \int_{\Omega\setminus\overline{\omega_\varepsilon}} |\nabla u_\varepsilon^0|^2 \, dx + C(f,\varphi)^2\varepsilon , \end{aligned}$$

where we used (6.3) and the uniform energy estimate of Lemma 7. Similarly, one has

$$\left| \int_\Omega f z_\varepsilon \, dx \right| \leq C(f,\varphi)^2\varepsilon ,$$

because of our assumptions about f, and the estimate (6.3), in combination with the fact that z_ε vanishes on $\partial\Omega$. Concerning the terms on ω_1,

$$\begin{aligned} \frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1} \frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}} \left(\frac{\partial v_\varepsilon^0}{\partial\tau}\right)^2 dx &\leq \frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1} (1+\kappa d_{\Omega^-}) \left(\frac{\partial v_\varepsilon^0}{\partial\tau}\right)^2 dx + C(f,\varphi)^2\varepsilon \\ &= \frac{\varepsilon a_\varepsilon}{3}\int_\sigma \left(\left(\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right)^2 + \left(\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right)^2 + \frac{\partial u_\varepsilon^{0+}}{\partial\tau}\frac{\partial u_\varepsilon^{0-}}{\partial\tau} \right) ds + C(f,\varphi)^2\varepsilon , \end{aligned}$$

where the first line is a consequence of the uniform energy estimates of Lemma 7, and the second line follows by the exact same calculation that we performed in Section 4.1.1. Similarly, we obtain

$$\frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\left(\frac{\partial v_\varepsilon^0}{\partial n}\right)^2\,dx \le \frac{a_\varepsilon}{4\varepsilon}\int_\sigma\left(u_\varepsilon^{0+}-u_\varepsilon^{0-}\right)^2\,ds + C(f,\varphi)^2\varepsilon\ .$$

To conclude, let $\overline{u}\in H^1(\Omega)$ denote the function

$$\overline{u} = \begin{cases} u_\varepsilon^0 + z_\varepsilon\ , & \text{in }\ \Omega\setminus\omega_\varepsilon\ ,\\ v_\varepsilon^0\circ H_\varepsilon^{-1}\ , & \text{in }\ \omega_\varepsilon\ .\end{cases}$$

Combining all these estimates, we finally get

$$E_\varepsilon(u_\varepsilon) - E_\varepsilon^0(u_\varepsilon^0) \le E_\varepsilon(\overline{u}) - E_\varepsilon^0(u_\varepsilon^0) = \overline{F_\varepsilon^0}(u_\varepsilon^0+z_\varepsilon, v_\varepsilon^0) - E_\varepsilon^0(u_\varepsilon^0) \le C(f,\varphi)^2\varepsilon\ .$$

6.2. **Proof of the lower bound: $E_\varepsilon^0(u_\varepsilon^0) - E_\varepsilon(u_\varepsilon) \le C(f,\varphi)^2\varepsilon$, and end of proof of Theorem 9.**

In order to prove the lower bound, we rely on the use of the dual energies associated to E_ε and E_ε^0. More precisely, based on the equivalent, rescaled form (4.15) of the dual problem to E_ε,

$$E_\varepsilon^0(u_\varepsilon^0) - E_\varepsilon(u_\varepsilon) \le E_\varepsilon^0(u_\varepsilon^0) - \overline{F_\varepsilon^{c0}}(\xi,\eta),$$

for every vector couple (ξ,η) in the space $\overline{V_\varepsilon^{c0}}$ defined by (4.16), and satisfying $-\mathrm{div}(\xi)=f$, $-\mathrm{div}(\eta)=0$. Using the definition of $\overline{F_\varepsilon^{c0}}$ and the alternative expression (5.7) for $E_\varepsilon^0(u_\varepsilon^0)$, we may rewrite this as

$$\begin{aligned}(6.4)\quad E_\varepsilon^0(u_\varepsilon^0) - E_\varepsilon(u_\varepsilon) \le{}& \frac12\int_{\Omega\setminus\overline{\omega_\varepsilon}}|\xi|^2\,dx + \frac{1}{2\varepsilon a_\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\eta_\tau^2\,dx + \frac{\varepsilon}{2a_\varepsilon}\int_{\omega_1}\frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\eta_n^2\,dx\\ &-\int_{\partial\Omega}\xi\cdot n\varphi\,ds + \int_{\partial\Omega}\frac{\partial u_\varepsilon^0}{\partial n}\varphi\,ds - \frac12\int_{\Omega\setminus\sigma}|\nabla u_\varepsilon^0|^2\,dx\\ &-\frac{\varepsilon a_\varepsilon}{3}\int_\sigma\left(\left(\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right)^2 + \left(\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right)^2 + \frac{\partial u_\varepsilon^{0+}}{\partial\tau}\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right)ds - \frac{a_\varepsilon}{4\varepsilon}\int_\sigma\left(u_\varepsilon^{0+}-u_\varepsilon^{0-}\right)^2\,ds\ .\end{aligned}$$

In light of the discussions in Section 4.1.2 and 5.2, and particularly due to the formulas (4.21), it is tempting to define a test flux $\xi\in H_{\mathrm{div}}(\Omega\setminus\sigma)$ by $\xi=\nabla u_\varepsilon^0$, and $\eta\in H_{\mathrm{div}}(\omega_1)$ in such a way that, for $x\in\sigma$, $t\in(-1,1)$

$$\frac{\partial}{\partial\tau}(\eta_\tau(x+tn(x))) = -\frac12\left[\frac{\partial u_\varepsilon^0}{\partial n}\right](x),\ \text{and}$$

$$\eta_n(x+tn(x)) = \frac12\left(\frac{t}{1+t\kappa(x)}\left[\frac{\partial u_\varepsilon^0}{\partial n}\right](x) + \frac{1}{1+t\kappa(x)}\left(\frac{\partial u_\varepsilon^{0+}}{\partial n}(x) + \frac{\partial u_\varepsilon^{0-}}{\partial n}(x)\right)\right),$$

and insert (ξ,η) into (6.4). Using the pointwise expression (5.4) for the boundary conditions for u_ε^0, we are led to

$$\begin{pmatrix}\eta_\tau(x+tn(x))\\ \eta_n(x+tn(x))\end{pmatrix} = \begin{pmatrix}\frac{\varepsilon a_\varepsilon}{2}\left(\frac{\partial u_\varepsilon^{0+}}{\partial\tau}+\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right)\\ \frac12\frac{1}{1+t\kappa}\left(-t\varepsilon a_\varepsilon\left(\frac{\partial^2 u_\varepsilon^{0+}}{\partial\tau^2}+\frac{\partial^2 u_\varepsilon^{0-}}{\partial\tau^2}\right) - \frac{\varepsilon a_\varepsilon}{3}\left(\frac{\partial^2 u_\varepsilon^{0+}}{\partial\tau^2}-\frac{\partial^2 u_\varepsilon^{0-}}{\partial\tau^2}\right) + \frac{a_\varepsilon}{\varepsilon}(u_\varepsilon^{0+}-u_\varepsilon^{0-})\right)\end{pmatrix}.$$

Unfortunately, such a choice of "test couple" is not admissible, since it does not belong to the space $\overline{V_\varepsilon^{c0}}$. Nevertheless, it "almost" belongs to this space, and we may use a "small" additive correction to remedy that situation. We define $z_\varepsilon\in H^1(\Omega\setminus\overline{\omega_\varepsilon})$ as the unique solution (up to a constant) to the problem

$$\begin{cases} -\Delta z_\varepsilon = 0 & \text{in }\Omega\setminus\overline{\omega_\varepsilon}\ ,\\ \frac{\partial z_\varepsilon}{\partial n} = 0 & \text{on }\partial\Omega\ ,\\ \frac{\partial z_\varepsilon}{\partial n} = g_\varepsilon^+ & \text{on }\partial\omega_\varepsilon^+\ ,\\ \frac{\partial z_\varepsilon}{\partial n} = g_\varepsilon^- & \text{on }\partial\omega_\varepsilon^-\ .\end{cases}$$

Recall that in the last two boundary conditions, n stands for the normal vector to $\partial\omega_\varepsilon^\pm$, oriented in the direction from Ω^- to Ω^+. The function g_ε^+ is defined by

$$\begin{aligned}\forall x\in\sigma,\ \ g_\varepsilon^+(x+\varepsilon n(x)) &= \tfrac{1+\kappa(x)}{1+\varepsilon\kappa(x)}\eta_n(x+n(x))-\xi_n(x+\varepsilon n(x))\\ &= (1+\kappa(x))\left(\tfrac{1}{1+\varepsilon\kappa(x)}-1\right)\eta_n(x+n(x))\\ &\quad +(1+\kappa(x))\eta_n(x+n(x))-\xi_n^+(x)+\xi_n^+(x)-\xi_n(x+\varepsilon n(x))\\ &= (1+\kappa(x))\left(\tfrac{1}{1+\varepsilon\kappa(x)}-1\right)\eta_n(x+n(x))+\xi_n^+(x)-\xi_n(x+\varepsilon n(x))\ ,\end{aligned}$$

and g_ε^- is defined by the similar formula

$$\begin{aligned}\forall x\in\sigma,\ \ g_\varepsilon^-(x-\varepsilon n(x)) &= \tfrac{1-\kappa(x)}{1-\varepsilon\kappa(x)}\eta_n(x-n(x))-\xi_n(x-\varepsilon n(x))\\ &= (1-\kappa(x))\left(\tfrac{1}{1-\varepsilon\kappa(x)}-1\right)\eta_n(x-n(x))\\ &\quad +(1-\kappa(x))\eta_n(x-n(x))-\xi_n^-(x)+\xi_n^-(x)-\xi_n(x-\varepsilon n(x))\\ &= (1-\kappa(x))\left(\tfrac{1}{1-\varepsilon\kappa(x)}-1\right)\eta_n(x-n(x))+\xi_n^-(x)-\xi_n(x-\varepsilon n(x))\ ,\end{aligned}$$

so that the couple $(\xi+\nabla z_\varepsilon,\eta)$ belongs to $\overline{V_\varepsilon^{c0}}$. The requirement that $\int_{\partial\omega_\varepsilon^+} g_\varepsilon^+\,ds=\int_{\partial\omega_\varepsilon^-} g_\varepsilon^-\,ds=0$ is guaranteed by the identity (5.5) and the fact that f vanishes in ω_ε, so that $\int_{\partial\omega_\varepsilon^\pm}\frac{\partial u_\varepsilon^0}{\partial n}\,ds=0$ as well. Using the uniform regularity estimates of Theorem 8 (and Remark 8), we obtain that

$$||g_\varepsilon^\pm||_{\mathcal{C}^{1,\alpha}(\partial\omega_\varepsilon^\pm)}\le C(f,\varphi)\varepsilon,$$

and a standard regularity argument (as for the Dirichlet problem in the previous section) now gives

$$||\nabla z_\varepsilon||_{L^2(\Omega\setminus\overline{\omega_\varepsilon})}\ \le\ C\left(||g_\varepsilon^+||_{\mathcal{C}^{1,\alpha}(\partial\omega_\varepsilon^+)}+||g_\varepsilon^-||_{\mathcal{C}^{1,\alpha}(\partial\omega_\varepsilon^-)}\right)\ \le\ C(f,\varphi)\varepsilon\ .$$

It is now possible to use $(\xi+\nabla z_\varepsilon,\eta)$ as a test couple in (6.4). Doing so, we obtain first

$$\begin{aligned}\tfrac{1}{2}\int_{\Omega\setminus\overline{\omega_\varepsilon}}|\xi+\nabla z_\varepsilon|^2\,dx &= \frac{1}{2}\int_{\Omega\setminus\overline{\omega_\varepsilon}}|\nabla u_\varepsilon^0|^2\,dx+\int_{\Omega\setminus\overline{\omega_\varepsilon}}\nabla u_\varepsilon^0\cdot\nabla z_\varepsilon\,dx+\frac{1}{2}\int_{\Omega\setminus\overline{\omega_\varepsilon}}|\nabla z_\varepsilon|^2\,dx\\ &\le \frac{1}{2}\int_{\Omega\setminus\overline{\omega_\varepsilon}}|\nabla u_\varepsilon^0|^2\,dx+C(f,\varphi)^2\varepsilon\ ,\end{aligned}\tag{6.5}$$

and

$$\int_{\partial\Omega}(\xi+\nabla z_\varepsilon)\cdot n\varphi\,ds=\int_{\partial\Omega}\frac{\partial u_\varepsilon^0}{\partial n}\varphi\,ds\ .\tag{6.6}$$

Besides,

$$\begin{aligned}\frac{1}{2\varepsilon a_\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\eta_\tau^2\,dx &\le (1+C\varepsilon)\frac{1}{2\varepsilon a_\varepsilon}\int_{\omega_1}\frac{1}{1+\kappa d_{\Omega^-}}\eta_\tau^2\,dx\\ &= (1+C\varepsilon)\frac{1}{2\varepsilon a_\varepsilon}\int_\sigma\int_{-1}^1\eta_\tau^2(x+tn(x))\,dt\,ds\\ &= (1+C\varepsilon)\frac{\varepsilon a_\varepsilon}{4}\int_\sigma\left(\frac{\partial u_\varepsilon^{0+}}{\partial\tau}+\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right)^2\,ds\\ &\le (1+C\varepsilon)\frac{\varepsilon a_\varepsilon}{3}\int_\sigma\left(\left(\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right)^2+\left(\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right)^2+\left(\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right)\left(\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right)\right)\,ds\ ,\end{aligned}$$

where for the last estimate we used the algebraic inequality

$$\forall a,b\in\mathbb{R},\ \ \frac{1}{4}(a+b)^2=\frac{1}{3}(a^2+b^2+ab)-\frac{1}{12}(a-b)^2\le\frac{1}{3}(a^2+b^2+ab)\ .$$

Using the uniform energy estimates of Lemma 7, we conclude

$$\frac{1}{2\varepsilon a_\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\eta_\tau^2\,dx\le\frac{\varepsilon a_\varepsilon}{3}\int_\sigma\left(\left(\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right)^2+\left(\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right)^2+\left(\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right)\left(\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right)\right)\,ds+C(f,\varphi)^2\varepsilon\ .\tag{6.7}$$

On the other hand,

$$
\begin{aligned}
&\frac{\varepsilon}{2a_\varepsilon}\int_{\omega_1}\frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\eta_n^2\,dx\\
&\quad\leq\ (1+C\varepsilon)\frac{\varepsilon}{2a_\varepsilon}\int_{\omega_1}(1+\kappa d_{\Omega^-})\eta_n^2\,dx\\
&\quad=\ (1+C\varepsilon)\frac{\varepsilon}{2a_\varepsilon}\int_\sigma\int_{-1}^{1}(1+t\kappa(x))^2\eta_n^2(x+tn(x))\,dt\,ds\\
&\quad=\ (1+C\varepsilon)\frac{\varepsilon}{8a_\varepsilon}\int_\sigma\int_{-1}^{1}\left(-t\varepsilon a_\varepsilon\left(\frac{\partial^2u_\varepsilon^{0+}}{\partial\tau^2}+\frac{\partial^2u_\varepsilon^{0-}}{\partial\tau^2}\right)-\frac{\varepsilon a_\varepsilon}{3}\left(\frac{\partial^2u_\varepsilon^{0+}}{\partial\tau^2}-\frac{\partial^2u_\varepsilon^{0-}}{\partial\tau^2}\right)+\frac{a_\varepsilon}{\varepsilon}(u_\varepsilon^{0+}-u_\varepsilon^{0-})\right)^2dt\,ds\\
&\quad=\ (1+C\varepsilon)\frac{\varepsilon^3a_\varepsilon}{12}\int_\sigma\left(\frac{\partial^2u_\varepsilon^{0+}}{\partial\tau^2}+\frac{\partial^2u_\varepsilon^{0-}}{\partial\tau^2}\right)^2ds+(1+C\varepsilon)\frac{\varepsilon}{4a_\varepsilon}\int_\sigma\left(-\frac{\varepsilon a_\varepsilon}{3}\left(\frac{\partial^2u_\varepsilon^{0+}}{\partial\tau^2}-\frac{\partial^2u_\varepsilon^{0-}}{\partial\tau^2}\right)+\frac{a_\varepsilon}{\varepsilon}(u_\varepsilon^{0+}-u_\varepsilon^{0-})\right)^2ds\\
&\quad=\ (1+C\varepsilon)\frac{\varepsilon^3a_\varepsilon}{12}\int_\sigma\left(\frac{\partial^2u_\varepsilon^{0+}}{\partial\tau^2}+\frac{\partial^2u_\varepsilon^{0-}}{\partial\tau^2}\right)^2ds+(1+C\varepsilon)\frac{\varepsilon^3a_\varepsilon}{36}\int_\sigma\left(\frac{\partial^2u_\varepsilon^{0+}}{\partial\tau^2}-\frac{\partial^2u_\varepsilon^{0-}}{\partial\tau^2}\right)^2ds\\
&\qquad-(1+C\varepsilon)\frac{\varepsilon a_\varepsilon}{6}\int_\sigma\left(\frac{\partial^2u_\varepsilon^{0+}}{\partial\tau^2}-\frac{\partial^2u_\varepsilon^{0-}}{\partial\tau^2}\right)(u_\varepsilon^{0+}-u_\varepsilon^{0-})\,ds+(1+C\varepsilon)\frac{a_\varepsilon}{4\varepsilon}\int_\sigma(u_\varepsilon^{0+}-u_\varepsilon^{0-})^2\,ds\ .
\end{aligned}
$$

Due to the uniform energy estimates of Theorem 8, the first two integrals in the last expression are easily controlled by $C(f,\varphi)^2\varepsilon^2$. When it comes to the third integral, one has

$$
\begin{aligned}
\left|\frac{\varepsilon a_\varepsilon}{6}\int_\sigma\left(\frac{\partial^2u_\varepsilon^{0+}}{\partial\tau^2}-\frac{\partial^2u_\varepsilon^{0-}}{\partial\tau^2}\right)(u_\varepsilon^{0+}-u_\varepsilon^{0-})\,ds\right|\ &\leq\ \frac{\varepsilon a_\varepsilon}{6}\left(\int_\sigma\left(\frac{\partial^2u_\varepsilon^{0+}}{\partial\tau^2}-\frac{\partial^2u_\varepsilon^{0-}}{\partial\tau^2}\right)^2ds\right)^{\frac12}\left(\int_\sigma(u_\varepsilon^{0+}-u_\varepsilon^{0-})^2\,ds\right)^{\frac12}\\
&\leq\ \frac{\varepsilon}{6}\left(\varepsilon a_\varepsilon\int_\sigma\left(\frac{\partial^2u_\varepsilon^{0+}}{\partial\tau^2}-\frac{\partial^2u_\varepsilon^{0-}}{\partial\tau^2}\right)^2ds\right)^{\frac12}\left(\frac{a_\varepsilon}{\varepsilon}\int_\sigma(u_\varepsilon^{0+}-u_\varepsilon^{0-})^2\,ds\right)^{\frac12}\\
&\leq\ C(f,\varphi)^2\varepsilon\ ,
\end{aligned}
$$

since the integral terms in the product are each bounded by $C(f,\varphi)$. We thus obtain the estimate

$$
\begin{aligned}
\frac{\varepsilon}{2a_\varepsilon}\int_{\omega_1}\frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\eta_n^2\,dx\ &\leq\ (1+C\varepsilon)\frac{a_\varepsilon}{4\varepsilon}\int_\sigma(u_\varepsilon^{0+}-u_\varepsilon^{0-})^2\,ds+C(f,\varphi)^2\varepsilon\\
&\leq\ \frac{a_\varepsilon}{4\varepsilon}\int_\sigma(u_\varepsilon^{0+}-u_\varepsilon^{0-})^2\,ds+C(f,\varphi)^2\varepsilon\ ,
\end{aligned}
\tag{6.8}
$$

where we have again made use of the uniform energy estimate in Lemma 7. Application of the auxiliary estimates (6.5)-(6.8) to (6.4) with the test couple $(\xi+\nabla z_\varepsilon,\eta)$ finally yields

$$E_\varepsilon^0(u_\varepsilon^0)-E_\varepsilon(u_\varepsilon)\leq C(f,\varphi)^2\varepsilon\ ,$$

which is the desired lower bound on $E_\varepsilon(u_\varepsilon)-E_\varepsilon^0(u_\varepsilon^0)$.

End of proof of Theorem 9. By a combination of the upper bound of the previous subsection and the lower bound of this subsection we obtain

$$-C(f,\varphi)^2\varepsilon\leq E_\varepsilon(u_\varepsilon)-E_\varepsilon^0(u_\varepsilon^0)\leq C(f,\varphi)^2\varepsilon\ ,$$

or

$$\left|E_\varepsilon(u_\varepsilon)-E_\varepsilon^0(u_\varepsilon^0)\right|\leq C\left(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}\right)^2\varepsilon\ .$$

Insertion of this estimate into (6.1) and (6.2) now finally gives

$$||u_\varepsilon-u_\varepsilon^0||_{L^2(\Omega^+\setminus\overline{\omega_\delta})}\leq C\left(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}\right)\varepsilon\ ,$$

$$||u_\varepsilon-u_\varepsilon^0||_{L^2_0(\Omega^-\setminus\overline{\omega_\delta})}\leq C\left(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}\right)\varepsilon\ ,$$

and this completes the proof of Theorem 9. □

Remark 9. The 0^{th} order uniform approximation to u_ε is only unique modulo a function that is of the order $\mathcal{O}(\varepsilon)$, uniformly in ε and a_ε. As a reflection of this, the energetic expression E_ε^0 (of (4.12) is not unique either; a proof very similar to the one presented above (together with corresponding uniform regularity and energy estimates) would reveal that the unique minimizer to

$$\widetilde{E^0_\varepsilon}(v) := \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla v|^2\, dx + \frac{\varepsilon a_\varepsilon}{2}\int_\sigma \left(\left(\frac{\partial v^+}{\partial \tau}\right)^2 + \left(\frac{\partial v^-}{\partial \tau}\right)^2 \right) ds + \frac{a_\varepsilon}{4\varepsilon}\int_\sigma (v^+ - v^-)^2\, ds - \int_\Omega f v\, dx$$

is also a uniform 0^{th} order approximation of u_ε.

7. Limit behavior of u^0_ε

So far, we have only discussed the approximation of u_ε in terms of the solution u^0_ε to another, simpler minimization problem, which, however, still depends on ε and a_ε. When the behavior of the sequence a_ε is known more precisely as $\varepsilon \to 0$, then explicit, ε and a_ε independent limit behaviors of u^0_ε (and thus of u_ε) can be derived.

7.1. The general case.

Let us assume that $\varepsilon a_\varepsilon$ and $\frac{a_\varepsilon}{\varepsilon}$ both have a limit as $\varepsilon \to 0$, including possible limits of 0 and ∞. Remark that, in the general case, there always exists a subsequence $\varepsilon_n \to 0$ such that this is achieved. Since $\varepsilon a_\varepsilon \ll \frac{a_\varepsilon}{\varepsilon}$ the limiting pair $\left(\lim_{\varepsilon\to 0} \varepsilon a_\varepsilon, \lim_{\varepsilon \to 0}\frac{a_\varepsilon}{\varepsilon}\right)$ has one of the five possible forms (∞,∞), (a_0,∞), $(0,\infty)$, $(0,b_0)$, and $(0,0)$, where $0 < a_0 < \infty$ and $0 < b_0 < \infty$ are arbitrary constants. The following result describes the precise limiting behaviour of u^0_ε (and thus of u_ε) in each of these five cases.

Proposition 10. *Let a_ε be any sequence of positive real numbers, and $u^0_\varepsilon \in V_\sigma$ be the unique solution to the minimization problem (4.10). Suppose $f \in \mathcal{F}_\delta$, for some $\delta > 0$, and $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$, and suppose $\varepsilon a_\varepsilon$ and $\frac{a_\varepsilon}{\varepsilon}$ both have a limit as $\varepsilon \to 0$, including possible limits of 0 and ∞. The following five cases describe the associated limiting behaviour of u^0_ε.*

Case 1: $\varepsilon a_\varepsilon \to \infty$ (thus $\frac{a_\varepsilon}{\varepsilon} \to \infty$). The limit of u^0_ε is $u^\infty_\infty \in H^1_{c,\sigma}(\Omega) := \left\{u \in H^1(\Omega),\ u = cst \text{ on } \sigma\right\}$, the unique solution to the minimization problem

$$\min_{\substack{u\in H^1_{c,\sigma}(\Omega)\\ u=\varphi \text{ on } \partial\Omega}} E^\infty_\infty(u),\ E^\infty_\infty(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega f u\, dx\ , \tag{7.1}$$

and there exists a constant C independent of ε and a_ε such that

$$||u^0_\varepsilon - u^\infty_\infty||_{L^2(\Omega)} \leq \frac{C}{\varepsilon a_\varepsilon}\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)\ .$$

Case 2: $\varepsilon a_\varepsilon \to a_0$ for a certain real value $0 < a_0 < \infty$ (thus $\frac{a_\varepsilon}{\varepsilon} \to \infty$). The limit of u^0_ε is $u^\infty_{a_0} \in H^1(\Omega)\cap V_\sigma = \left\{u \in H^1(\Omega),\ u|_\sigma \in H^1(\sigma)\right\}$, the unique solution to the minimization problem

$$\min_{\substack{u\in H^1(\Omega)\cap V_\sigma\\ u=\varphi \text{ on } \partial\Omega}} E^\infty_{a_0}(u),\ E^\infty_{a_0}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + a_0\int_\sigma \left(\frac{\partial u}{\partial\tau}\right)^2 ds - \int_\Omega f u\, dx\ , \tag{7.2}$$

and there exists a constant C independent of ε and a_ε such that

$$||u^0_\varepsilon - u^\infty_{a_0}||_{L^2(\Omega)} \leq C\left(\left|\frac{\varepsilon a_\varepsilon}{a_0} - 1\right| + \left|\frac{a_0}{\varepsilon a_\varepsilon} - 1\right| + \frac{\varepsilon}{a_\varepsilon}\right)\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)\ .$$

Case 3: $\varepsilon a_\varepsilon \to 0$ and $\frac{a_\varepsilon}{\varepsilon} \to \infty$. The limit of u^0_ε is $u^\infty_0 \in H^1(\Omega)$, the unique solution to the minimization problem

$$\min_{\substack{u\in H^1(\Omega)\\ u=\varphi \text{ on } \partial\Omega}} E^\infty_0(u),\ E^\infty_0(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \int_\Omega f u\, dx\ , \tag{7.3}$$

and there exists a constant C independent of ε and a_ε such that

$$||u^0_\varepsilon - u^\infty_0||_{L^2(\Omega)} \leq C\left(\varepsilon a_\varepsilon + \frac{\varepsilon}{a_\varepsilon}\right)\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)\ .$$

Case 4: $\frac{a_\varepsilon}{\varepsilon} \to b_0$ *for a certain real value* $0 < b_0 < \infty$ *(thus* $\varepsilon a_\varepsilon \to 0$*). The limit of* u^0_ε *is* $u^{b_0}_0 \in H^1(\Omega \setminus \sigma)$*, the unique solution to the minimization problem*

$$\min_{\substack{u\in H^1(\Omega\setminus\sigma)\\ u=\varphi \text{ on } \partial\Omega}} E^{b_0}_0(u),\;\; E^{b_0}_0(u) := \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u|^2\,dx + \frac{b_0}{4}\int_\sigma (u^+ - u^-)^2\,ds - \int_\Omega fu\,dx\;, \tag{7.4}$$

and there exists a constant C *independent of* ε *and* a_ε *such that*

$$||u^0_\varepsilon - u^{b_0}_0||_{L^2(\Omega^+)} + ||u^0_\varepsilon - u^{b_0}_0||_{L^2_0(\Omega^-)} \le C\left(\varepsilon a_\varepsilon + \left|\frac{a_\varepsilon}{\varepsilon b_0} - 1\right| + \left|\frac{\varepsilon b_0}{a_\varepsilon} - 1\right|\right)\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)\;.$$

Case 5: $\frac{a_\varepsilon}{\varepsilon} \to 0$ *(thus* $\varepsilon a_\varepsilon \to 0$*). The limit of* u^0_ε *is* $u^0_0 \in H^1(\Omega \setminus \sigma)$*, a solution to the minimization problem*

$$\min_{\substack{u\in H^1(\Omega\setminus\sigma)\\ u=\varphi \text{ on } \partial\Omega}} E^0_0(u),\;\; E^0_0(u) := \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u|^2\,dx - \int_\Omega fu\,dx\;. \tag{7.5}$$

This solution is unique up to an additive constant on Ω^-*. There exists a constant* C *independent of* ε *and* a_ε *such that*

$$||u^0_\varepsilon - u^0_0||_{L^2(\Omega^+)} + ||u^0_\varepsilon - u^0_0||_{L^2_0(\Omega^-)} \le C\left(\varepsilon a_\varepsilon + \frac{a_\varepsilon}{\varepsilon}\right)\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)\;.$$

The proof of this proposition again relies on Lemma 4. It is in many ways very similar to the proof of Theorem 9, but simpler, so we only provide a sketch. A complete proof would notably involve uniform estimates for the limit problems in the spirit of Theorem 8. Before we proceed to the sketch of the proof, some remarks are in order

- The functional spaces involved in the minimization problems (7.1),(7.2), and (7.3) feature functions that belong (at least) to $H^1(\Omega)$, and thus do not jump across σ. As a consequence, the derivation of uniform energy estimates in the spirit of Lemma 7 does not require any assumption about f other than $f \in L^2(\Omega)$. The natural choice for the space H in the application of Lemma 4 is then $L^2(\Omega)$, and so we obtain $L^2(\Omega)$ estimates of the discrepancy between u^0_ε and its limits. The assumption $\int_{\Omega^-} f = 0$ is not necessary in order to establish the results of Proposition 10 in cases 1 through 3 .
- In case 4, the assumption $\int_{\Omega^-} f = 0$ is not required to ensure that the minimization problem (7.4) has a unique solution. It is needed in order to insure that one may obtain energy estimates for $u^0_{b_0}$ that are uniform with respect to b_0 (see the proof of Lemma 7). Lemma 4 then provides a uniform estimate for $(u^0_\varepsilon - u^0_{b_0})$ on Ω^+, and a uniform estimate for the same difference on Ω^-, modulo a constant .
- In case 5, the assumption $\int_{\Omega^-} f = 0$ is required to ensure that the minimization problem (7.5) has a unique solution, which is defined up to a constant in Ω^-. Note that the convergence result expressed in this case is independent of this constant.

Proof. (1)*:* We use Lemma 4 with $V_\varepsilon = V_\sigma$, $W_\varepsilon = H^1_{c,\sigma}(\Omega)$ and $H = L^2(\Omega)$, and proceed to estimate the difference $|E^0_\varepsilon(u^0_\varepsilon) - E^\infty_\infty(u^\infty_\infty)|$. Since $H^1_{c,\sigma}(\Omega) \subset V_\sigma$, we have

$$\begin{aligned} E^0_\varepsilon(u^0_\varepsilon) - E^\infty_\infty(u^\infty_\infty) &\le E^0_\varepsilon(u^\infty_\infty) - E^\infty_\infty(u^\infty_\infty) \\ &= \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u^\infty_\infty|^2\,dx + \frac{\varepsilon a_\varepsilon}{3}.0 + \frac{a_\varepsilon}{4\varepsilon}.0 - \int_\Omega f u^\infty_\infty\,dx - \frac{1}{2}\int_\Omega |\nabla u^\infty_\infty|^2\,dx + \int_\Omega f u^\infty_\infty\,dx \\ &= 0 \end{aligned}\;.$$

To obtain an upper bound for $\left(E^\infty_\infty(u^\infty_\infty) - E^0_\varepsilon(u^0_\varepsilon)\right)$, we first rewrite $E^\infty_\infty(u^\infty_\infty)$ as

$$\begin{aligned} E^\infty_\infty(u^\infty_\infty) &= \frac{1}{2}\int_\Omega |\nabla u^\infty_\infty|^2\,dx - \int_\Omega f u^\infty_\infty\,dx \\ &= \int_{\partial\Omega} \frac{\partial u^\infty_\infty}{\partial n}\varphi\,ds - \frac{1}{2}\int_\Omega |\nabla u^\infty_\infty|^2\,dx - \int_\sigma \left[\frac{\partial u^\infty_\infty}{\partial n}\right] u^\infty_\infty\,ds\;. \end{aligned}$$

Since u_∞^∞ amounts to a constant on σ, and since $\int_\sigma \left[\frac{\partial u_\infty^\infty}{\partial n}\right] ds = 0$ (which is easily derived from the fact that u_∞^∞ is the minimizer to (7.1)), we conclude that

$$E_\infty^\infty(u_\infty^\infty) = \int_{\partial\Omega} \frac{\partial u_\infty^\infty}{\partial n}\varphi\, ds - \frac{1}{2}\int_\Omega |\nabla u_\infty^\infty|^2\, dx\ .$$

Now, introducing the dual energy principle for u_ε^0 established in Section 5.2, we obtain

$$E_\infty^\infty(u_\infty^\infty) - E_\varepsilon^0(u_\varepsilon^0) \le \int_{\partial\Omega} \frac{\partial u_\infty}{\partial n}\varphi\, ds - \frac{1}{2}\int_\Omega |\nabla u_\infty|^2 - \int_{\partial\Omega} \xi\cdot n\varphi\, ds + \frac{1}{2}\int_{\Omega\setminus\sigma} |\xi|^2\, dx$$
$$+ \frac{\varepsilon a_\varepsilon}{3}\int_\sigma \left(w^{+2} + w^{-2} + w^+w^-\right) ds + \frac{a_\varepsilon}{4\varepsilon}\int_\sigma z^2\, ds,$$

for any $\xi \in L^2(\Omega\setminus\sigma)^2$ and $w^+, w^-, z \in L^2(\sigma)$ satisfying the relations (5.6). Insertion of $\xi = \nabla u_\infty^\infty$, $z = 0$, and

$$\begin{aligned} w^+ &= -\frac{1}{\varepsilon a_\varepsilon}\left(2\int \frac{\partial u_\infty^{\infty +}}{\partial n}\, ds + \int \frac{\partial u_\infty^{\infty -}}{\partial n}\, ds\right)\ , \\ w^- &= \frac{1}{\varepsilon a_\varepsilon}\left(2\int \frac{\partial u_\infty^{\infty -}}{\partial n}\, ds + \int \frac{\partial u_\infty^{\infty +}}{\partial n}\, ds\right) \end{aligned}$$

in the above relation yields

$$E_\infty^\infty(u_\infty^\infty) - E_\varepsilon^0(u_\varepsilon^0) \le \frac{1}{\varepsilon a_\varepsilon}\int_\sigma \left(\left(\int \frac{\partial u_\infty^{\infty +}}{\partial n}\, ds\right)^2 + \left(\int \frac{\partial u_\infty^{\infty -}}{\partial n}\, ds\right)^2 + \left(\int \frac{\partial u_\infty^{\infty +}}{\partial n}\, ds\right)\left(\int \frac{\partial u_\infty^{\infty -}}{\partial n}\, ds\right)\right) ds\ .$$

The result follows by using energy estimates for u_∞^∞.

(2)*:* We rely again on Lemma 4 with $V_\varepsilon = V_\sigma$, $W_\varepsilon = H^1(\Omega)\cap V_\sigma$ and $H = L^2(\Omega)$. As $H^1(\Omega)\cap V_\sigma \subset V_\sigma$, we have on the one hand

$$\begin{aligned} E_\varepsilon^0(u_\varepsilon^0) - E_{a_0}^\infty(u_{a_0}^\infty) &\le E_\varepsilon^0(u_{a_0}^\infty) - E_{a_0}^\infty(u_{a_0}^\infty) \\ &= \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u_{a_0}^\infty|^2\, dx + \varepsilon a_\varepsilon \int_\sigma \left(\frac{\partial u_{a_0}^\infty}{\partial\tau}\right)^2 ds - \frac{1}{2}\int_\Omega |\nabla u_{a_0}^\infty|^2\, dx - a_0\int_\sigma \left(\frac{\partial u_{a_0}^\infty}{\partial\tau}\right)^2 ds \\ &\le \left|\frac{\varepsilon a_\varepsilon}{a_0} - 1\right|\, a_0\int_\sigma \left(\frac{\partial u_{a_0}^\infty}{\partial\tau}\right)^2 ds\ . \end{aligned}$$

The factor $a_0\int_\sigma \left(\frac{\partial u_{a_0}^\infty}{\partial\tau}\right)^2 ds$ is bounded by $C(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)})^2$, uniformly with respect to a_0 (as follows easily from standard energy estimates for the problem (7.2)).

On the other hand, the dual energy maximization principle for $E_{a_0}^\infty(u_{a_0}^\infty)$ reads

$$E_{a_0}(u_{a_0}^\infty) = \max\left(\int_{\partial\Omega} \xi\cdot n\varphi\, ds - \frac{1}{2}\int_\Omega |\xi|^2 - \frac{1}{a_0}\int_\sigma w^2\, ds\right)\ ,$$

where the maximum is taken over the set of functions $\xi \in L^2(\Omega)^2$, $w \in L^2(\sigma)$ such that

$$\text{-div}(\xi) = f \text{ in } \Omega^+ \text{ and in } \Omega^-\ ,\ \text{and } [\xi_n] + 2\frac{\partial w}{\partial\tau} = 0 \text{ on } \sigma\ . \tag{7.6}$$

The maximum is uniquely attained at $\xi = \nabla u_{a_0}^\infty$ and $w = a_0\frac{\partial u_{a_0}^\infty}{\partial\tau}$. We thus obtain

$$E_{a_0}^\infty(u_{a_0}^\infty) - E_\varepsilon^0(u_\varepsilon^0) \le \int_{\partial\Omega} \frac{\partial u_{a_0}^\infty}{\partial n}\varphi\, ds - \frac{1}{2}\int_\Omega |\nabla u_{a_0}^\infty|^2 - a_0\int_\sigma \left(\frac{\partial u_{a_0}^\infty}{\partial\tau}\right)^2 ds - \int_{\partial\Omega} \xi\cdot n\varphi\, ds$$
$$+ \frac{1}{2}\int_{\Omega\setminus\sigma} |\xi|^2\, dx + \frac{\varepsilon a_\varepsilon}{3}\int_\sigma \left(w^{+2} + w^{-2} + w^+w^-\right) ds + \frac{a_\varepsilon}{4\varepsilon}\int_\sigma z^2\, ds\ ,$$

for any $\xi \in L^2(\Omega\setminus\sigma)^2$ and $w^+, w^-, z \in L^2(\sigma)$ satisfying (5.6). We now insert $\xi = \nabla u_{a_0}^\infty$, together with

$$w^+ = w^- = \frac{a_0}{\varepsilon a_\varepsilon}\frac{\partial u_{a_0}^\infty}{\partial\tau}\ ,$$

and z given by

$$\frac{a_\varepsilon}{2\varepsilon} z = \frac{\partial u_{a_0}^{\infty +}}{\partial n} + a_0 \frac{\partial^2 u_{a_0}^{\infty}}{\partial \tau^2} = \frac{\partial u_{a_0}^{\infty -}}{\partial n} - a_0 \frac{\partial^2 u_{a_0}^{\infty}}{\partial \tau^2} \ .$$

The last identity holds true because of (7.6), and it insures that this choice of $\xi, w^{\pm}, z$ satisfies (5.6). As a result

$$\begin{aligned} E_{a_0}^{\infty}(u_{a_0}^{\infty}) - E_\varepsilon^0(u_\varepsilon^0) &\leq \varepsilon a_\varepsilon \int_\sigma w^{+2}\, ds + \frac{a_\varepsilon}{4\varepsilon}\int_\sigma z^2\, ds - a_0 \int_\sigma \left(\frac{\partial u_{a_0}^{\infty}}{\partial \tau}\right)^2 ds \\ &\leq \left|\frac{a_0}{\varepsilon a_\varepsilon} - 1\right| a_0 \int_\sigma \left(\frac{\partial u_{a_0}^{\infty}}{\partial \tau}\right)^2 ds + \frac{\varepsilon}{a_\varepsilon}\int_\sigma \left(\frac{\partial u_{a_0}^{\infty +}}{\partial n} + a_0 \frac{\partial^2 u_{a_0}^{\infty}}{\partial \tau^2}\right)^2 ds \ . \end{aligned}$$

These upper and lower bounds for $E_\varepsilon^0(u_\varepsilon^0) - E_{a_0}^{\infty}(u_{a_0}^{\infty})$, in combination with the appropriate apriori estimate for $u_{a_0}^{\infty}$, lead to the desired conclusion.

(3) is in every aspect simpler to handle than the other cases, and is left to the reader.

(4)*:* Here we take $V_\varepsilon = V_\sigma$, $W_\varepsilon = H^1(\Omega \setminus \sigma)$ and

$$H = \left\{ f \in L^2(\Omega),\ \int_{\Omega^-} f = 0 \right\} \ .$$

We obtain an upper bound for $(E_\varepsilon^0(u_\varepsilon^0) - E_0^{b_0}(u_0^{b_0}))$ by using $v = u_0^{b_0}$ as a "test function" in the minimization of E_ε^0:

$$\begin{aligned} E_\varepsilon^0(u_\varepsilon^0) - E_0^{b_0}(u_0^{b_0}) &\leq E_\varepsilon^0(u_0^{b_0}) - E_0^{b_0}(u_0^{b_0}) \\ &\leq \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u_0^{b_0}|^2\, dx + \frac{\varepsilon a_\varepsilon}{2}\int_\sigma \left(\left(\frac{\partial u_0^{b_0+}}{\partial \tau}\right)^2 + \left(\frac{\partial u_0^{b_0-}}{\partial \tau}\right)^2\right) ds + \frac{a_\varepsilon}{4\varepsilon}\int_\sigma (u_0^{b_0+} - u_0^{b_0-})^2\, ds \\ &\quad -\frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u_0^{b_0}|^2\, dx - \frac{b_0}{4}\int_\sigma (u_0^{b_0+} - u_0^{b_0-})^2\, ds \\ &\leq C\left(\varepsilon a_\varepsilon + \left|\tfrac{a_\varepsilon}{\varepsilon b_0} - 1\right|\right)\left(||\varphi||_{H^{1/2}(\partial\Omega)} + ||f||_{L^2(\Omega)}\right)^2 \ , \end{aligned}$$

for a constant C, which does not depend on b_0, and ε, a_ε . Here we used the fact that

$$\frac{1}{3}(a^2 + b^2 + ab) = \frac{1}{2}(a^2 + b^2) - \frac{1}{6}(a - b)^2 \leq \frac{1}{2}(a^2 + b^2) \ ,$$

and an appropriate apriori estimate for $u_0^{b_0}$. In order to establish a satisfactory lower bound on $E_\varepsilon^0(u_\varepsilon^0) - E_0^{b_0}(u_0^{b_0}))$, we first observe that, as an immediate consequence of the variational problem satisfied by $u_0^{b_0}$, one has

(7.7) $$\frac{\partial u_0^{b_0+}}{\partial n} = \frac{\partial u_0^{b_0-}}{\partial n} = \frac{b_0}{2}(u_0^{b_0+} - u_0^{b_0-}) \text{ on } \sigma \ .$$

Now, using the dual energy maximization principle for E_ε^0 (see Section 5.2), and the fact that

(7.8) $$\begin{aligned} &\frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u_0^{b_0}|^2\, dx + \frac{b_0}{4}\int_\sigma (u_0^{b_0+} - u_0^{b_0-})^2\, ds - \int_\Omega f u_0^{b_0}\, dx \\ &= \int_{\partial\Omega} \frac{\partial u_0^{b_0}}{\partial n}\varphi\, ds - \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u_0^{b_0}|^2\, dx - \frac{b_0}{4}\int_\sigma (u_0^{b_0+} - u_0^{b_0-})^2\, ds \ , \end{aligned}$$

we obtain

$$\begin{aligned} E_0^{b_0}(u_0^{b_0}) - E_\varepsilon^0(u_\varepsilon^0) \leq \int_{\partial\Omega} \frac{\partial u_0^{b_0}}{\partial n}\varphi\, ds - \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u_0^{b_0}|^2\, dx - \frac{b_0}{4}\int_\sigma (u_0^{b_0+} - u_0^{b_0-})^2\, ds - \int_{\partial\Omega} \xi\cdot n\varphi\, ds \\ + \frac{1}{2}\int_{\Omega\setminus\sigma} |\xi|^2\, dx + \frac{\varepsilon a_\varepsilon}{2}\int_\sigma \left(w^{+2} + w^{-2}\right) ds + \frac{a_\varepsilon}{4\varepsilon}\int_\sigma z^2\, ds \ , \end{aligned}$$

for any $\xi \in L^2(\Omega \setminus \sigma)^2$ and $w^+, w^-, z \in L^2(\sigma)$ satisfying (5.6). Due to (7.7), we may choose $\xi = \nabla u_0^{b_0}$, $w^+ = w^- = 0$ and $z = \frac{\varepsilon b_0}{a_\varepsilon}(u_0^{b_0+} - u_0^{b_0-})$ for insertion into the last line of the previous inequality. This yields

$$
\begin{aligned}
E_0^{b_0}(u_0^{b_0}) - E_\varepsilon^0(u_\varepsilon^0) &\leq \frac{a_\varepsilon}{4\varepsilon}\int_\sigma z^2\,ds - \frac{b_0}{4}\int_\sigma (u_0^{b_0+} - u_0^{b_0-})^2\,ds \\
&\leq \left(\tfrac{\varepsilon b_0}{a_\varepsilon} - 1\right)\tfrac{b_0}{4}\int_\sigma (u_0^{b_0+} - u_0^{b_0-})^2\,ds \\
&\leq C\left(\tfrac{\varepsilon b_0}{a_\varepsilon} - 1\right)\left(||\varphi||_{H^{1/2}(\partial\Omega)} + ||f||_{L^2(\Omega)}\right)^2 ,
\end{aligned}
$$

for some constant C which is independent of b_0, and ε, a_ε. Here we used the same algebraic inequality as before, and an appropriate apriori estimate for $u_0^{b_0}$. In summary we have proved

$$
|E_\varepsilon^0(u_\varepsilon^0) - E_0^{b_0}(u_0^{b_0})| \leq C\left(\varepsilon a_\varepsilon + \left|\frac{a_\varepsilon}{\varepsilon b_0} - 1\right| + \left|\frac{\varepsilon b_0}{a_\varepsilon} - 1\right|\right)\left(||\varphi||_{H^{1/2}(\partial\Omega)} + ||f||_{L^2(\Omega)}\right)^2 ,
$$

and by Lemma 4 this yields the desired estimate for $||u_\varepsilon^0 - u_0^{b_0}||_{L^2(\Omega^+)} + ||u_\varepsilon^0 - u_0^{b_0}||_{L_0^2(\Omega^-)}$.

(5): In this last case, we take $V_\varepsilon = V_\sigma$, $W_\varepsilon = \left\{v \in H^1(\Omega \setminus \sigma),\ \int_{\Omega^-} v\,dx = 0\right\}$ (a set over which the minimization problem (7.5) has a unique solution), and

$$
H = \left\{f \in L^2(\Omega),\ \int_{\Omega^-} f = 0\right\}.
$$

The proof proceeds along the same lines as in the previous case(s), and is left to the reader. □

7.2. A closer look at the case $a_\varepsilon = a$, independently of ε.

In this section we make some observations pertaining to the case when the coefficient a_ε is independent of ε, in other words when

$$
a_\varepsilon = a, \text{ where } a > 0 \text{ is a fixed real number.}
$$

Following the discussions in Sections 6 and 7.1, two 0^{th}-order approximations of the solution u_ε to (2.2) are available in this case, namely

$$
u_\varepsilon = u_\varepsilon^0 + \mathcal{O}(\varepsilon) , \tag{7.9}
$$

which we shall refer to as the 0^{th} order *uniform expansion* of u_ε, and

$$
u_\varepsilon = u_0^\infty + \mathcal{O}(a\varepsilon + \frac{\varepsilon}{a}) , \tag{7.10}
$$

which we shall refer to as the 0^{th} order *"natural asymptotic" expansion* of u_ε. The latter is just the one term Taylor expansion of u_ε with respect to ε (at zero). u_0^∞ is the unique solution to

$$
-\Delta u_0^\infty = f \ \text{ in } \Omega\ ,\ u_0^\infty = \varphi \ \text{ on } \partial\Omega\ .
$$

The particular form of the remainder term in (7.10) follows from (7.9) and case 3 of Proposition 10. We recall that $u_\varepsilon^0 \in V_\sigma$ is the unique solutions to (4.10) (or (5.4)).

From Proposition 10 we know that

$$
u_\varepsilon^0 = u_0^\infty + \mathcal{O}(a\varepsilon + \frac{\varepsilon}{a}) ,
$$

and so a Taylor expansion of u_ε^0 with respect to ε also starts with the term u_0^∞. We would like to understand a little better the answer to the following question "in the process of correcting u_0^∞ to make it into a uniform approximation to u_ε in terms of the conductivity coefficient a, will it suffice to add just a finite number of terms in the Taylor series (of u_ε^0) ?". For that purpose we now derive the specific form of the first-order Taylor expansion

$$
u_\varepsilon^0 = u_0^\infty + \varepsilon u_1 + \mathcal{O}_a(\varepsilon^2) .
$$

To this end, we follow the strategy employed before: as a first step, we define the (ε-dependent) function $\overline{u_1} \in V_\sigma$ by the relation $u_\varepsilon^0 = u_0^\infty + \varepsilon\overline{u_1}$, and write a minimization problem satisfied by $\overline{u_1}$. We then approximate this problem using heuristic arguments, and define u_1 as the solution to this simplified problem. In spite of the heuristic nature of our derivation it is possible to prove that $\overline{u_1} = u_1 + \mathcal{O}(\varepsilon)$ – we shall, however, omit the proof here.

1^{st} *step: Derivation of a minimization problem for* $\overline{u_1}$. Due to the definition of u_ε^0, $\overline{u_1}$ arises as the unique minimizer in $V_{\sigma,0}$ of the following energy

$$J_\varepsilon^1(u) = \frac{1}{\varepsilon}\left(E_\varepsilon^0(u_0^\infty + \varepsilon u) - E_\varepsilon^0(u_0^\infty)\right) \ .$$

A simple calculation gives
(7.11)

$$\begin{aligned}
J_\varepsilon^1(u) &= \varepsilon\left(\frac{1}{2}\int_{\Omega\setminus\sigma}|\nabla u|^2\,dx + \frac{\varepsilon a}{3}\int_\sigma\left(\left(\frac{\partial u^+}{\partial\tau}\right)^2 + \left(\frac{\partial u^-}{\partial\tau}\right)^2 + \frac{\partial u^+}{\partial\tau}\frac{\partial u^-}{\partial\tau}\right)ds + \frac{a}{4\varepsilon}\int_\sigma\left(u^+ - u^-\right)^2\,ds\right)\\
&\quad + \int_{\Omega\setminus\sigma}\nabla u_0^\infty\cdot\nabla u\,dx + \varepsilon a\int_\sigma\frac{\partial u_0^\infty}{\partial\tau}\left(\frac{\partial u^+}{\partial\tau} + \frac{\partial u^-}{\partial\tau}\right)ds - \int_\Omega fu\,dx\\
&= \varepsilon\left(\frac{1}{2}\int_{\Omega\setminus\sigma}|\nabla u|^2\,dx + \frac{\varepsilon a}{3}\int_\sigma\left(\left(\frac{\partial u^+}{\partial\tau}\right)^2 + \left(\frac{\partial u^-}{\partial\tau}\right)^2 + \frac{\partial u^+}{\partial\tau}\frac{\partial u^-}{\partial\tau}\right)ds - a\int_\sigma\frac{\partial^2 u_0^\infty}{\partial\tau^2}(u^+ + u^-)\,ds\right)\\
&\quad + \frac{a}{4}\int_\sigma\left(u^+ - u^-\right)^2\,ds - \int_\sigma\frac{\partial u_0^\infty}{\partial n}(u^+ - u^-)\,ds \ .
\end{aligned}$$

2^{nd} *step: Simplification of the minimization problem of* $J_\varepsilon^1(u)$. It seems reasonable to assume that the minimization process of $J_\varepsilon^1(u)$ will principally seek to minimize the terms of order 0 as $\varepsilon\to 0$, that is, the two terms

$$\frac{a}{4}\int_\sigma(u^+ - u^-)^2\,ds - \int_\sigma\frac{\partial u_0^\infty}{\partial n}(u^+ - u^-)\,ds \ .$$

The minimum of this last expression is achieved when $(u^+ - u^-) = \frac{2}{a}\frac{\partial u_0^\infty}{\partial n}$ on σ. Subject to this relation, the minimization process should then concentrate on the first order terms

$$\varepsilon\left(\frac{1}{2}\int_{\Omega\setminus\sigma}|\nabla u|^2\,dx - a\int_\sigma\frac{\partial^2 u_0^\infty}{\partial\tau^2}(u^+ + u^-)\,ds\right) \ .$$

Using the corresponding Euler-Lagrange equations, we are led to a candidate $u_1\in V_{\sigma,0}$ (for the 0^{th}-order approximation to $\overline{u_1}$) that is characterized as the solution to the following problem

$$\text{(7.12)}\qquad \begin{cases} -\Delta u_1 = 0 & \text{in } \Omega\setminus\sigma \ ,\\ u_1 = 0 & \text{on } \partial\Omega \ ,\\ [u_1] = \frac{2}{a}\frac{\partial u_0^\infty}{\partial n} & \text{on } \sigma \ ,\\ \left[\frac{\partial u_1}{\partial n}\right] = -2a\frac{\partial^2 u_0^\infty}{\partial\tau^2} & \text{on } \sigma \ . \end{cases}$$

It is indeed possible to prove

Proposition 11. *Let* $u_1\in H^1(\Omega\setminus\sigma)$ *be the unique solution to (7.12). There exists a constant* C*, which only depends on* Ω*,* σ *and* a*, such that*

$$||\nabla(u_\varepsilon^0 - u_0^\infty - \varepsilon u_1)||_{L^2(\Omega\setminus\sigma)^2}\le C\varepsilon^2\left(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}\right) \ .$$

The proof of this is fairly straightforward, and follows by carefully considering the boundary value problem satisfied by $u_\varepsilon^0 - u_0^\infty - \varepsilon u_1 = \varepsilon(\overline{u}_1 - u_1)$. We leave the details to the reader. The fact that u_1 degenerates like a and $1/a$ when a tends to ∞ and 0 respectively, strongly indicates that the estimate $u_\varepsilon^0 - u_0^\infty = \mathcal{O}(a\varepsilon + \frac{\varepsilon}{a})$ is the best possible. Higher order terms in the Taylor series of u_ε^0 could be calculated, and they too would degenerate when a tends to ∞ and 0. This would strongly indicate that *no* finite Taylor expansion of u_ε^0 (at zero) would achieve a uniform approximation to u_ε^0 – uniform with respect to a that is.

It is interesting to compare the above calculation of the first two terms in the Taylor Series of u_ε^0 to the calculation carried out in [7]. In that paper, the authors consider the Neumann version of Problem (2.2) in the case that $a_\varepsilon = a$, and they calculate the first two terms in the $\varepsilon\to 0$ asymptotic expansion of the solution to the problem

$$\begin{cases} -\text{div}(\gamma_\varepsilon\nabla u_\varepsilon) = f & \text{in } \Omega \ ,\\ \gamma_\varepsilon\frac{\partial u_\varepsilon}{\partial n} = \psi & \text{on } \partial\Omega \ , \end{cases}$$

which we shall also call u_ε, since the difference in the type of boundary conditions on $\partial\Omega$ plays no role for the discussion here. γ_ε is as before defined by (2.1). The result in [7] is

$$\forall y \in \partial\Omega,\ \ u_\varepsilon(y) = u_0^\infty(y) + \varepsilon\widetilde{u_1}(y) + o(\varepsilon)\ . \tag{7.13}$$

In this formula, the function $\widetilde{u_1}$ is defined in terms of the Neumann function $N(x,y)$ of Ω, a polarization tensor $\mathcal{M}(x)$, and the harmonic function u_0^∞:

$$\widetilde{u_1}(y) = 2\int_\sigma (a-1)\mathcal{M}(x)\nabla u_0^\infty(x)\cdot\nabla_x N(x,y)\,ds(x);\ \ y\notin\sigma.$$

The polarization tensor $\mathcal{M}(x)$ is for $x\in\sigma$ given by $\mathcal{M}(x) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{a}\end{pmatrix}$ in the local basis $(\tau(x), n(x))$, and the Neumann function is the solution to:

$$\left\{\begin{array}{ll} \Delta_x N(x,y) = \delta_y & \text{in } \Omega, \\ \frac{\partial}{\partial n_x}N(x,y) = \frac{1}{|\partial\Omega|} & \text{on } \partial\Omega, \end{array}\right.$$

where δ_y is the Dirac distribution centred at $x=y$. Equivalently, due to the jump relations for single and double layer potentials (see e.g. [16], Chap. 3), $\widetilde{u_1}\in H^1(\Omega\setminus\sigma)$ is the unique solution (modulo a constant) to the following problem:

$$\left\{\begin{array}{ll} -\Delta\widetilde{u_1} = 0 & \text{in } \Omega\setminus\sigma\ , \\ \frac{\partial\widetilde{u_1}}{\partial n} = 0 & \text{on } \partial\Omega\ , \\ [\widetilde{u_1}] = -2\left(1-\frac{1}{a}\right)\frac{\partial u_0^\infty}{\partial n} & \text{on } \sigma\ , \\ \left[\frac{\partial\widetilde{u_1}}{\partial n}\right] = -2\,(a-1)\,\frac{\partial^2 u_0^\infty}{\partial\tau^2} & \text{on } \sigma\ . \end{array}\right. \tag{7.14}$$

We immediately notice that the boundary value problems satisfied by u_1 and $\tilde{u}_1$ imply that the difference $u_1-\tilde{u}_1$ is uniformly bounded with respect to a. If the same thing were to happen for higher terms in the Taylor Series, then it would be very consistent with the fact that the difference $u_\varepsilon - u_\varepsilon^0$ is uniformly bounded with respect to a; it would also strongly suggest that no finite Taylor expansion of u_ε would lead to a uniform approximation (uniform in a, that is).

8. Derivation of the 1$^{\text{st}}$ order approximation of u_ε

In the previous sections, we have derived a uniform 0^{th}-order approximation $(u_\varepsilon^0, v_\varepsilon^0)\in V_\sigma\times H^1(\omega_1)$ to the couple $(u_\varepsilon|_{\Omega\setminus\overline{\omega_\varepsilon}}, u_\varepsilon\circ H_\varepsilon)\in H^1(\Omega\setminus\overline{\omega_\varepsilon})\times H^1(\omega_1)$. Properly speaking, we only proved that u_ε^0 is a uniform approximation of $u_\varepsilon|_{\Omega\setminus\overline{\omega_\varepsilon}}$ "far away from the curve σ", that is, on subsets of Ω of the form $\Omega\setminus\overline{\omega_\delta}$, for some fixed $\delta>0$. However, the proof of this fact made use of the heuristic approximate guess v_ε^0 for the potential $(u_\varepsilon\circ H_\varepsilon)$ inside the rescaled inhomogeneity.

Relying on the same strategy, we now briefly outline the derivation of a uniform first-order approximation result for the solution u_ε of (2.2). We note that the 0^{th}- and first-order analyses turn out to share a lot of common features; we shall thus for the sake of brevity omit some of the very tedious calculations related to the latter.

We start from the rescaled form of problem (4.1) as established in Section 4.1.1: the couple $(u_\varepsilon|_{\Omega\setminus\overline{\omega_\varepsilon}}, u_\varepsilon\circ H_\varepsilon)$ is the unique minimizer of the energy

$$\overline{F_\varepsilon^0}(u,v) = \frac{1}{2}\int_{\Omega\setminus\overline{\omega_\varepsilon}}|\nabla u|^2\,dx + \frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1}\frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\left(\frac{\partial v}{\partial\tau}\right)^2 dx + \frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\left(\frac{\partial v}{\partial n}\right)^2 dx - \int_\Omega fu\,dx,$$

among the elements of the space

$$\overline{V_\varepsilon^0} = \left\{(u,v),\ u\in H^1(\Omega\setminus\overline{\omega_\varepsilon}),\ v\in H^1(\omega_1),\ \forall x\in\sigma\ \begin{array}{l} v(x+n(x)) = u(x+\varepsilon n(x)) \\ v(x-n(x)) = u(x-\varepsilon n(x)) \end{array}\right\}\ ,$$

that additionnally satisfy $u=\varphi$ on $\partial\Omega$. We have seen that a uniform 0^{th}-order approximation of this couple (in the sense described above) is $(u_\varepsilon^0, v_\varepsilon^0)\in V^0$, where V^0 is defined in (4.6), u_ε^0 is defined as the solution to the minimization problem (4.10), and v_ε^0 is given by (4.13). For technical convenience, we define the couple $(\overline{u_\varepsilon}, \overline{v_\varepsilon})\in H^1(\Omega\setminus\overline{\omega_\varepsilon})\times H^1(\omega_1)$ by the identity

$$(u_\varepsilon|_{\Omega\setminus\overline{\omega_\varepsilon}}, u_\varepsilon\circ H_\varepsilon) = (u_\varepsilon^0 + \varepsilon(y_\varepsilon + \overline{u_\varepsilon}), v_\varepsilon^0 + \varepsilon(w_\varepsilon + \overline{v_\varepsilon}))\ , \tag{8.1}$$

where $y_\varepsilon \in H^1(\Omega \setminus \overline{\omega_\varepsilon})$ denotes the unique solution to the problem

$$\begin{cases} -\Delta y_\varepsilon = 0 & \text{in } \Omega \setminus \overline{\omega_\varepsilon} \ , \\ y_\varepsilon = 0 & \text{on } \partial\Omega \ , \\ y_\varepsilon(x+\varepsilon n(x)) = \frac{\partial u_\varepsilon^{0+}}{\partial n}(x) - \frac{1}{\varepsilon}\left(u_\varepsilon^0(x+\varepsilon n(x)) - u_\varepsilon^{0+}(x)\right) & x \in \sigma \ , \\ y_\varepsilon(x-\varepsilon n(x)) = -\frac{\partial u_\varepsilon^{0-}}{\partial n}(x) - \frac{1}{\varepsilon}\left(u_\varepsilon^0(x-\varepsilon n(x)) - u_\varepsilon^{0-}(x)\right) & x \in \sigma \ , \end{cases}$$

and $w_\varepsilon \in H^1(\omega_1)$ is given by the formula

$$\forall x \in \sigma,\ \forall t \in (-1,1),\ w_\varepsilon(x+tn(x)) = \frac{t}{2}\left(\frac{\partial u_\varepsilon^{0+}}{\partial n}(x) + \frac{\partial u_\varepsilon^{0-}}{\partial n}(x)\right) + \frac{1}{2}\left(\frac{\partial u_\varepsilon^{0+}}{\partial n}(x) - \frac{\partial u_\varepsilon^{0-}}{\partial n}(x)\right). \tag{8.2}$$

We note that $(x \pm \varepsilon n(x))$ describes $\partial\omega_\varepsilon^\pm$ as x runs through σ. Due to the introduction of these two auxiliary functions y_ε and w_ε, the "unknown" couple $(\overline{u_\varepsilon}, \overline{v_\varepsilon})$ has no "jump" from $\partial\omega_\varepsilon$ to $\partial\omega_1$, i.e., $(\overline{u_\varepsilon}, \overline{v_\varepsilon})$ lies in $\overline{V_\varepsilon^0}$. Note that, using the uniform regularity estimates of Theorem 8 and arguing as we did for the study of the function z_ε in Section 6.1, we may easily prove that

$$||y_\varepsilon||_{L^2(\Omega^+\setminus\overline{\omega_\delta})} + ||y_\varepsilon||_{L^2_0(\Omega^-\setminus\overline{\omega_\delta})} + ||\nabla y_\varepsilon||_{L^2(\Omega\setminus\overline{\omega_\delta})} \leq C\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)\varepsilon \ . \tag{8.3}$$

From its definition, $(\overline{u_\varepsilon}, \overline{v_\varepsilon})$ is the unique minimizer of the functional

$$\overline{F_\varepsilon^1}(u,v) := \frac{1}{\varepsilon}\left(\overline{F_\varepsilon^0}(u_\varepsilon^0 + \varepsilon(y_\varepsilon + u), v_\varepsilon^0 + \varepsilon(w_\varepsilon + v)) - \overline{F_\varepsilon^0}(u_\varepsilon^0 + \varepsilon y_\varepsilon, v_\varepsilon^0 + \varepsilon w_\varepsilon)\right),$$

among the couples $(u,v) \in \overline{V_\varepsilon^0}$ such that $u = 0$ on $\partial\Omega$. To find a uniform 0^{th}-order approximation to $(\overline{u_\varepsilon}, \overline{v_\varepsilon})$ we expand the functional $\overline{F_\varepsilon^1}(u,v)$ as follows:

$$\begin{aligned} \overline{F_\varepsilon^1}(u,v) = &\left(\frac{1}{2}\int_{\Omega\setminus\overline{\omega_\varepsilon}} |\nabla u|^2\,dx + \frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1} \frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\left(\frac{\partial v}{\partial\tau}\right)^2 dx + \frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\left(\frac{\partial v}{\partial n}\right)^2 dx\right. \\ &+ \left.\int_{\Omega\setminus\overline{\omega_\varepsilon}} \nabla y_\varepsilon\cdot\nabla u\,dx + \varepsilon a_\varepsilon\int_{\omega_1}\frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\frac{\partial w_\varepsilon}{\partial\tau}\frac{\partial v}{\partial\tau}\,dx + \frac{a_\varepsilon}{\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\frac{\partial w_\varepsilon}{\partial n}\frac{\partial v}{\partial n}\,dx\right)\varepsilon \\ &+ \int_{\Omega\setminus\overline{\omega_\varepsilon}}\nabla u_\varepsilon^0\cdot\nabla u\,dx + \varepsilon a_\varepsilon\int_{\omega_1}\frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\frac{\partial v_\varepsilon^0}{\partial\tau}\frac{\partial v}{\partial\tau}\,dx + \frac{a_\varepsilon}{\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\frac{\partial v_\varepsilon^0}{\partial n}\frac{\partial v}{\partial n}\,dx - \int_\Omega fu\,dx\ . \end{aligned} \tag{8.4}$$

We observe that the quadratic part of this energy is the same as that of the 0^{th}-order energy $\overline{F_\varepsilon^0}$ (modulo a factor of ε). The linear part has two components, correponding to the second line and the third line of (8.4) respectively. Following this splitting of the linear part we decompose $(\overline{u_\varepsilon}, \overline{v_\varepsilon})$ as

$$(\overline{u_\varepsilon}, \overline{v_\varepsilon}) = (\overline{u_{1,\varepsilon}}, \overline{v_{1,\varepsilon}}) + (\overline{u_{2,\varepsilon}}, \overline{v_{2,\varepsilon}})\ , \tag{8.5}$$

where $(\overline{u_{1,\varepsilon}}, \overline{v_{1,\varepsilon}})$ and $(\overline{u_{2,\varepsilon}}, \overline{v_{2,\varepsilon}}) \in \overline{V_\varepsilon^0}$ are the unique minimizers of the respective energies $\overline{F_\varepsilon^{1,1}}(u,v)$ and $\overline{F_\varepsilon^{1,2}}(u,v)$, defined by:

$$\begin{aligned} \overline{F_\varepsilon^{1,1}}(u,v) = &\frac{1}{2}\int_{\Omega\setminus\overline{\omega_\varepsilon}}|\nabla u|^2\,dx + \frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1}\frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\left(\frac{\partial w_\varepsilon}{\partial\tau} + \frac{\partial v}{\partial\tau}\right)^2 dx \\ &+ \frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\left(\frac{\partial w_\varepsilon}{\partial n} + \frac{\partial v}{\partial n}\right)^2 dx + \int_{\Omega\setminus\overline{\omega_\varepsilon}}\nabla y_\varepsilon\cdot\nabla u\,dx\ , \end{aligned} \tag{8.6}$$

and

$$\begin{aligned} \overline{F_\varepsilon^{1,2}}(u,v) = &\left(\frac{1}{2}\int_{\Omega\setminus\overline{\omega_\varepsilon}}|\nabla u|^2\,dx + \frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1}\frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\left(\frac{\partial v}{\partial\tau}\right)^2 dx + \frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\left(\frac{\partial v}{\partial n}\right)^2 dx\right)\varepsilon \\ &+ \int_{\Omega\setminus\overline{\omega_\varepsilon}}\nabla u_\varepsilon^0\cdot\nabla u\,dx + \varepsilon a_\varepsilon\int_{\omega_1}\frac{1+\kappa d_{\Omega^-}}{1+\varepsilon\kappa d_{\Omega^-}}\frac{\partial v_\varepsilon^0}{\partial\tau}\frac{\partial v}{\partial\tau}\,dx + \frac{a_\varepsilon}{\varepsilon}\int_{\omega_1}\frac{1+\varepsilon\kappa d_{\Omega^-}}{1+\kappa d_{\Omega^-}}\frac{\partial v_\varepsilon^0}{\partial n}\frac{\partial v}{\partial n}\,dx - \int_\Omega fu\,dx\ . \end{aligned} \tag{8.7}$$

Note that the definition of $\overline{F_\varepsilon^{1,1}}$ slightly differs from the sum of the first two lines of (8.4) by an additive term that only depends on u_ε^0 (and a factor of ε), which has no effect on the solution to the corresponding minimization problem.

8.1. 0^{th}-order approximation of the couple $(\overline{u_{1,\varepsilon}}, \overline{v_{1,\varepsilon}})$.

To obtain a 0^{th}-order approximation $(u_{1,\varepsilon}, v_{1,\varepsilon})$ of $(\overline{u_{1,\varepsilon}}, \overline{v_{1,\varepsilon}})$, we follow the same strategy as in Section 4. We use a heuristic argument to build an approximate two-scale minimization problem

$$\min_{\substack{(u,v)\in V^0 \\ u=0 \text{ on } \partial\Omega}} F_\varepsilon^{1,1}(u,v). \tag{8.8}$$

This problem can now (heuristically) be solved for v in terms of u, leading to a minimization problem featuring only u. This process yields a candidate $(u_{1,\varepsilon}, v_{1,\varepsilon})$ for a uniform 0^{th}-order approximation of $(\overline{u_{1,\varepsilon}}, \overline{v_{1,\varepsilon}})$. Then we can rigorously prove a uniform approximation estimate, using arguments similar to those of Section 6. This estimate would assert that

$$||\overline{u_{1,\varepsilon}} - u_{1,\varepsilon}||_{L^2(\Omega^+\setminus\overline{\omega_\delta})} + ||\overline{u_{1,\varepsilon}} - u_{1,\varepsilon}||_{L^2_0(\Omega^-\setminus\overline{\omega_\delta})} \le C\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)\sqrt{\varepsilon}\ ,$$

with C independent of ε and a_ε. For brevity we shall not present the proof of this estimate here, instead we limit ourselves to describing the heuristic derivation of the approximate energy $F_\varepsilon^{1,1}$.

Arguing as in Section 4, and relying on the estimate (8.3), we approximate the quantity $\overline{F_\varepsilon^{1,1}}(u,v)$ by

$$F_\varepsilon^{1,1}(u,v) := \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u|^2\,dx + \frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1} (1+\kappa d_{\Omega^-})\left(\frac{\partial w_\varepsilon}{\partial\tau} + \frac{\partial v}{\partial\tau}\right)^2 dx + \frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1} \frac{1}{1+\kappa d_{\Omega^-}}\left(\frac{\partial w_\varepsilon}{\partial n} + \frac{\partial v}{\partial n}\right)^2 dx\ . \tag{8.9}$$

Problem (8.8) can now be rewritten

$$\min_{\substack{u\in V_\sigma \\ u=0 \text{ on } \partial\Omega}} \left\{\frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u|^2\,dx + G_\varepsilon^1(u)\right\}\ ,$$

where we have defined

$$G_\varepsilon^1(u) := \min_{\substack{v\in H^1(\omega_1) \\ v(x+n(x))=u^+(x),\ x\in\sigma \\ v(x-n(x))=u^-(x),\ x\in\sigma}} \left\{\frac{\varepsilon a_\varepsilon}{2}\int_{\omega_1}(1+\kappa d_{\Omega^-})\left(\frac{\partial w_\varepsilon}{\partial\tau} + \frac{\partial v}{\partial\tau}\right)^2 dx + \frac{a_\varepsilon}{2\varepsilon}\int_{\omega_1}\frac{1}{1+\kappa d_{\Omega^-}}\left(\frac{\partial w_\varepsilon}{\partial n} + \frac{\partial v}{\partial n}\right)^2 dx\right\}. \tag{8.10}$$

We (heuristically) solve this minimization problem to get an explicit approximate expression for $G_\varepsilon^1(u)$ in terms of u. To this end, we notice that $G_\varepsilon^1(u)$ features two terms with different behavior as $\varepsilon \to 0$. Intuitively, the minimizer v_u of this composite energy will to lowest order be determined by the term $\int_{\omega_1} \frac{1}{1+\kappa d_{\Omega^-}} \left(\frac{\partial w_\varepsilon}{\partial n} + \frac{\partial v}{\partial n}\right)^2 dx$. The corresponding Euler-Lagrange equation asserts that v_u must satisfy

$$\forall w \in H_0^1(\omega_1),\ \int_{\omega_1} \frac{1}{1+\kappa d_{\Omega^-}}\left(\frac{\partial v_u}{\partial n} + \frac{\partial w_\varepsilon}{\partial n}\right)\frac{\partial w}{\partial n}\,dx = 0\ .$$

Arguing as in Section 4.1.1 (that is, taking $w(x+tn(x)) = \phi(x)\psi(t)$ with arbitrary $\phi \in C^\infty(\sigma)$ and $\psi \in C_c^\infty(-1,1)$, and using Proposition 1) we conclude that the function $t \mapsto v_u(x+tn(x))$ is affine for any fixed $x\in\sigma$. The boundary conditions of problem (8.10) now give

$$\forall x\in\sigma,\ t\in(-1,1),\ \ v_u(x+tn(x)) = \frac{t}{2}[u](x) + \frac{1}{2}(u^+(x)+u^-(x))\ .$$

Inserting this expression into (8.10), and using (8.2) as well as Proposition 1, we arrive at the minimization problem

$$\min_{\substack{u\in V_\sigma \\ u=0 \text{ on } \partial\Omega}} E_\varepsilon^1(u), \tag{8.11}$$

where $E^1_\varepsilon(u) := F^{1,1}_\varepsilon(u, v_u)$ has the following expression

$$E^1_\varepsilon(u) := \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u|^2\, dx + \frac{a_\varepsilon}{4\varepsilon}\int_\sigma \left(u^+ + \frac{\partial u^{0+}_\varepsilon}{\partial n} - \left(u^- - \frac{\partial u^{0-}_\varepsilon}{\partial n}\right)\right)^2 ds$$

$$+\frac{\varepsilon a_\varepsilon}{3}\int_\sigma \left(\frac{\partial}{\partial \tau}\left(u^+ + \frac{\partial u^{0+}_\varepsilon}{\partial n}\right)\right)^2 + \left(\frac{\partial}{\partial \tau}\left(u^- - \frac{\partial u^{0-}_\varepsilon}{\partial n}\right)\right)^2 + \left(\frac{\partial}{\partial \tau}\left(u^+ + \frac{\partial u^{0+}_\varepsilon}{\partial n}\right)\right)\left(\frac{\partial}{\partial \tau}\left(u^- - \frac{\partial u^{0-}_\varepsilon}{\partial n}\right)\right) ds\,.$$

The solution $u_{1,\varepsilon}$ to this minimization problem is our candidate for a uniform approximation to $\overline{u_{1,\varepsilon}}$. The function $v_{1,\varepsilon} \in H^1(\omega_1)$ defined in the rescaled inhomogeneity by

$$\forall x \in \sigma,\ t \in (-1,1),\ \ v_{1,\varepsilon}(x + tn(x)) = \frac{t}{2}[u_{1,\varepsilon}](x) + \frac{1}{2}(u^+_{1,\varepsilon}(x) + u^-_{1,\varepsilon}(x))$$

is our candidate for an approximation to $\overline{v_{1,\varepsilon}}$.

8.2. 0^{th}-order approximation of the couple $(\overline{u_{2,\varepsilon}}, \overline{v_{2,\varepsilon}})$ and the uniform first order approximation result.

Let us now turn our attention to the uniform approximation of the solution $(\overline{u_{2,\varepsilon}}, \overline{v_{2,\varepsilon}})$ to the problem

$$\min_{\substack{(u,v)\in \overline{V^0_\varepsilon}\\ u=0 \text{ on } \partial\Omega}} \overline{F^{1,2}_\varepsilon}(u,v)\ , \tag{8.12}$$

where the energy $\overline{F^{1,2}_\varepsilon}(u,v)$ is given by (8.7). Performing calculations somewhat more complicated than those in the previous section it is possible heuristically to arrive at a candidate $(u_{2,\varepsilon}, v_{2,\varepsilon})$ for a uniform approximation. We shall not present these calculations here, but only state the result:

The function $u_{2,\varepsilon}$ is the solution to the problem

$$\min_{\substack{u\in V_\sigma\\ u=0 \text{ on } \partial\Omega}} E^2_\varepsilon(u)\ , \tag{8.13}$$

where the functional E^2_ε is given by

$$E^2_\varepsilon(u) = \frac{1}{2}\int_{\Omega\setminus\sigma} |\nabla u|^2\, dx + \frac{\varepsilon a_\varepsilon}{3}\int_\sigma \left(\left(\frac{\partial u^+}{\partial \tau}\right)^2 + \left(\frac{\partial u^-}{\partial \tau}\right)^2 + \frac{\partial u^+}{\partial \tau}\frac{\partial u^-}{\partial \tau}\right) ds + \frac{a_\varepsilon}{4\varepsilon}\int_\sigma (u^+ - u^-)^2\, ds$$
$$+\int_\sigma \frac{\partial^2 u^{0+}_\varepsilon}{\partial \tau^2} u^+\, ds + \int_\sigma \frac{\partial^2 u^{0-}_\varepsilon}{\partial \tau^2} u^-\, ds + \frac{1}{4}\int_\sigma \kappa \left[\frac{\partial u^0_\varepsilon}{\partial n}\right](u^+ - u^-)\, ds\ . \tag{8.14}$$

The function $v_{2,\varepsilon} \in H^1(\omega_1)$ is defined as

$$\forall x \in \sigma, \forall t \in (-1,1),\ \ v_{2,\varepsilon}(x + tn(x)) = \frac{t}{2}\left[u_{2,\varepsilon}\right](x) + \frac{1}{2}\left(u^+_{2,\varepsilon}(x) + u^-_{2,\varepsilon}(x)\right) + w_{2,\varepsilon}\ , \tag{8.15}$$

the function $w_{2,\varepsilon} \in H^1(\omega_1)$ being given by

$$\forall x \in \sigma,\ \begin{cases} w_{2,\varepsilon}(x + tn(x)) = t^2 a^+(x) + tb(x) + c(x),\ \forall t \in (0,1),\ x \in \sigma\ ,\\ w_{2,\varepsilon}(x + tn(x)) = t^2 a^-(x) + tb(x) + c(x),\ \forall t \in (-1,0),\ x \in \sigma\ , \end{cases} \tag{8.16}$$

with

$$a^\pm(x) = -\frac{\varepsilon\kappa(x)}{2a_\varepsilon}\frac{\partial u^{0\pm}_\varepsilon}{\partial n}(x) + \frac{\varepsilon}{4}\left(\frac{\partial^2 u^{0+}_\varepsilon}{\partial \tau^2} + \frac{\partial^2 u^{0-}_\varepsilon}{\partial \tau^2}\right)(x), \quad b(x) = \frac{\varepsilon\kappa(x)}{4a_\varepsilon}\left[\frac{\partial u^0_\varepsilon}{\partial n}\right](x),$$

$$c(x) = \frac{\varepsilon\kappa(x)}{4a_\varepsilon}\left(\frac{\partial u^{0+}_\varepsilon}{\partial n}(x) + \frac{\partial u^{0-}_\varepsilon}{\partial n}(x)\right) - \frac{\varepsilon}{4}\left(\frac{\partial^2 u^{0+}_\varepsilon}{\partial \tau^2} + \frac{\partial^2 u^{0-}_\varepsilon}{\partial \tau^2}\right)(x)\ .$$

It is then possible to prove that

$$||\overline{u_{2,\varepsilon}} - u_{2,\varepsilon}||_{L^2(\Omega^+\setminus\overline{\omega_\delta})} + ||\overline{u_{2,\varepsilon}} - u_{2,\varepsilon}||_{L^2_0(\Omega^-\setminus\overline{\omega_\delta})} \le C\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)\sqrt{\varepsilon}\ ,$$

with C independent of ε and a_ε. Combining the decompositions (8.1) and (8.5) with (8.3) and the above estimates for $\overline{u_{1,\varepsilon}} - u_{1,\varepsilon}$ and $\overline{u_{2,\varepsilon}} - u_{2,\varepsilon}$ we would now arrive at the following theorem

Theorem 12. *In the situation described in Section 2.1, let $\delta > 0$ be a fixed positive real number, $f \in \mathcal{F}_\delta$, and $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$. Let $u_\varepsilon \in H^1(\Omega)$, be the unique solution of the minimization problem (4.1), let u_ε^0 be the unique solution to (4.10)), and $u_{1,\varepsilon}, u_{2,\varepsilon}$ be the unique solutions of (8.11) and (8.13). Then the following estimates hold for $\varepsilon > 0$ sufficiently small*

$$||u_\varepsilon - u_\varepsilon^0 \ - \varepsilon(u_{1,\varepsilon} + u_{2,\varepsilon})||_{L^2(\Omega^+ \setminus \overline{\omega_\delta})} \leq C\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)\varepsilon^{3/2} \ ,$$

$$||u_\varepsilon - u_\varepsilon^0 - \varepsilon(u_{1,\varepsilon} + u_{2,\varepsilon})||_{L_0^2(\Omega^- \setminus \overline{\omega_\delta})} \leq C\left(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}\right)\varepsilon^{3/2} \ ,$$

where the constant C depends only on Ω and σ, and is independent of f, φ, ε and the sequence a_ε.

Remark 10. In view of these results it is interesting to expand a little on the discussion of Section 7.2, concerning the comparison between the uniform asymptotic expansion of u_ε (uniform, with respect to the conductivity a_ε) and the "natural" asymptotic expansion (7.13) in the particular case where a_ε is a fixed real number $a > 0$ independent of ε.

For fixed $a_\varepsilon = a$, arguing as in Section 7.2, one may show that the following expansion holds for the first-order term $(u_{1,\varepsilon} + u_{2,\varepsilon})$ of the uniform asymptotic expansion of u_ε as $\varepsilon \to 0$

$$u_{1,\varepsilon} + u_{2,\varepsilon} = U_1 + \mathcal{O}_a(\varepsilon) \ ,$$

where $U_1 \in V_{\sigma,0}$ is characterized by the following equations

$$\begin{cases} -\Delta U_1 = 0 & \text{in } \Omega \setminus \sigma \ , \\ U_1 = 0 & \text{on } \partial\Omega \ , \\ [U_1] = -2\frac{\partial u_0^\infty}{\partial n} & \text{on } \sigma \ , \\ \left[\frac{\partial U_1}{\partial n}\right] = 2\frac{\partial^2 u_0^\infty}{\partial \tau^2} & \text{on } \sigma \ . \end{cases}$$

Hence, it is verified exactly how the first-order term $\widetilde{u_1}$ of the a-dependent "natural" asymptotic expansion of u_ε (defined as in (7.14), but with a homogeneous Dirichlet boundary condition on $\partial\Omega$) decomposes as the sum of the first-order term u_1 of the principal uniform expansion u_ε^0 (defined by (7.12)) and of the leading term U_1 of the first-order term $(u_{1,\varepsilon} + u_{2,\varepsilon})$ in the uniform expansion of u_ε.

Appendix A. Proof of the uniform regularity estimates for u_ε^0

This appendix is devoted to the proof of Theorem 8. For the reader's convenience, let us first recall a useful characterization of $W^{1,p}$ spaces. Let $\Omega \subset \mathbb{R}^2$ be an open set, and suppose $1 < p \leq \infty$; define $1 \leq p' < \infty$ by the relation $\frac{1}{p} + \frac{1}{p'} = 1$. For any function $u \in L^p(\Omega)$, any open subset $V \Subset \Omega$ and any vector $h \in \mathbb{R}^2$ with $|h| < \text{dist}(V, \partial\Omega)$, we define the *difference quotient* $D_h u \in L^p(V)$ by

$$\forall x \in V, \ \ D_h u(x) = \frac{u(x+h) - u(x)}{|h|} \ .$$

If Ω and V are both convex, then it is fairly simple to prove that

$$||D_h u||_{L^p(V)} \leq ||\nabla u||_{L^p(\Omega)} \ ,$$

for any vector $h \in \mathbb{R}^2$ with $|h| < \text{dist}(V, \partial\Omega)$. The related complete characterization of $W^{1,p}$ spaces we have in mind is the following (see [8], Prop. 9.3)

Proposition 13. *Let $u \in L^p(\Omega)$. Then the following assertions are equivalent*

- (i) *u belongs to $W^{1,p}(\Omega)$,*
- (ii) *there exists a constant $C > 0$ such that*

$$\left|\int_\Omega u\frac{\partial v}{\partial x_i}\,dx\right| \leq C||v||_{L^{p'}(\Omega)}, \ \textit{for any } v \in \mathcal{C}_c^\infty(\Omega), \ \forall i = 1,2 \ ,$$

- (iii) *there exists a constant $C > 0$ such that, for any open subset $V \Subset \Omega$,*

$$\limsup_{h\to 0} ||D_h u||_{L^p(V)} \leq C \ .$$

Furthermore, the smallest constant C satisfying (ii) or (iii) is $C = ||\nabla u||_{L^p(\Omega)}$.

We are now in position to prove the desired result.

Proof of Theorem 8. The proof of this result is an adaptation of that of Theorem 9.25 in [8], and relies on the method of translations. First we observe that, by a standard argument of partition of unity, it is enough to prove that u_ε^0 belongs to $H^2(V\setminus\sigma)$ and that the estimate (5.11) holds with $V\setminus\sigma$ instead of $\Omega\setminus\sigma$, where V is a sufficiently small (convex) neighborhood in Ω of an arbitrary point $x_0\in\overline{\Omega}$. Three cases must be distinguished:

(*i*) x_0 belongs to $\Omega\setminus\sigma$,
(*ii*) x_0 lies on $\partial\Omega$,
(*iii*) x_0 lies on σ .

The uniform estimate (5.12) arises as a consequence of the treatment of case (*iii*).

• *Case (i):* Let V and W be open convex subsets of Ω^+ (or Ω^-) with $V\Subset W\Subset\Omega^+$ (or Ω^-). Let $\chi\in\mathcal{C}_c^\infty(\Omega\setminus\sigma)$ be a smooth cutoff function with

$$0\le\chi\le 1,\ \ \chi\equiv 1 \text{ on } V, \ \text{ and } \chi\equiv 0 \text{ on } \Omega\setminus\overline{W}\ .$$

Then, for any test function $v\in H^1(\Omega\setminus\sigma)$,

$$\text{(A.1)}\qquad \begin{aligned}\int_W\nabla\left(\chi u_\varepsilon^0\right)\cdot\nabla v\,dx &= \int_W\chi\nabla u_\varepsilon^0\cdot\nabla v\,dx+\int_W u_\varepsilon^0\nabla\chi\cdot\nabla v\,dx\\ &= \int_W\nabla u_\varepsilon^0\cdot\nabla\left(\chi v\right)\,dx-\int_W v\nabla u_\varepsilon^0\cdot\nabla\chi\,dx+\int_W u_\varepsilon^0\nabla\chi\cdot\nabla v\,dx\ ,\\ &= \int_W f\chi v\,dx-\int_W v\nabla u_\varepsilon^0\cdot\nabla\chi\,dx+\int_W u_\varepsilon^0\nabla\chi\cdot\nabla v\,dx\ ,\end{aligned}$$

where we used the variational formulation (5.1) with a test function whose support is compact in $\Omega\setminus\sigma$. Let us now define $w_\varepsilon:=\chi u_\varepsilon^0$. Our goal is to use the method of translations to show that ∇w_ε belongs to $H^1(\Omega\setminus\sigma)$. Let $h\in\mathbb{R}^2$ be any vector of sufficiently small length, and let us insert $D_{-h}D_hw_\varepsilon\in H^1(\Omega\setminus\sigma)$ as a test function in (A.1). The result is

$$\text{(A.2)}\qquad \begin{aligned}\int_{\Omega\setminus\sigma}|\nabla D_hw_\varepsilon|^2\,dx &= \int_{\Omega\setminus\sigma}D_h(\chi f)D_hw_\varepsilon\,dx-\int_{\Omega\setminus\sigma}(D_{-h}D_hw_\varepsilon)\nabla u_\varepsilon^0\cdot\nabla\chi\,dx\\ &\quad+\int_{\Omega\setminus\sigma}D_hu_\varepsilon^0\nabla\chi(x+h)\cdot\nabla D_hw_\varepsilon\,dx+\int_{\Omega\setminus\sigma}u_\varepsilon^0\nabla D_h\chi\cdot\nabla D_hw_\varepsilon\,dx\ .\end{aligned}$$

Here we have used the following formula for the difference quotient of a product

$$D_h(uv)(x)=D_hu(x)v(x+h)+u(x)D_hv(x)\ ,$$

as well as "discrete integration by parts" for the difference quotients (which is nothing but change of variables in the corresponding integrals). We recall that for h sufficiently small (less than $\frac{1}{2}\mathrm{dist}(W,\partial(\Omega\setminus\sigma))$), D_hw_ε has compact support in some convex $\widetilde{W}$, with $W\Subset\widetilde{W}\Subset\Omega^+$ (or Ω^-). From (A.2) we now obtain

$$\text{(A.3)}\qquad \begin{aligned}\limsup_{h\to 0}\|\nabla D_hw_\varepsilon\|^2_{L^2(\widetilde{W})} &\le C\limsup_{h\to 0}\|D_h(\chi f)\|_{H^{-1}(\widetilde{W})}\ \limsup_{h\to 0}\|D_hw_\varepsilon\|_{H^1(\widetilde{W})}\\ &\quad+\ C\limsup_{h\to 0}\|D_{-h}D_hw_\varepsilon\|_{L^2(\widetilde{W})}\|\nabla u_\varepsilon^0\|_{L^2(\widetilde{W})}\\ &\quad+\ C\ (\limsup_{h\to 0}\|D_hu_\varepsilon^0\|_{L^2(\widetilde{W})}+\|u_\varepsilon^0\|_{L^2(\widetilde{W})})\ \limsup_{h\to 0}\|\nabla D_hw_\varepsilon\|_{L^2(\widetilde{W})}\\ &\le C(\|u_\varepsilon^0\|_{L^2(\widetilde{W})}+\|\nabla u_\varepsilon^0\|_{L^2(\widetilde{W})})\limsup_{h\to 0}\|\nabla D_hw_\varepsilon\|_{L^2(\widetilde{W})}\\ &\quad+\ C\|f\|_{L^2(\Omega)}\limsup_{h\to 0}\|D_hw_\varepsilon\|_{H^1(\widetilde{W})}\ .\end{aligned}$$

Using the Poincaré inequality for $H^1(\widetilde{W})$ functions vanishing on $\partial\widetilde{W}$, we have that there exists a constant C which only depends on $\widetilde{W}$ such that

$$\|D_hw_\varepsilon\|_{H^1(\widetilde{W})}\le C\|\nabla D_hw_\varepsilon\|_{L^2(\widetilde{W})}\ .$$

From (A.3) we conclude that

$$\text{(A.4)}\qquad \limsup_{h\to 0}\|\nabla D_hw_\varepsilon\|^2_{L^2(\widetilde{W})}\le C(\|f\|_{L^2(\Omega)}+\|u_\varepsilon^0\|_{L^2(\widetilde{W})}+\|\nabla u_\varepsilon^0\|_{L^2(\widetilde{W})})\ \limsup_{h\to 0}\|\nabla D_hw_\varepsilon\|_{L^2(\widetilde{W})}\ .$$

If $\widetilde{W} \Subset \Omega^+$ then, due to Lemma 7,

$$\begin{aligned} ||u_\varepsilon^0||_{L^2(\widetilde{W})}+||\nabla u_\varepsilon^0||_{L^2(\widetilde{W})} &\leq ||u_\varepsilon^0||_{L^2(\Omega^+)}+||\nabla u_\varepsilon^0||_{L^2(\Omega^+)} \\ &\leq C(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}) \ . \end{aligned}$$

On the other hand, if $\widetilde{W}$ is a subset of Ω^-, then we have a priori no bound on $||u_\varepsilon^0||_{L^2(\Omega^-)}$. To circumvent this we note that from the very beginning, we could re-write the entire argument by replacing u_ε^0 in the various integral inequalities by $u_\varepsilon^0 - m$, where m is an arbitrary constant; this includes the very definition of w_ε which now becomes $w_\varepsilon = \chi(u_\varepsilon^0 - m)$. We select $m = \frac{1}{|\Omega^-|}\int_{\Omega^-} u_\varepsilon^0 \, dx$, and from the "revised" version of (A.4) we now obtain

$$\begin{aligned} \limsup_{h\to 0}||\nabla D_h w_\varepsilon||_{L^2(\widetilde{W})} &\leq C(||f||_{L^2(\Omega)}+||u_\varepsilon^0 - m||_{L^2(\Omega^-)}+||\nabla u_\varepsilon^0||_{L^2(\Omega^-)}) \\ &\leq C(||f||_{L^2(\Omega)}+||\nabla u_\varepsilon^0||_{L^2(\Omega^-)}) \\ &\leq C(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}) \ , \end{aligned}$$

owing to the Poincaré-Wirtinger inequality and Lemma 7. Whether $\widetilde{W}$ is a subset of Ω^+ or Ω^-, Proposition 13 now allows us to conclude that all the entries of the Hessian matrix $\nabla^2 w_\varepsilon$ belong to $L^2(W)$, and that the following inequality holds

$$|u_\varepsilon^0|_{H^2(V)} \leq |w_\varepsilon|_{H^2(W)} \leq C(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}) \ .$$

(ii) The proof in this case is similar to that of (i), modulo the usual changes of the method of translation due to the presence of the boundary (again, see [8], Theorem 9.25). We omit the details and concentrate instead on those of case (iii).

(iii) Let $V \Subset \Omega$ be a sufficiently small convex neighborhood of the point $x_0 \in \sigma$. Let W be another convex open subset of Ω such that $V \Subset W \Subset \Omega$, and let $\chi \in \mathcal{C}_c^\infty(\Omega)$ be a smooth cutoff function such that

$$0 \leq \chi \leq 1, \ \ \chi \equiv 1 \text{ on } V, \ \ \chi \equiv 0 \text{ on } \Omega \setminus \overline{W} \ .$$

To simplify notations, we assume that $\sigma \cap W$ is flat (the general case being no more difficult, but more involved as far as notations are concerned): the tangent vector τ to σ is the coordinate vector e_x, and the normal vector n, pointing outward from Ω^-, is e_y. Following the steps of the proof of (i), let $w_\varepsilon = \chi(u_\varepsilon^0 - m)$, for some constant m to be specified later. A simple calculation reveals that w_ε satisfies

$$\begin{aligned} \text{(A.5)} \quad &\int_{\Omega\setminus\sigma} \nabla w_\varepsilon \cdot \nabla v \, dx + \frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \left(\frac{\partial w_\varepsilon^+}{\partial\tau}\frac{\partial v^+}{\partial\tau} + \frac{\partial w_\varepsilon^-}{\partial\tau}\frac{\partial v^-}{\partial\tau} + \frac{1}{2}\left(\frac{\partial w_\varepsilon^+}{\partial\tau}\frac{\partial v^-}{\partial\tau} + \frac{\partial w_\varepsilon^-}{\partial\tau}\frac{\partial v^+}{\partial\tau} \right)\right) ds \\ &\qquad + \frac{a_\varepsilon}{2\varepsilon}\int_\sigma (w_\varepsilon^+ - w_\varepsilon^-)(v^+ - v^-) \, ds = \\ &\int_{\Omega\setminus\sigma} g_\varepsilon v \, dx + \int_{\Omega\setminus\sigma} h_\varepsilon \cdot \nabla v \, dx - \frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \frac{\partial\chi}{\partial\tau}\left(v^+\frac{\partial u_\varepsilon^{0+}}{\partial\tau} + v^-\frac{\partial u_\varepsilon^{0-}}{\partial\tau} + \frac{1}{2}\left(v^+\frac{\partial u_\varepsilon^{0-}}{\partial\tau} + v^-\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right)\right) ds \\ &\quad + \frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \frac{\partial\chi}{\partial\tau}\left((u_\varepsilon^{0+} - m)\frac{\partial v^+}{\partial\tau} + (u_\varepsilon^{0-} - m)\frac{\partial v^-}{\partial\tau} + \frac{1}{2}\left((u_\varepsilon^{0+} - m)\frac{\partial v^-}{\partial\tau} + (u_\varepsilon^{0-} - m)\frac{\partial v^+}{\partial\tau}\right)\right) ds \ , \end{aligned}$$

for all $v \in V_{\sigma,0}$. Here $g_\varepsilon = f\chi - \nabla u_\varepsilon^0 \cdot \nabla\chi$ and $h_\varepsilon = (u_\varepsilon^0 - m)\nabla\chi$.

Let us introduce $m_0 = \frac{1}{|\sigma|}\int_\sigma u_\varepsilon^{0-} \, ds$ and $m_1 = \frac{1}{|\sigma|}\int_\sigma u_\varepsilon^{0+} \, ds$, and let w_ε^i be defined as $w_\varepsilon^i = \chi(u_\varepsilon^0 - m_i)$, $i = 0, 1$. We now use the method of translations to show that the tangential derivatives $\frac{\partial w_\varepsilon^0}{\partial\tau}$ and $\frac{\partial w_\varepsilon^1}{\partial\tau}$ belong to $H^1(W^-)$ and $H^1(W^+)$, respectively. To this end, let $h = t\tau = te_x$, for $t > 0$ sufficiently small, and choose $v = D_{-h}D_h w_\varepsilon^0$ in W^- and $v = 0$ in W^+, and then $v = 0$ in W^- and $v = D_{-h}D_h w_\varepsilon^1$ in W^+ as test functions in (A.5). This yields

$$\text{(A.6)}\quad \int_{\Omega^-} |\nabla D_h w_\varepsilon^0|^2\,dx + \frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \left(\left(\frac{\partial D_h w_\varepsilon^{0-}}{\partial\tau}\right)^2 + \frac{1}{2}\frac{\partial D_h w_\varepsilon^{0+}}{\partial\tau}\frac{\partial D_h w_\varepsilon^{0-}}{\partial\tau}\right) ds$$
$$+ \frac{a_\varepsilon}{2\varepsilon}\int_\sigma (D_h w_\varepsilon^{0+} - D_h w_\varepsilon^{0-})(-D_h w_\varepsilon^{0-})\,ds = -\frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \frac{\partial\chi}{\partial\tau}(x+h) D_h w_\varepsilon^{0-}\left(\frac{\partial D_h u_\varepsilon^{0-}}{\partial\tau} + \frac{1}{2}\frac{\partial D_h u_\varepsilon^{0+}}{\partial\tau}\right) ds$$
$$-\frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \frac{\partial D_h\chi}{\partial\tau} D_h w_\varepsilon^{0-}\left(\frac{\partial u_\varepsilon^{0-}}{\partial\tau} + \frac{1}{2}\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right) ds + \frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \frac{\partial\chi}{\partial\tau}(x+h)\frac{\partial D_h w_\varepsilon^{0-}}{\partial\tau}\left(D_h u_\varepsilon^{0-} + \frac{1}{2} D_h u_\varepsilon^{0+}\right) ds$$
$$+\frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \frac{\partial D_h\chi}{\partial\tau}\frac{\partial D_h w_\varepsilon^{0-}}{\partial\tau}\left((u_\varepsilon^{0-} - m_0) + \frac{1}{2}(u_\varepsilon^{0+} - m_0)\right) ds + \int_{\Omega^-} D_h g_\varepsilon D_h w_\varepsilon^0\,dx + \int_{\Omega^-} D_h h_\varepsilon^0 \cdot \nabla D_h w_\varepsilon^0\,dx\,,$$

where $h_\varepsilon^0 = (u_\varepsilon^0 - m_0)\nabla\chi$, and

$$\text{(A.7)}\quad \int_{\Omega^+} |\nabla D_h w_\varepsilon^1|^2\,dx + \frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \left(\left(\frac{\partial D_h w_\varepsilon^{1+}}{\partial\tau}\right)^2 + \frac{1}{2}\frac{\partial D_h w_\varepsilon^{1+}}{\partial\tau}\frac{\partial D_h w_\varepsilon^{1-}}{\partial\tau}\right) ds$$
$$+ \frac{a_\varepsilon}{2\varepsilon}\int_\sigma (D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-}) D_h w_\varepsilon^{1+}\,ds = -\frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \frac{\partial\chi}{\partial\tau}(x+h) D_h w_\varepsilon^{1+}\left(\frac{\partial D_h u_\varepsilon^{0+}}{\partial\tau} + \frac{1}{2}\frac{\partial D_h u_\varepsilon^{0-}}{\partial\tau}\right) ds$$
$$-\frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \frac{\partial D_h\chi}{\partial\tau} D_h w_\varepsilon^{1+}\left(\frac{\partial u_\varepsilon^{0+}}{\partial\tau} + \frac{1}{2}\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right) ds + \frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \frac{\partial\chi}{\partial\tau}(x+h)\frac{\partial D_h w_\varepsilon^{1+}}{\partial\tau}\left(D_h u_\varepsilon^{0+} + \frac{1}{2} D_h u_\varepsilon^{0-}\right) ds$$
$$+\frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \frac{\partial D_h\chi}{\partial\tau}\frac{\partial D_h w_\varepsilon^{1+}}{\partial\tau}\left((u_\varepsilon^{0+} - m_1) + \frac{1}{2}(u_\varepsilon^{0-} - m_1)\right) ds + \int_{\Omega^+} D_h g_\varepsilon D_h w_\varepsilon^1\,dx + \int_{\Omega^+} D_h h_\varepsilon^1 \cdot \nabla D_h w_\varepsilon^1\,dx\,,$$

where $h_\varepsilon^1 = (u_\varepsilon^0 - m_1)\nabla\chi$. Note that, by performing an integration by parts on the first integral in the right-hand side of (A.6), we can rewrite

(A.8)
$$\int_\sigma \frac{\partial\chi}{\partial\tau}(x+h) D_h w_\varepsilon^{0-}\left(\frac{\partial D_h u_\varepsilon^{0-}}{\partial\tau} + \frac{1}{2}\frac{\partial D_h u_\varepsilon^{0+}}{\partial\tau}\right) ds = -\int_\sigma \frac{\partial^2\chi}{\partial\tau^2}(x+h) D_h w_\varepsilon^{0-}\left(D_h u_\varepsilon^{0-} + \frac{1}{2} D_h u_\varepsilon^{0+}\right) ds$$
$$-\int_\sigma \frac{\partial\chi}{\partial\tau}(x+h)\frac{\partial D_h w_\varepsilon^{0-}}{\partial\tau}\left(D_h u_\varepsilon^{0-} + \frac{1}{2} D_h u_\varepsilon^{0+}\right) ds\,;$$

a similar identity holds for the first integral in the right-hand side of (A.7). Combining (A.6), (A.7) and (A.8), we obtain

$$\text{(A.9)}\quad \int_{\Omega^-} |\nabla D_h w_\varepsilon^0|^2\,dx + \frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \left(\left(\frac{\partial D_h w_\varepsilon^{0-}}{\partial\tau}\right)^2 + \frac{1}{2}\frac{\partial D_h w_\varepsilon^{0+}}{\partial\tau}\frac{\partial D_h w_\varepsilon^{0-}}{\partial\tau}\right) ds$$
$$+ \frac{a_\varepsilon}{2\varepsilon}\int_\sigma (D_h w_\varepsilon^{0+} - D_h w_\varepsilon^{0-})(-D_h w_\varepsilon^{0-})\,ds \le$$
$$C\varepsilon a_\varepsilon ||D_h w_\varepsilon^{0-}||_{L^2(\sigma)}\left(||D_h u_\varepsilon^{0-}||_{L^2(\sigma\cap W)} + ||D_h u_\varepsilon^{0+}||_{L^2(\sigma\cap W)} + \left|\left|\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right|\right|_{L^2(\sigma)} + \left|\left|\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right|\right|_{L^2(\sigma)}\right)$$
$$+ C\varepsilon a_\varepsilon \left|\left|\frac{\partial D_h w_\varepsilon^{0-}}{\partial\tau}\right|\right|_{L^2(\sigma)}\left(||D_h u_\varepsilon^{0-}||_{L^2(\sigma\cap W)} + ||D_h u_\varepsilon^{0+}||_{L^2(\sigma\cap W)} + ||u_\varepsilon^{0-} - m_0||_{L^2(\sigma)} + ||u_\varepsilon^{0+} - m_0||_{L^2(\sigma)}\right)$$
$$+ ||D_h g_\varepsilon||_{H^{-1}(W^-)}||D_h w_\varepsilon^0||_{H^1(W^-)} + ||D_h h_\varepsilon^0||_{L^2(W^-)}||\nabla D_h w_\varepsilon^0||_{L^2(W^-)}\,,$$

and

$$\text{(A.10)}\quad \int_{\Omega^+} |\nabla D_h w_\varepsilon^1|^2\,dx + \frac{2\varepsilon a_\varepsilon}{3}\int_\sigma \left(\left(\frac{\partial D_h w_\varepsilon^{1+}}{\partial\tau}\right)^2 + \frac{1}{2}\frac{\partial D_h w_\varepsilon^{1+}}{\partial\tau}\frac{\partial D_h w_\varepsilon^{1-}}{\partial\tau}\right) ds$$

$$+ \frac{a_\varepsilon}{2\varepsilon}\int_\sigma (D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-}) D_h w_\varepsilon^{1+}\,ds \leq$$

$$C\varepsilon a_\varepsilon ||D_h w_\varepsilon^{1+}||_{L^2(\sigma)} \left(\left|\left|D_h u_\varepsilon^{0-}\right|\right|_{L^2(\sigma\cap W)} + \left|\left|D_h u_\varepsilon^{0+}\right|\right|_{L^2(\sigma\cap W)} + \left|\left|\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right|\right|_{L^2(\sigma)} + \left|\left|\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right|\right|_{L^2(\sigma)}\right)$$

$$+ C\varepsilon a_\varepsilon \left|\left|\frac{\partial D_h w_\varepsilon^{1+}}{\partial\tau}\right|\right|_{L^2(\sigma)} \left(\left|\left|D_h u_\varepsilon^{0-}\right|\right|_{L^2(\sigma\cap W)} + \left|\left|D_h u_\varepsilon^{0+}\right|\right|_{L^2(\sigma\cap W)} + \left|\left|u_\varepsilon^{0-} - m_1\right|\right|_{L^2(\sigma)} + \left|\left|u_\varepsilon^{0+} - m_1\right|\right|_{L^2(\sigma)}\right)$$

$$+ ||D_h g_\varepsilon||_{H^{-1}(W^+)} ||D_h w_\varepsilon^1||_{H^1(W^+)} + ||D_h h_\varepsilon^1||_{L^2(W^+)} ||\nabla D_h w_\varepsilon^1||_{L^2(W^+)}.$$

Some of the terms in the right hand sides of the above inequalities can be estimated further. Owing to Poincaré's inequality, there exists a constant C (which only depends on W and σ) such that for any function $u \in H^1(W \setminus \sigma)$ with $u = 0$ on ∂W,

$$\text{(A.11)}\qquad ||u||_{H^1(W^\pm)} \leq C ||\nabla u||_{L^2(W^\pm)}\ .$$

Similarly, there exists a constant C (still depending only on W and σ) such that for any function $u \in H^1(\sigma)$ with $u = 0$ on $\partial W \cap \sigma$,

$$\text{(A.12)}\qquad ||u||_{L^2(W\cap\sigma)} \leq C \left|\left|\frac{\partial u}{\partial\tau}\right|\right|_{L^2(W\cap\sigma)}\ .$$

From Proposition 13 (and the equivalent for σ) we conclude that

$$\text{(A.13)}\qquad \forall u \in H^1(\sigma),\ \limsup_{\substack{h=te_x\\ t\to 0}} ||D_h u||_{L^2(\sigma\cap W)} \leq \left|\left|\frac{\partial u}{\partial\tau}\right|\right|_{L^2(\sigma)}\ ,$$

$$\forall u \in H^1(\Omega\setminus\sigma),\ \limsup_{\substack{h=te_x\\ t\to 0}} ||D_h u||_{L^2(W\setminus\sigma)} \leq ||\nabla u||_{L^2(\Omega\setminus\sigma)}\ .$$

In particular we deduce from (A.13) that

$$\text{(A.14)}\qquad \limsup_{\substack{h=te_x\\ t\to 0}} ||D_h u_\varepsilon^{0\pm}||_{L^2(\sigma\cap W)} \leq \left|\left|\frac{\partial u_\varepsilon^{0\pm}}{\partial\tau}\right|\right|_{L^2(\sigma)}\ .$$

Using (A.11), we obtain that there exists a constant C, independent of ε and a_ε, such that

$$\text{(A.15)}\qquad ||D_h w_\varepsilon^0||_{H^1(W^-)} \leq C ||\nabla D_h w_\varepsilon^0||_{L^2(\widetilde{W}^-)}\ ,\ \text{and}\ \ ||D_h w_\varepsilon^1||_{H^1(W^+)} \leq C ||\nabla D_h w_\varepsilon^1||_{L^2(\widetilde{W}^+)}\ .$$

From the a priori estimates of Lemma 7 it also follows that

$$\text{(A.16)}\qquad \limsup_{\substack{h=te_x\\ t\to 0}} \left[||D_h g_\varepsilon||_{H^{-1}(W\setminus\sigma)} + ||D_h h_\varepsilon^0||_{L^2(W^-)} + ||D_h h_\varepsilon^1||_{L^2(W^+)}\right] \leq C(||\varphi||_{H^{1/2}(\partial\Omega)} + ||f||_{L^2(\Omega)})\ .$$

From the Poincaré-Wirtinger inequality on σ, we have

$$\text{(A.17)}\qquad ||u_\varepsilon^{0-} - m_0||_{L^2(\sigma)} \leq C \left|\left|\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right|\right|_{L^2(\sigma)}\ ,\ \text{and}\ \ ||u_\varepsilon^{0+} - m_1||_{L^2(\sigma)} \leq C \left|\left|\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right|\right|_{L^2(\sigma)}\ .$$

We now sum (A.9) and (A.10), noticing that $(D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-}) = (D_h w_\varepsilon^{0+} - D_h w_\varepsilon^{0-})$ on σ. Taking into account (A.15), (A.16), (A.17) and (A.14), we arrive at

$$\text{(A.18)}\quad \limsup_{\substack{h=te_x\\ t\to 0}} \Bigg[\int_{\Omega^-} |\nabla D_h w_\varepsilon^0|^2\, dx + \int_{\Omega^+} |\nabla D_h w_\varepsilon^1|^2\, dx + \frac{a_\varepsilon}{2\varepsilon} \int_\sigma (D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-})(D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{0-})\, ds$$
$$+ \frac{2\varepsilon a_\varepsilon}{3} \int_\sigma \left(\left(\frac{\partial D_h w_\varepsilon^{0-}}{\partial \tau} \right)^2 + \left(\frac{\partial D_h w_\varepsilon^{1+}}{\partial \tau} \right)^2 + \frac{1}{2} \left(\frac{\partial D_h w_\varepsilon^{0+}}{\partial \tau} \frac{\partial D_h w_\varepsilon^{0-}}{\partial \tau} + \frac{\partial D_h w_\varepsilon^{1+}}{\partial \tau} \frac{\partial D_h w_\varepsilon^{1-}}{\partial \tau} \right) \right) ds \Bigg] \le$$
$$C\varepsilon a_\varepsilon \limsup_{\substack{h=te_x\\ t\to 0}} \left(\left\| \frac{\partial D_h w_\varepsilon^{0-}}{\partial \tau} \right\|_{L^2(\sigma)} + \left\| \frac{\partial D_h w_\varepsilon^{1+}}{\partial \tau} \right\|_{L^2(\sigma)} \right)$$
$$\times \left(\left\| \frac{\partial u_\varepsilon^{0-}}{\partial \tau} \right\|_{L^2(\sigma)} + \left\| \frac{\partial u_\varepsilon^{0+}}{\partial \tau} \right\|_{L^2(\sigma)} + ||u_\varepsilon^{0+} - m_0||_{L^2(\sigma)} + ||u_\varepsilon^{0-} - m_1||_{L^2(\sigma)} \right)$$
$$+ C(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}) \limsup_{\substack{h=te_x\\ t\to 0}} \left(||\nabla D_h w_\varepsilon^0||_{L^2(\widetilde{W}^-)} + ||\nabla D_h w_\varepsilon^1||_{L^2(\widetilde{W}^+)} \right) .$$

Some terms in this last expression still need to be rewritten. We observe that

$$\text{(A.19)}\quad \begin{aligned} (\tfrac{a_\varepsilon}{\varepsilon})^{\frac12} |m_1 - m_0| &\le C(\tfrac{a_\varepsilon}{\varepsilon})^{\frac12} ||u_\varepsilon^{0+} - u_\varepsilon^{0-}||_{L^2(\sigma)} \\ &\le C(||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}) , \end{aligned}$$

where we used the uniform a priori estimates of Lemma 7. This inequality, in combination with the fact that

$$D_h w_\varepsilon^{0+} - D_h w_\varepsilon^{1+} = (m_1 - m_0) D_h \chi, \text{ and } D_h w_\varepsilon^{1-} - D_h w_\varepsilon^{0-} = (m_0 - m_1) D_h \chi ,$$

allows us to rewrite the last integral in the left-hand side of (A.18) as follows

$$\int_\sigma \left(\left(\frac{\partial D_h w_\varepsilon^{0-}}{\partial \tau} \right)^2 + \left(\frac{\partial D_h w_\varepsilon^{1+}}{\partial \tau} \right)^2 + \frac{1}{2} \left(\frac{\partial D_h w_\varepsilon^{0+}}{\partial \tau} \frac{\partial D_h w_\varepsilon^{0-}}{\partial \tau} + \frac{\partial D_h w_\varepsilon^{1+}}{\partial \tau} \frac{\partial D_h w_\varepsilon^{1-}}{\partial \tau} \right) \right) ds =$$
$$\int_\sigma \left(\left(\frac{\partial D_h w_\varepsilon^{0-}}{\partial \tau} \right)^2 + \left(\frac{\partial D_h w_\varepsilon^{1+}}{\partial \tau} \right)^2 + \frac{\partial D_h w_\varepsilon^{1+}}{\partial \tau} \frac{\partial D_h w_\varepsilon^{0-}}{\partial \tau} \right) ds + \frac{1}{2}(m_1 - m_0) \int_\sigma \frac{\partial D_h \chi}{\partial \tau} \left(\frac{\partial D_h w_\varepsilon^{0-}}{\partial \tau} - \frac{\partial D_h w_\varepsilon^{1+}}{\partial \tau} \right) ds .$$

It follows, using the algebraic identity (5.2) and (A.19), that there exist two positive constants C_1 and C_2, which do not depend on ε or a_ε, such that

$$\text{(A.20)}\quad \varepsilon a_\varepsilon \int_\sigma \left(\left(\frac{\partial D_h w_\varepsilon^{0-}}{\partial \tau} \right)^2 + \left(\frac{\partial D_h w_\varepsilon^{1+}}{\partial \tau} \right)^2 + \frac{1}{2} \left(\frac{\partial D_h w_\varepsilon^{0+}}{\partial \tau} \frac{\partial D_h w_\varepsilon^{0-}}{\partial \tau} + \frac{\partial D_h w_\varepsilon^{1+}}{\partial \tau} \frac{\partial D_h w_\varepsilon^{1-}}{\partial \tau} \right) \right) ds \ge$$
$$C_1 \varepsilon a_\varepsilon \left(\left\| \frac{\partial D_h w_\varepsilon^{0-}}{\partial \tau} \right\|_{L^2(\sigma)}^2 + \left\| \frac{\partial D_h w_\varepsilon^{1+}}{\partial \tau} \right\|_{L^2(\sigma)}^2 \right)$$
$$- C_2 (\varepsilon^3 a_\varepsilon)^{\frac12} (||f||_{L^2(\Omega)} + ||\varphi||_{H^{1/2}(\partial\Omega)}) \left(\left\| \frac{\partial D_h w_\varepsilon^{0-}}{\partial \tau} \right\|_{L^2(\sigma)} + \left\| \frac{\partial D_h w_\varepsilon^{1+}}{\partial \tau} \right\|_{L^2(\sigma)} \right) .$$

We now estimate the next to last integral in the left-hand side of (A.18). It may be rewritten

$$\frac{a_\varepsilon}{2\varepsilon} \int_\sigma (D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-})(D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{0-})\, ds =$$
$$\frac{a_\varepsilon}{2\varepsilon} ||D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-}||_{L^2(\sigma)}^2 + \frac{a_\varepsilon}{2\varepsilon} \int_\sigma (D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-})(D_h w_\varepsilon^{1-} - D_h w_\varepsilon^{0-})\, ds ,$$

with

$$\left|\frac{a_\varepsilon}{2\varepsilon}\int_\sigma (D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-})(D_h w_\varepsilon^{1-} - D_h w_\varepsilon^{0-})\,ds\right|$$
$$= \frac{a_\varepsilon}{2\varepsilon}|m_1 - m_0|\left|\int_\sigma (D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-}) D_h\chi\,ds\right|$$
$$\leq C(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)})(\frac{a_\varepsilon}{\varepsilon})^{\frac{1}{2}}||D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-}||_{L^2(\sigma)}\ ,$$

and so

$$\frac{a_\varepsilon}{2\varepsilon}\int_\sigma (D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-})(D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{0-})\,ds \geq \frac{a_\varepsilon}{2\varepsilon}||D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-}||^2_{L^2(\sigma)} - C(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)})(\frac{a_\varepsilon}{\varepsilon})^{\frac{1}{2}}||D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-}||_{L^2(\sigma)}\ . \tag{A.21}$$

Turning to the right-hand side of (A.18), we have

$$\begin{aligned}
&(\varepsilon a_\varepsilon)^{\frac{1}{2}}\left(||u_\varepsilon^{0+} - m_0||_{L^2(\sigma)}+||u_\varepsilon^{0-} - m_1||_{L^2(\sigma)}\right)\\
&\quad\leq\ C(\varepsilon a_\varepsilon)^{\frac{1}{2}}\left(||u_\varepsilon^{0+} - m_1||_{L^2(\sigma)}+||u_\varepsilon^{0-} - m_0||_{L^2(\sigma)}+|m_1 - m_0|\right)\\
&\quad\leq\ C(\varepsilon a_\varepsilon)^{\frac{1}{2}}\left(\left|\left|\frac{\partial u_\varepsilon^{0+}}{\partial\tau}\right|\right|_{L^2(\sigma)}+\left|\left|\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right|\right|_{L^2(\sigma)}\right)+C(\tfrac{a_\varepsilon}{\varepsilon})^{\frac{1}{2}}|m_1 - m_0|\\
&\quad\leq\ C(||f||_{L^2(\Omega)}+||\varphi||_{H^{1/2}(\partial\Omega)}),
\end{aligned}\tag{A.22}$$

due to (A.19) and the uniform a priori estimates of Lemma 7. Here we have also used that $\varepsilon a_\varepsilon \leq \frac{a_\varepsilon}{\varepsilon}$. Combining (A.18), (A.20), (A.21), (A.22), and using Lemma 7 we finally get

$$\limsup_{h=te_x\ ,\ t\to 0}\begin{pmatrix} ||\nabla D_h w_\varepsilon^0||^2_{L^2(\Omega^-)^2}+||\nabla D_h w_\varepsilon^1||^2_{L^2(\Omega^+)^2}\\ +\varepsilon a_\varepsilon\left(\left|\left|\dfrac{\partial D_h w_\varepsilon^{1+}}{\partial\tau}\right|\right|^2_{L^2(\sigma)}+\left|\left|\dfrac{\partial D_h w_\varepsilon^{0-}}{\partial\tau}\right|\right|^2_{L^2(\sigma)}\right)\\ +\dfrac{a_\varepsilon}{2\varepsilon}||D_h w_\varepsilon^{1+} - D_h w_\varepsilon^{1-}||^2_{L^2(\sigma)}\end{pmatrix}^{\frac{1}{2}} \leq C(||\varphi||_{H^{1/2}(\partial\Omega)}+||f||_{L^2(\Omega)})\ . \tag{A.23}$$

In particular

$$\limsup_{h=te_x\ ,\ t\to 0}\left(||\nabla D_h w_\varepsilon^0||_{L^2(\Omega^-)}+||\nabla D_h w_\varepsilon^1||_{L^2(\Omega^+)}\right) \leq C(||\varphi||_{H^{1/2}(\partial\Omega)}+||f||_{L^2(\Omega)}),$$

from which Proposition 13 allows us to conclude that $\frac{\partial w_\varepsilon^0}{\partial x} = \frac{\partial w_\varepsilon^0}{\partial\tau} \in H^1(W^-)$ and $\frac{\partial w_\varepsilon^1}{\partial x} = \frac{\partial w_\varepsilon^1}{\partial\tau} \in H^1(W^+)$, with the estimate

$$\left|\left|\frac{\partial u_\varepsilon^0}{\partial x}\right|\right|_{H^1(V\setminus\sigma)} \leq \left|\left|\frac{\partial w_\varepsilon^0}{\partial x}\right|\right|_{H^1(W^-)}+\left|\left|\frac{\partial w_\varepsilon^1}{\partial x}\right|\right|_{H^1(W^+)} \leq C(||\varphi||_{H^{1/2}(\partial\Omega)}+||f||_{L^2(\Omega)})\ ,$$

the constant C being independent of ε and a_ε.
We have to obtain the corresponding estimate for $\frac{\partial u_\varepsilon^0}{\partial y}$. First

$$\left|\left|\frac{\partial^2 u_\varepsilon^0}{\partial x\partial y}\right|\right|_{L^2(V\setminus\sigma)^2} \leq \left|\left|\frac{\partial u_\varepsilon^0}{\partial x}\right|\right|_{H^1(V\setminus\sigma)^2} \leq C(||\varphi||_{H^{1/2}(\partial\Omega)}+||f||_{L^2(\Omega)})\ .$$

To get control of $\frac{\partial^2 u_\varepsilon^0}{\partial y^2}$, we go back to the original equation (5.4) satisfied by u_ε^0

$$\frac{\partial^2 u_\varepsilon^0}{\partial y^2} = -f - \frac{\partial^2 u_\varepsilon^0}{\partial x^2}\ \text{in the sense of distributions on } V\setminus\sigma\ .$$

These two observations lead to a uniform $H^1(V\setminus\sigma)$ estimate for $\frac{\partial u_\varepsilon^0}{\partial y}$, and thus to the desired uniform $H^2(V\setminus\sigma)$ seminorm estimate for u_ε^0. From (A.23) it also follows that

$$\varepsilon a_\varepsilon\left(\left|\left|\frac{\partial^2 u_\varepsilon^{0+}}{\partial\tau^2}\right|\right|^2_{L^2(\sigma\cap V)}+\left|\left|\frac{\partial^2 u_\varepsilon^{0-}}{\partial\tau^2}\right|\right|^2_{L^2(\sigma\cap V)}\right)+\frac{a_\varepsilon}{\varepsilon}\left|\left|\frac{\partial u_\varepsilon^{0+}}{\partial\tau}-\frac{\partial u_\varepsilon^{0-}}{\partial\tau}\right|\right|^2_{L^2(\sigma\cap V)} \leq C(||\varphi||_{H^{1/2}(\partial\Omega)}+||f||_{L^2(\Omega)})^2\ ,$$

and this completes the proof of Theorem 8. □

Remark 11. In this proof, we relied in a crucial way on the ordering $\varepsilon a_\varepsilon \leq \frac{a_\varepsilon}{\varepsilon}$ between the coefficients appearing in the approximate energy (4.12). We do not know whether the similar uniform regularity estimate holds in other regimes of coefficients.

Acknowledgements.

CD was partially supported by NSF grant DMS-12-11330 while being a postdoctoral visitor at Rutgers University. The work of MV was supported by the NSF IR/D program while serving at the National Science Foundation. Any opinion, findings, and conclusions or recommendations expressed in this paper are those of the authors, and do not necessarily reflect the views of the National Science Foundation.

References

[1] E. Acerbi and G. Buttazzo, *Reinforcement problems in the calculus of variations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 4, (1986), pp. 273–284.

[2] G. Allaire, C. Dapogny, G. Delgado and G. Michailidis, *Multi-phase optimization via a level set method*, ESAIM: Control, Optimization and Calculus of Variations, 20, 2 (2014), pp. 576–611.

[3] H. Ammari, E. Beretta, and E. Francini, *Reconstruction of thin conductivity imperfections.* Appl. Anal., 83 (2004), pp. 63–76.

[4] H. Ammari and H. Kang, *Reconstruction of Small Inhomogeneities from Boundary Measurements*, Lecture Notes in Mathematics Vol. 1846, Springer, (2004).

[5] H. Attouch, *Variational Convergence for Functions and Operators*, Applicable Mathematics Series, Pitman, London (1984).

[6] E. Beretta and E. Francini, *Asymptotic formulas for perturbations in the electromagnetic fields due to the presence of thin inhomogeneities*, in Inverse Problems: Theory and Applications, Contemp. Math. 333, American Mathematical Society, Providence, RI, (2003).

[7] E. Beretta, E. Francini and M. Vogelius, *Asymptotic formulas for steady state voltage potentials in the presence of thin inhomogeneities. A rigorous error analysis*, J. Math. Pures Appl., 82, (2003), pp. 1277-1301.

[8] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer (2000).

[9] G. Buttazzo and R.V. Kohn, *Reinforcement by a Thin Layer with Oscillating Thickness*, Appl. Math. Optim., 16, (1987), pp. 247–261.

[10] Y. Capdeboscq and M.S. Vogelius, *A general representation formula for boundary voltage perturbations caused by internal conductivity inhomogeneities of low volume fraction*, ESAIM: Math. Mod. Numer. Anal., 37 (2003), pp. 159–173.

[11] D.J. Cedio-Fengya, S. Moskow, and M.S. Vogelius, *Identification of conductivity imperfections of small diameter by boundary measurements. Continuous dependence and computational reconstruction*, Inverse Problems, 14 (1998), pp. 553–595.

[12] I. Chavel, *Riemannian Geometry, a modern introduction*, 2nd Edition, Cambridge University Press (2006).

[13] M.C. Delfour and J.-P. Zolesio, *Shapes and Geometries: Metrics, Analysis, Differential Calculus, and Optimization*, SIAM, Philadelphia 2nd ed. (2011).

[14] I. Ekeland and R. Temam, *Convex Analysis and Variational Problems*, North-Holland, (1976).

[15] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*, CRC Press (1992).

[16] G. B. Folland, *Introduction to Partial Differential Equations*, 2nd Edition, Princeton University Press (1995).

[17] A. Friedman and M. Vogelius, *Determining cracks by boundary measurements*, Indiana Univ. Math. J., 38, (1989), pp. 527–556.

[18] A. Henrot and M. Pierre, *Variation et optimisation de formes, une analyse géométrique*, Springer (2005).

[19] R.V. Kohn and G.W. Milton, *On bounding the effective conductivity of anisotropic composites. Homogenization and Effective Moduli of Materials and Media*, Eds. J.L. Ericksen, D. Kinderlehrer, R. Kohn, and J.-L. Lions, IMA Volumes in Mathematics and its Applications, 1, Springer Verlag, (1986), pp. 97–125.

[20] D. Morgenstern and I. Szabo, *Vorlesungen über theoretische Mechanik*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, (112), Springer-Verlag, Berlin - Göttingen - Heidelberg (1961).

[21] H.-M. Nguyen and M. Vogelius, *A representation formula for the voltage perturbations caused by diametrically small conductivity inhomogeneities. Proof of uniform validity*, Annales de l'Institut Henri Poincaré - Non Linear Analysis, 26, (2009), pp. 2283–2315.

[22] R. Perrussel and C. Poignard, *Asymptotic Expansion of Steady-State Potential in a High Contrast Medium with a Thin Resistive Layer*, Applied Mathematics and Computation, 221, (2013), pp. 48–65.