

Analysis of an Enhanced Approximate Cloaking Scheme for the Conductivity Problem

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Abstract

We extend and analyse an enhanced approximate cloaking scheme, which was recently introduced by Ammari, Kang, Lee, and Lim [3] to cloak a domain with a fixed, homogeneous Neumann boundary condition. Subject to the solvability of a finite set of algebraic equations we construct an approximate cloak for the two dimensional transmission case, which achieves invisibility of the order ρ^{2N+2} while maintaining the same level of local anisotropy as earlier schemes of order ρ^2 [10]. The approximate cloak and the invisibility estimate is independent of the objects being cloaked. Finally, we present analytical as well as numerical evidence for the solvability of the required algebraic equations.

1 Introduction

The central objective of cloaking is to create a domain in space, the presence of which, and the contents of which is invisible or nearly invisible to any outside observer. In the approach referred to as "cloaking by mapping" this is achieved by surrounding the domain one wants to hide by a material layer with very special properties. The material with the appropriate properties is designed by a "push forward" strategy, using a mapping that typically has a very simple description. Cloaking by mapping schemes may be divided into two different categories (1) those that achieve "perfect" invisibility, at the cost of having to use materials with extreme aspect ratios [8, 18], and (2) those that achieve only "approximate" invisibility, but use materials with finite aspect ratios [7, 9, 10, 13, 14]. This paper is entirely devoted to schemes of the second (approximate) kind. For the present discussion we shall limit ourselves to the case in which the measurements available to the outside observer are those of steady state voltages and currents. There is a vast, and rapidly growing literature on cloaking (by mapping, or by other means) – we mention for instance [4, 6, 9, 13, 16] and the references therein.

A key observation that lies at the basis of "cloaking by mapping" is the following invariance of solutions to second order elliptic boundary value problems. Suppose Ω is a

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bounded, simply connected, smooth domain in \mathbb{R}^d , $d \geq 2$, and F is a one-to-one Lipschitz mapping of Ω onto Ω , with $F|_{\partial\Omega} = id$. Let $x \rightarrow a(x)$ be a positive definite, symmetric matrix valued function, with

$$c_0|\xi|^2 \leq \langle a(x)\xi, \xi \rangle \leq C_0|\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \quad a.e. \quad x \in \Omega,$$

for some positive constants c_0, C_0 , and let $u \in H^1(\Omega)$ be the solution to

$$\nabla \cdot (a\nabla u) = 0 \quad \text{in } \Omega, \quad \text{with } u = \phi \text{ on } \partial\Omega,$$

for some given $\phi \in H^{1/2}(\partial\Omega)$. Then $v = u \circ F^{-1}$ is the solution to

$$\nabla \cdot (F_*a\nabla v) = 0 \quad \text{in } \Omega, \quad \text{with } v = \phi \text{ on } \partial\Omega,$$

where F_*a denotes the "push forward of the coefficient a by F "

$$F_*a = \frac{DFaDF^t}{|\det DF|} \circ F^{-1},$$

and at the same time

$$(a\nabla u) \cdot \nu = (F_*a\nabla v) \cdot \nu \quad \text{on } \partial\Omega,$$

where ν denotes the (outward) unit normal to $\partial\Omega$. If we use Λ_a to denote the Dirichlet to Neumann data operator associated with a , then the previous identity expresses that

$$\Lambda_a = \Lambda_{F_*a}.$$

We also note that if B is a subdomain of Ω , and $F = id$ in $\Omega \setminus B$, then the solutions u and v agree in $\Omega \setminus B$, and $\Lambda_a^{(B)} = \Lambda_{F_*a}^{(B)}$, where $\Lambda^{(B)}$ refers to the Dirichlet to Neumann data maps on ∂B . These observations were originally made by Luc Tartar in connection with discussions about the so-called Calderon problem, see [11, 12] for more details.

Consider the situation where Ω contains the ball of radius 2, and is mapped one-to-one onto itself by the mapping

$$F_\rho(x) = \begin{cases} x & x \in \Omega \setminus B_2, \\ \frac{1}{2-\rho}x + \frac{2(1-\rho)}{2-\rho} \frac{x}{|x|} & x \in B_2 \setminus B_\rho, \\ \frac{1}{\rho}x & x \in B_\rho. \end{cases} \quad (1)$$

This piecewise smooth Lipschitz mapping has the properties that $F_\rho(B_\rho) = B_1$, $F_\rho(B_2) = B_2$. In the "physical domain" we seek to hide the contents of the unit ball B_1 (represented by the conductivity distribution a_{obj}^*). This may approximately be accomplished by placing the conductivity distribution $(F_\rho)_*I$ in $B_2 \setminus B_1$. In the "physical domain" we thus have conductivity distribution

$$A_\rho(x) = \begin{cases} I & x \in \Omega \setminus B_2, \\ (F_\rho)_*I & x \in B_2 \setminus B_1, \\ a_{obj}^*(x) & x \in B_1, \end{cases} \quad (2)$$

By a pull back to the "non-physical domain" (the one with the small inclusion B_ρ) we obtain the conductivity distribution

$$a_\rho(x) = \begin{cases} I & x \in \Omega \setminus B_\rho , \\ a_{obj}(x) & x \in B_\rho , \end{cases}$$

where a_{obj} is given by $a_{obj} = (F_\rho^{-1})_* a_{obj}^*$. The relation between a_ρ and A_ρ is that

$$A_\rho = (F_\rho)_* a_\rho .$$

The solutions u_ρ and v_ρ , corresponding to coefficients a_ρ and A_ρ , respectively, and the common boundary data ϕ , completely agree in $\Omega \setminus B_2$ (where the mapping F_ρ is the identity). In other words, the outside observer (an observer in $\Omega \setminus B_2$) views the identical effect of a_ρ and A_ρ . To assess how nearly we have cloaked a_{obj}^* (*i.e.*, how closely it resembles the uniform conductivity 1 to the outside observer), it thus suffices to estimate the effect of the small inhomogeneity B_ρ with contents a_{obj} . Let K denote a compact subset of $\Omega \setminus B_2$, and let U denote the solution to $\Delta U = 0$ in Ω , with $U = \phi$ on $\partial\Omega$. In [10] it was proven that $\|U - u_\rho\|_{H^1(K)} \leq C\rho^d \|\phi\|_{H^{1/2}(\partial\Omega)}$, with a constant C that is independent of a_{obj} (the same estimate thus holds for $\|U - v_\rho\|_{H^1(K)}$, with a constant that is independent of the object we seek to hide, a_{obj}^*)¹.

For the cloak (the region $B_2 \setminus B_1$ in the "physical domain") described by (2) we calculate

$$\begin{aligned} (F_\rho)_* I(x) &= \frac{DF_\rho DF_\rho^t}{|\det DF_\rho|} \circ F^{-1}(x) \\ &= (2 - \rho) \left(2 - \rho - \frac{2(1 - \rho)}{|x|} \right)^{d-3} \left[\left(I - \frac{xx^t}{|x|^2} \right) + \frac{\left(2 - \rho - \frac{2(1 - \rho)}{|x|} \right)^2}{(2 - \rho)^2} \frac{xx^t}{|x|^2} \right] , \end{aligned}$$

which has eigenvalues

$$\lambda_{min} = (2 - \rho)^{-1} \left(2 - \rho - \frac{2(1 - \rho)}{|x|} \right)^{d-1} , \text{ and } \lambda_{max} = (2 - \rho) \left(2 - \rho - \frac{2(1 - \rho)}{|x|} \right)^{d-3} ,$$

the latter of multiplicity $d - 1$. We may introduce, as a measure of the anisotropy of the cloak, the number

$$\chi_{an} := \max_{x \in \bar{B}_2 \setminus B_1} \frac{\lambda_{max}}{\lambda_{min}}(x) , \quad (3)$$

and, as measures of the degeneracy of the cloak,

$$\Lambda_{max} = \max_{x \in \bar{B}_2 \setminus B_1} \lambda_{max}(x) , \text{ and } \Lambda_{min} = \min_{x \in \bar{B}_2 \setminus B_1} \lambda_{min}(x) . \quad (4)$$

¹The estimate in [10] was stated in terms of the Neumann to Dirichlet data operator, but the $H^1(K)$ estimate is also a consequence of that analysis

For this particular cloak we arrive at

$$\chi_{an} = \max_{x \in \bar{B}_2 \setminus B_1} \frac{(2 - \rho)^2}{\left(2 - \rho - \frac{2(1-\rho)}{|x|}\right)^2} = \frac{(2 - \rho)^2}{\rho^2} , \quad (5)$$

and

$$\Lambda_{min} = (2 - \rho)^{-1} \rho^{d-1} , \quad \Lambda_{max} = \begin{cases} (2 - \rho)/\rho & d = 2 , \\ 2 - \rho & d \geq 3 . \end{cases} . \quad (6)$$

The minimal value Λ_{min} is always achieved at $|x| = 1$, whereas Λ_{max} is achieved at $|x| = 1$ for $d = 2, 3$, but at $|x| = 2$ for $d > 3$.

The focus of this paper is on cloaking strategies that will allow for enhanced invisibility, i.e., on strategies that lead to estimates that are strictly better than $C\rho^d$. A particular point of interest is to what extent these may be realized without significantly worsening the total anisotropy and/or the degeneracy of the cloak.

A trivial strategy would be to simply replace ρ by ρ^m , in which case the visibility estimate becomes ρ^{md} . At the same time the anisotropy measure becomes

$$\chi_{an} = \frac{(2 - \rho^m)^2}{\rho^{2m}} ,$$

and the degeneracy measures become

$$\Lambda_{min} = (2 - \rho^m)^{-1} \rho^{(d-1)m} , \quad \Lambda_{max} = \begin{cases} (2 - \rho^m)/\rho^m & d = 2 , \\ 2 - \rho^m & d \geq 3 . \end{cases} . \quad (7)$$

A natural goal is to try to understand to what extent we may do better.

There has recently been some very interesting work on enhanced cloaking of a domain with a fixed, homogeneous Neumann boundary condition, both in the context of the two dimensional conductivity-, and the two dimensional Helmholtz problem [2, 3, 15]. The approach has been to combine the mapping $F_{2\rho}$ with a finite number of radial layers of appropriately selected constant (finite and non-zero) conductivity, occupying the annulus $B_{2\rho} \setminus B_\rho$. The rationale behind this is that, in the "non-physical domain" it is well-known that the solution to $\Delta u_\rho = 0$ in $\Omega \setminus B_\rho$, $\frac{\partial u_\rho}{\partial n} = 0$ on ∂B_ρ , and $u_\rho = \phi$ on $\partial\Omega$, has an expansion in terms of powers of ρ , starting with ρ^d (see [1, 5]). The layered conductivity structure in $B_{2\rho} \setminus B_\rho$ is now selected so that a finite number of these powers vanish, and the corresponding solution starts with ρ^{d+N} , for some positive N . In the "physical domain" (after mapping by $F_{2\rho}$) the cloak now occupies $B_2 \setminus B_{1/2}$, and the objects being cloaked are inside $B_{1/2}$. Even though it should in principle be possible to achieve any power of ρ by adding sufficiently many layers, there is currently no proof of this. We discuss the structure of the appropriate conductivities in detail in Section 3. This discussion naturally builds on, and extends the work in [3].

The cloaking enhancement discussed so far only addresses the cloaking of a fixed domain with a fixed (say Neumann) boundary condition. A major goal of this paper is to

extend the enhancement strategy to the transmission setting, where we cloak arbitrary objects, inside $B_{1/4}$, by use of conducting materials occupying $B_2 \setminus B_{1/4}$ (and in such a way, that the enhanced cloak is independent of the objects). We achieve this goal by combining the previous enhancement strategy with the addition of a layer of very small conductivity occupying $B_\rho \setminus B_{\rho/2}$. Mathematically we then have to estimate how well this poorly conducting layer simulates a Neumann boundary condition, uniformly with respect to the conductivities selected for enhancement, and uniformly with respect to the objects we are seeking to cloak. This analysis is the focus of Section 2. Finally, in Section 4 we combine the enhancement estimates of Section 3 with this "simulated Neumann boundary condition estimate" to give an estimate of the effectivity of our enhanced approximate cloaking strategy. We conclude with a discussion of the degeneracy and anisotropy of the resulting approximate cloak.

2 Approximation of the homogeneous Neumann boundary condition

For our application to cloaking we need a very precise result, that estimates how the perfectly homogeneous Neumann boundary condition is approximated through the use of poorly conducting materials. We suppose Ω is a bounded, simply connected, smooth domain in \mathbb{R}^d , containing the origin, and we let B_ρ denote the ball of radius ρ , centered at the origin. Suppose ρ is sufficiently small that $B_{2\rho} \subset\subset \Omega$. The focus of our study in this section will be the conductivity distribution

$$a_{\epsilon,\rho} = \begin{cases} a_0 & \text{in } \Omega \setminus B_\rho , \\ \epsilon & \text{in } B_\rho \setminus B_{\rho/2} , \\ a_{obj} & \text{in } B_{\rho/2} , \end{cases} \quad (8)$$

where a_0 is an L^∞ -function that satisfies $0 < c_0 \leq a_0(x) \leq C_0 < \infty$ for a.e. $x \in \Omega \setminus B_\rho$ (for some fixed constants c_0 and C_0), ϵ is a positive number, and $a_{obj}(x)$ is an arbitrary L^∞ -function, that is bounded away from zero (a_{obj} represents the "pull-back" of the object we want to hide with our enhanced approximate cloak). By $u_{\epsilon,\rho}$ we denote the solution to

$$\nabla \cdot (a_{\epsilon,\rho} \nabla u_{\epsilon,\rho}) = 0 \text{ in } \Omega , \quad u_{\epsilon,\rho} = \phi \text{ on } \partial\Omega , \quad (9)$$

and by $u_{0,\rho}$ the solution to

$$\nabla \cdot (a_0 \nabla u_{0,\rho}) = 0 \text{ in } \Omega , \quad u_{0,\rho} = \phi \text{ on } \partial\Omega , \quad a_0 \frac{\partial u_{0,\rho}}{\partial \nu} = 0 \text{ on } \partial B_\rho . \quad (10)$$

The specific goal in this section is to establish an estimate for $\|u_{\epsilon,\rho} - u_{0,\rho}\|_{H^1(\Omega \setminus B_\rho)}$, that is explicit in terms of both ϵ and ρ , and *uniform* with respect to a_{obj} . Since the domain $\Omega \setminus B_\rho$ depends on ρ , and we seek to establish an estimate that is explicit in its dependence

on ρ , we must be precise in our definition of the $H^1(\Omega \setminus B_\rho)$ norm. We use

$$\|v\|_{H^1(\Omega \setminus B_\rho)} := \left(\int_{\Omega \setminus B_\rho} |\nabla v|^2 dx + \int_{\Omega \setminus B_\rho} |v|^2 dx \right)^{1/2} .$$

Remark 1. *At the final point in our analysis we shall use the fact that the expression*

$$\left(\int_{\Omega \setminus B_\rho} |\nabla v|^2 dx \right)^{1/2}$$

is indeed a ρ -uniformly equivalent norm on $H^1(\Omega \setminus B_\rho) \cap \{v : v = 0 \text{ on } \partial\Omega\}$. In other words, we shall use the fact that there exists a constant C , independent of ρ , so that

$$\int_{\Omega \setminus B_\rho} |\nabla v|^2 dx \leq \int_{\Omega \setminus B_\rho} |\nabla v|^2 dx + \int_{\Omega \setminus B_\rho} |v|^2 dx \leq C \int_{\Omega \setminus B_\rho} |\nabla v|^2 dx , \quad (11)$$

for all $v \in H^1(\Omega \setminus B_\rho)$, with v vanishing on $\partial\Omega$. We leave the proof of this simple fact to the reader.

For our analysis we shall need an estimate for the Dirichlet to Neumann data map associated with an elliptic operator which equals the Laplacian near the boundary. We formulate this as

Lemma 1. *Let a in $L^\infty(B_1)$ be given by*

$$a = \begin{cases} 1 & \text{in } B_1 \setminus B_{1/2} , \\ b & \text{in } B_{1/2} , \end{cases}$$

where b is in $L^\infty(B_{1/2})$, positive, and strictly bounded away from zero. Let $v \in H^1(B_1)$ be a solution to $\nabla \cdot (a \nabla v) = 0$ in B_1 . Then

$$\left\| \frac{\partial v}{\partial \nu} \right\|_{H^{-1/2}(\partial B_1)} \leq C \min_{k \in \mathbb{R}} \|v + k\|_{H^{1/2}(\partial B_1)} . \quad (12)$$

The constant C is independent of b and v .

Proof. Let w be such that $w = v$ on ∂B_1 , w vanishes identically in $B_{1/2}$ and $\|w\|_{H^1(B_1)} \leq C \|v\|_{H^{1/2}(\partial B_1)}$. By Dirichlet's principle

$$\begin{aligned} \int_{B_1 \setminus B_{1/2}} |\nabla v|^2 dx &\leq \int_{B_{1/2}} b |\nabla v|^2 dx + \int_{B_1 \setminus B_{1/2}} |\nabla v|^2 dx \\ &= \int_{B_1} a |\nabla v|^2 dx \leq \int_{B_1} a |\nabla w|^2 dx \\ &= \int_{B_1 \setminus B_{1/2}} |\nabla w|^2 dx \leq C \|v\|_{H^{1/2}(\partial B_1)}^2 , \end{aligned} \quad (13)$$

with C independent of b and v . It follows that

$$\|v\|_{H^1(B_1 \setminus B_{1/2})}^2 \leq C \left(\int_{B_1 \setminus B_{1/2}} |\nabla v|^2 dx + \|v\|_{L^2(\partial B_1)}^2 \right) \leq C \|v\|_{H^{1/2}(\partial B_1)}^2 ,$$

with C independent of b and v . Since $\Delta v = 0$ in $B_1 \setminus B_{1/2}$, a local elliptic energy estimate thus yields

$$\left\| \frac{\partial v}{\partial \nu} \right\|_{H^{-1/2}(\partial B_1)} \leq C \|v\|_{H^1(B_1 \setminus B_{1/2})} \leq C \|v\|_{H^{1/2}(\partial B_1)} ,$$

and insertion of $v + k$, $k \in \mathbb{R}$, in place of v now completes the proof of the lemma. \square

We are now ready to prove the following result, that estimates quite precisely how well $u_{\epsilon, \rho}$ approximates $u_{0, \rho}$, the solution subject to a homogeneous Neumann boundary condition on ∂B_ρ .

Proposition 1. *Let $u_{\epsilon, \rho}$ be the solution to (9) with $a_{\epsilon, \rho}$ given by (8), and let $u_{0, \rho}$ be the solution to (10), then*

$$\|u_{\epsilon, \rho} - u_{0, \rho}\|_{H^1(\Omega \setminus B_\rho)} \leq C \epsilon \|\phi\|_{H^{1/2}(\partial \Omega)} .$$

The constant C is independent of ϵ , ρ , and the function a_{obj} . C depends on c_0 and C_0 , but it is otherwise also independent of a_0 .

Proof. For any v in $H^1(\Omega \setminus B_\rho)$, let v_ρ denote the rescaled function $v_\rho(x) = v(\rho x)$, defined on $\rho^{-1}\Omega \setminus B_1$. The standard trace estimate

$$\|w\|_{H^{1/2}(\partial B_1)} \leq C \|w\|_{H^1(B_2 \setminus B_1)} ,$$

immediately leads to

$$\begin{aligned} \min_{k \in \mathbb{R}} \|v_\rho + k\|_{H^{1/2}(\partial B_1)} &\leq C \min_{k \in \mathbb{R}} \|v_\rho + k\|_{H^1(B_2 \setminus B_1)} \leq C \|\nabla v_\rho\|_{L^2(B_2 \setminus B_1)} \\ &\leq C \rho^{1-\frac{d}{2}} \|\nabla v\|_{L^2(B_{2\rho} \setminus B_\rho)} \leq C \rho^{1-\frac{d}{2}} \|\nabla v\|_{L^2(\Omega \setminus B_\rho)} . \end{aligned} \quad (14)$$

Let w be a function in $H^1(\Omega)$ that is selected so that $w = \phi$ on $\partial \Omega$, w vanishes on some fixed $B \subset \Omega$ that contains all B_ρ , and $\|w\|_{H^1(\Omega)} \leq C \|\phi\|_{H^{1/2}(\partial \Omega)}$. Using Dirichlet's principle we now calculate

$$\begin{aligned} \|\nabla u_{\epsilon, \rho}\|_{L^2(\Omega \setminus B_\rho)}^2 &\leq C \int_{\Omega \setminus B_\rho} a_0 |\nabla u_{\epsilon, \rho}|^2 dx \\ &\leq C \left(\int_{B_{\rho/2}} a_{obj} |\nabla u_{\epsilon, \rho}|^2 dx + \int_{B_\rho \setminus B_{\rho/2}} \epsilon |\nabla u_{\epsilon, \rho}|^2 dx + \int_{\Omega \setminus B_\rho} a_0 |\nabla u_{\epsilon, \rho}|^2 dx \right) \\ &= C \int_{\Omega} a_{\epsilon, \rho} |\nabla u_{\epsilon, \rho}|^2 dx \leq C \int_{\Omega} a_{\epsilon, \rho} |\nabla w|^2 dx \\ &= C \int_{\Omega \setminus B_\rho} a_0 |\nabla w|^2 dx \leq C \|w\|_{H^1(\Omega)}^2 \leq C \|\phi\|_{H^{1/2}(\partial \Omega)}^2 . \end{aligned}$$

By a combination with (14) (with $v = u_{\epsilon, \rho}$) we have thus established the bound

$$\min_{k \in \mathbb{R}} \|u_{\epsilon, \rho}(\rho \cdot) + k\|_{H^{1/2}(\partial B_1)} \leq C \rho^{1-\frac{d}{2}} \|\phi\|_{H^{1/2}(\partial \Omega)} \quad . \quad (15)$$

The function $v_{\epsilon, \rho}(x) = u_{\epsilon, \rho}(\rho x)$, $x \in B_1$, is in $H^1(B_1)$, it satisfies $\nabla \cdot (\epsilon^{-1} a_{\epsilon, \rho}(\rho x) \nabla v_{\epsilon, \rho}) = 0$, with $\epsilon^{-1} a_{\epsilon, \rho}(\rho \cdot) \in L^\infty(B_1)$ and $\epsilon^{-1} a_{\epsilon, \rho}(\rho \cdot)$ identically equal to 1 in $B_1 \setminus B_{1/2}$. Lemma 1 therefore applies to give the estimate

$$\left\| \left(\frac{\partial v_{\epsilon, \rho}}{\partial \nu} \right)^- \right\|_{H^{-1/2}(\partial B_1)} \leq C \min_{k \in \mathbb{R}} \|v_{\epsilon, \rho} + k\|_{H^{1/2}(\partial B_1)} \quad ,$$

which in combination with (15) leads to

$$\left\| \left(\frac{\partial v_{\epsilon, \rho}}{\partial \nu} \right)^- \right\|_{H^{-1/2}(\partial B_1)} \leq C \rho^{1-\frac{d}{2}} \|\phi\|_{H^{1/2}(\partial \Omega)} \quad . \quad (16)$$

The function $u_{\epsilon, \rho}$ satisfies the jump relation

$$\left(\epsilon \frac{\partial u_{\epsilon, \rho}}{\partial \nu} \right)^- = \left(a_0 \frac{\partial u_{\epsilon, \rho}}{\partial \nu} \right)^+ \quad \text{on } \partial B_\rho \quad ,$$

and since

$$\left(\frac{\partial v_{\epsilon, \rho}}{\partial \nu} \right)^- (x) = \rho \left(\frac{\partial u_{\epsilon, \rho}}{\partial \nu} \right)^- (\rho x) \quad ,$$

it now follows from (16) that

$$\left\| \left(a_0 \frac{\partial u_{\epsilon, \rho}}{\partial \nu} \right)^+ (\rho \cdot) \right\|_{H^{-1/2}(\partial B_1)} \leq C \epsilon \rho^{-\frac{d}{2}} \|\phi\|_{H^{1/2}(\partial \Omega)} \quad . \quad (17)$$

Integration by parts, in combination with the fact that

$$\int_{\partial B_1} \left(a_0 \frac{\partial u_{\epsilon, \rho}}{\partial \nu} \right)^+ (\rho x) ds_x = \epsilon \int_{\partial B_1} \left(\frac{\partial u_{\epsilon, \rho}}{\partial \nu} \right)^- (\rho x) ds_x = 0 \quad ,$$

therefore gives

$$\begin{aligned} & \int_{\Omega \setminus B_\rho} a_0 |\nabla(u_{\epsilon, \rho} - u_{0, \rho})|^2 dx \\ &= - \int_{\partial B_\rho} (u_{\epsilon, \rho} - u_{0, \rho}) \left(a_0 \frac{\partial(u_{\epsilon, \rho} - u_{0, \rho})}{\partial \nu} \right)^+ ds \\ &= -\rho^{d-1} \int_{\partial B_1} (u_{\epsilon, \rho} - u_{0, \rho})(\rho x) \left(a_0 \frac{\partial(u_{\epsilon, \rho} - u_{0, \rho})}{\partial \nu} \right)^+ (\rho x) ds_x \\ &= -\rho^{d-1} \int_{\partial B_1} [(u_{\epsilon, \rho} - u_{0, \rho})(\rho x) + k] \left(a_0 \frac{\partial u_{\epsilon, \rho}}{\partial \nu} \right)^+ (\rho x) ds_x \\ &\leq \rho^{d-1} \|(u_{\epsilon, \rho} - u_{0, \rho})(\rho \cdot) + k\|_{H^{1/2}(\partial B_1)} \left\| \left(a_0 \frac{\partial u_{\epsilon, \rho}}{\partial \nu} \right)^+ (\rho \cdot) \right\|_{H^{-1/2}(\partial B_1)} \quad , \end{aligned}$$

for any constant k . Minimization over k , and use of (14) and (17), the first with $v = u_{\epsilon,\rho} - u_{0,\rho}$, yields

$$\begin{aligned}
& \int_{\Omega \setminus B_\rho} a_0 |\nabla(u_{\epsilon,\rho} - u_{0,\rho})|^2 dx \\
& \leq \rho^{d-1} \min_{k \in \mathbb{R}} \|(u_{\epsilon,\rho} - u_{0,\rho})(\rho \cdot) + k\|_{H^{1/2}(\partial B_1)} \left\| \left(a_0 \frac{\partial u_{\epsilon,\rho}}{\partial \nu} \right)^+ (\rho \cdot) \right\|_{H^{-1/2}(\partial B_1)} \\
& \leq C\epsilon \|\nabla(u_{\epsilon,\rho} - u_{0,\rho})\|_{L^2(\Omega \setminus B_\rho)} \|\phi\|_{H^{1/2}(\partial\Omega)} \\
& \leq C\epsilon \left(\int_{\Omega \setminus B_\rho} a_0 |\nabla(u_{\epsilon,\rho} - u_{0,\rho})|^2 dx \right)^{1/2} \|\phi\|_{H^{1/2}(\partial\Omega)} . \tag{18}
\end{aligned}$$

The estimate (18) in combination with the norm equivalence (11) immediately leads to

$$\begin{aligned}
\|u_{\epsilon,\rho} - u_{0,\rho}\|_{H^1(\Omega \setminus B_\rho)} & \leq C \left(\int_{\Omega \setminus B_\rho} a_0 |\nabla(u_{\epsilon,\rho} - u_{0,\rho})|^2 dx \right)^{1/2} \\
& \leq C\epsilon \|\phi\|_{H^{1/2}(\partial\Omega)} ,
\end{aligned}$$

as desired. \square

3 Enhanced cloaking of the homogeneous Neumann boundary condition

Following the construction in [3] we introduce multiple, spherical layers of constant conductivity in the annulus $B_{2\rho} \setminus B_\rho$ in order to enhance the approximate cloaking effect of $B_2 \setminus B_\rho$ (when pushed into the "physical domain"). As in [3] we initially consider only the case of a fixed homogeneous Neumann boundary condition at the interface between the cloak and the cloaked area. However, by a combination with a low conductivity layer (and the uniform estimates of the previous section) we show that the the enhancement effect for this special case carries over to the general transmission case with an arbitrary conducting object inside the cloak. We note that in the transmission case, the authors in [3] consider only a constant conductivity object inside the cloak, and then the material properties of the enhancement layers depend on the constant inside the cloak. As in [3] we restrict our discussion to the case $\Omega \subset \mathbb{R}^2$ (*i.e.*, $d = 2$).

Let R be a fixed radius, with $2\rho < R$ and such that $B_R \subset \Omega$. Let a_0 be the piecewise constant conductivity distribution

$$a_0 = \begin{cases} 1 & \text{in } \Omega \setminus B_{\rho_0} \\ \sigma_1 & \text{in } B_{\rho_0} \setminus B_{\rho_1} =: S_1 \\ \dots & \\ \sigma_\ell & \text{in } B_{\rho_{\ell-1}} \setminus B_{\rho_\ell} =: S_\ell \\ \dots & \\ \sigma_L & \text{in } B_{\rho_{L-1}} \setminus B_{\rho_L} =: S_L \end{cases} \tag{19}$$

with $\rho = \rho_L < \rho_{L-1} < \dots < \rho_0 = 2\rho$, and $\sigma_1, \dots, \sigma_\ell, \dots, \sigma_L \in \mathbb{R}_+ = \mathbb{R} \cap (0, \infty)$ (we may think of there being a σ_0 , which equals 1). Note that the spherical layers are not necessarily of same thickness. As before, $u_{0,\rho}$ denotes the solution to (10). The trace $\varphi = u_{0,\rho}|_{\partial B_R}$ of this solution has the following Fourier representation

$$\varphi(\theta) = \sum_{n=0}^{\infty} R^n g_n^c \cos(n\theta) + \sum_{n=1}^{\infty} R^n g_n^s \sin(n\theta) . \quad (20)$$

It follows immediately from the standard Sobolev trace estimate and the H^1 energy estimate that

$$\|\varphi\|_{H^{1/2}(\partial B_R)} \leq C \|u_{0,\rho}\|_{H^1(\Omega \setminus B_R)} \leq C \|u_{0,\rho}\|_{H^{1/2}(\partial\Omega)} = C \|\phi\|_{H^{1/2}(\partial\Omega)} .$$

Here the constants C are independent of ρ and ϕ (but may depend on R). Let U denote the solution to

$$\Delta U = 0 \quad \text{in } \Omega , \quad U = \phi \quad \text{on } \partial\Omega . \quad (21)$$

By a combination of Proposition 1 (with $\epsilon = \rho^d$) and Corollary 2 of [17] we immediately arrive at ²

Lemma 2. *Let a_0 be as above, for a given choice of $\rho = \rho_L < \rho_{L-1} < \dots < \rho_1 < \rho_0 = 2\rho$ and $\{\sigma_\ell\}$, and let K be a fixed compact subdomain of $\Omega \setminus B_{2\rho}$. There exists a constant C independent of ρ and $\{\rho_\ell\}$ and L such that*

$$\|U - u_{0,\rho}\|_{H^1(K)} \leq C \rho^d \|\phi\|_{H^{1/2}(\partial\Omega)} .$$

C depends on the set K and on $\max_\ell \sigma_\ell$ and $\min_\ell \sigma_\ell$, but is otherwise also independent of the conductivities $\{\sigma_\ell\}$.

Proposition 2. *With a_0 as defined above, the solution $u_{0,\rho}$ to (10) has the following representation in $B_R \setminus B_{2\rho}$*

$$\begin{aligned} u_{0,\rho}(r, \theta) &= \sum_{n=0}^{\infty} \left((1 - R^{-2n} M_n) r^n + M_n r^{-n} \right) g_n^c \cos(n\theta) \\ &\quad + \sum_{n=1}^{\infty} \left((1 - R^{-2n} M_n) r^n + M_n r^{-n} \right) g_n^s \sin(n\theta) \end{aligned} \quad (22)$$

where $M_0 = 1$, and

$$M_n = \frac{\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} R^{-2n} \\ -1 \end{pmatrix}}, \quad (23)$$

²The result in Corollary 2 of [17] concerns the case of fixed Neumann boundary data on $\partial\Omega$, but the exact same method of proof applies to fixed Dirichlet boundary data

$n \geq 1$, with

$$\mathbf{R}_{n,i}(r) = \begin{pmatrix} r^n & r^{-n} \\ \sigma_i r^{n-1} & -\sigma_i r^{-n-1} \end{pmatrix} .$$

Proof. For simplicity of notation we assume that $g_n^s = 0$, for all n . The general case follows analogously. The solution $u_{0,\rho}$ has the expansion:

$$u_{0,\rho}(r, \theta) = g_0^c + \begin{cases} \sum_{n=1}^{\infty} (c_n^0 r^n + d_n^0 r^{-n}) \cos(n\theta) & \text{in } B_R \setminus B_{2\rho} \\ \sum_{n=1}^{\infty} (c_n^1 r^n + d_n^1 r^{-n}) \cos(n\theta) & \text{in } S_1 \\ \dots \\ \sum_{n=1}^{\infty} (c_n^L r^n + d_n^L r^{-n}) \cos(n\theta) & \text{in } S_L \end{cases}$$

The usual transmission conditions at the interfaces $\partial B_{\rho_0}, \dots, \partial B_{\rho_\ell}, \dots, \partial B_{\rho_{L-1}}$, and the boundary condition at $\partial B_\rho = \partial B_{\rho_L}$ yield the following linear system for the coefficients:

$$\begin{aligned} \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} c_n^0 \\ d_n^0 \end{pmatrix} &= \mathbf{R}_{n,1}(\rho_0) \begin{pmatrix} c_n^1 \\ d_n^1 \end{pmatrix} \\ \mathbf{R}_{n,1}(\rho_1) \begin{pmatrix} c_n^1 \\ d_n^1 \end{pmatrix} &= \mathbf{R}_{n,2}(\rho_1) \begin{pmatrix} c_n^2 \\ d_n^2 \end{pmatrix} \\ &\dots \\ \mathbf{R}_{n,L-1}(\rho_{L-1}) \begin{pmatrix} c_n^{L-1} \\ d_n^{L-1} \end{pmatrix} &= \mathbf{R}_{n,L}(\rho_{L-1}) \begin{pmatrix} c_n^L \\ d_n^L \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \begin{pmatrix} c_n^L \\ d_n^L \end{pmatrix} &= 0 . \end{aligned} \tag{24}$$

After elimination of $(c_n^1, d_n^1) \dots (c_n^L, d_n^L)$ this gives

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} c_n^0 \\ d_n^0 \end{pmatrix} = 0 .$$

In terms of the Dirichlet data at ∂B_R we have

$$R^n g_n^c = R^n c_n^0 + R^{-n} d_n^0 ,$$

and hence

$$d_n^0 = g_n^c M_n ,$$

and

$$c_n^0 = g_n^c - g_n^c R^{-2n} M_n ,$$

which immediately leads to the desired representation (22). \square

Lemma 3. *With notation as above we have that*

$$\begin{aligned} & \mathbf{R}_{n,L}^{-1}(\rho_{L-1})\mathbf{R}_{n,L-1}(\rho_{L-1})\dots\mathbf{R}_{n,1}(\rho_1)\mathbf{R}_{n,1}^{-1}(\rho_0)\mathbf{R}_{n,0}(\rho_0) \\ &= \prod_{\ell=0}^{L-1} \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \begin{pmatrix} 1 + \sum_{|I|=2,4,\dots,L} \Lambda_I \rho_I^{2nA(I)} & \sum_{|I|=1,3,\dots,L} \Lambda_I \rho_I^{-2nA(I)} \\ \sum_{|I|=1,3,\dots,L} \Lambda_I \rho_I^{2nA(I)} & 1 + \sum_{|I|=2,4,\dots,L} \Lambda_I \rho_I^{-2nA(I)} \end{pmatrix} \end{aligned} \quad (25)$$

Here $I = (I_0, \dots, I_{L-1}) \in \{0, 1\}^L$ is an arbitrary ordered multi-index, and $|I|$ denotes the number of its non-zero entries. Furthermore $\lambda_\ell = \frac{\sigma_{\ell+1} - \sigma_\ell}{\sigma_{\ell+1} + \sigma_\ell}$, $0 \leq \ell \leq L-1$, and

$$\Lambda_I = \lambda_{s_1(I)} \lambda_{s_2(I)} \lambda_{s_3(I)} \dots$$

where $0 \leq s_1(I), s_2(I), s_3(I) \dots s_{|I|}(I) \leq L-1$ are the ordered indices of the non-zero entries of I . $A(I) \in \{-1, +1\}^{|I|}$ is the multi-index that alternates between $+1$ and -1 , starting with $+1$, and $\rho_I = (\rho_{s_1(I)}, \rho_{s_2(I)}, \dots, \rho_{s_{|I|}(I)}) \in \mathbb{R}^{|I|}$, so that

$$\rho_I^{2nA(I)} = \rho_{s_1(I)}^{2n} \rho_{s_2(I)}^{-2n} \rho_{s_3(I)}^{2n} \dots \quad (26)$$

Proof. We prove this formula by induction in the number of layers L . First we recall

$$\mathbf{R}_{n,\ell+1}^{-1}(\rho_\ell)\mathbf{R}_{n,\ell}(\rho_\ell) = \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \begin{pmatrix} 1 & \lambda_\ell \rho_\ell^{-2n} \\ \lambda_\ell \rho_\ell^{2n} & 1 \end{pmatrix}$$

which (with $\ell = 0$) verifies the base case, $L = 1$. The induction step follows from simple matrix multiplication and the definition of Λ_I , $A(I)$ and ρ_I . Suppose (25) holds

for L layers, then

$$\begin{aligned}
& \left(\mathbf{R}_{n,L+1}^{-1}(\rho_L) \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \right)_{11} \\
&= \prod_{\ell=0}^L \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \left(1 + \sum_{|I|=2,4,\dots,L} \Lambda_I \rho_I^{2nA(I)} + \lambda_L \rho_L^{-2n} \sum_{|I|=1,3,\dots,L} \Lambda_I \rho_I^{2nA(I)} \right) \\
&= \prod_{\ell=0}^L \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \left(1 + \sum_{|I^*|=2,4,\dots,L+1} \Lambda_{I^*} \rho_{I^*}^{2nA(I^*)} \right) \\
& \left(\mathbf{R}_{n,L+1}^{-1}(\rho_L) \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \right)_{12} \\
&= \prod_{\ell=0}^L \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \left(\sum_{|I|=1,3,\dots,L} \Lambda_I \rho_I^{-2nA(I)} + \lambda_L \rho_L^{-2n} + \lambda_L \rho_L^{-2n} \sum_{|I|=2,4,\dots,L} \Lambda_I \rho_I^{-2nA(I)} \right) \\
&= \prod_{\ell=0}^L \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \left(\sum_{|I^*|=1,3,\dots,L+1} \Lambda_{I^*} \rho_{I^*}^{-2nA(I^*)} \right) \\
& \left(\mathbf{R}_{n,L+1}^{-1}(\rho_L) \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \right)_{21} \\
&= \prod_{\ell=0}^L \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \left(\lambda_L \rho_L^{2n} + \lambda_L \rho_L^{2n} \sum_{|I|=2,4,\dots,L} \Lambda_I \rho_I^{2nA(I)} + \sum_{|I|=1,3,\dots,L} \Lambda_I \rho_I^{2nA(I)} \right) \\
&= \prod_{\ell=0}^L \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \left(\sum_{|I^*|=1,3,\dots,L+1} \Lambda_{I^*} \rho_{I^*}^{2nA(I^*)} \right) \\
& \left(\mathbf{R}_{n,L+1}^{-1}(\rho_L) \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \right)_{22} \\
&= \prod_{\ell=0}^L \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \left(\lambda_L \rho_L^{2n} \sum_{|I|=1,3,\dots,L} \Lambda_I \rho_I^{-2nA(I)} + 1 + \sum_{|I|=2,4,\dots,L} \Lambda_I \rho_I^{-2nA(I)} \right) \\
&= \prod_{\ell=0}^L \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \left(1 + \sum_{|I^*|=2,4,\dots,L+1} \Lambda_{I^*} \rho_{I^*}^{-2nA(I^*)} \right)
\end{aligned}$$

Here we have used the notation I for a multi-index from $\{0, 1\}^L$, and I^* for a multi-index from $\{0, 1\}^{L+1}$. These four identities together verify (25) for the case of $L + 1$ layers, and this completes the proof of the lemma. \square

The following lemma will be needed in order to estimate the constants M_n in the representation formula (22) for $u_{0,\rho}$.

Lemma 4. *The expression*

$$\sum_{|I|=1,3,\dots,L} \Lambda_I t_I^{-2nA(I)} - 1 - \sum_{|I|=2,4,\dots,L} \Lambda_I t_I^{-2nA(I)} \quad (27)$$

is different from zero for all choices of $n, L, 1 = t_L < t_{L-1} < \dots < t_1 < t_0 = 2$, and $\{\sigma_\ell\}_{\ell=1}^L$. Here $t_I^{-2nA(I)}$ is defined analogously to $\rho_I^{-2nA(I)}$ in Lemma 3.

Proof. Suppose the expression (27) vanished for some particular choice of $n, L, 1 = t_L < t_{L-1} < \dots < t_1 < t_0 = 2$, and $\{\sigma_\ell\}_{\ell=1}^L$. A simple calculation with $\rho_\ell = \rho t_\ell$ gives

$$\begin{aligned} & \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= \rho^{-n-1} \sigma_L \prod_{\ell=0}^{L-1} \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \left(\sum_{|I|=1,3,\dots,L} \Lambda_I t_I^{-2nA(I)} - 1 - \sum_{|I|=2,4,\dots,L} \Lambda_I t_I^{-2nA(I)} \right). \end{aligned}$$

It would thus follow that

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0, \quad ,$$

for all ρ and this choice of $n, L, 1 = t_L < t_{L-1} < \dots < t_1 < t_0 = 2$ and $\{\sigma_\ell\}_{\ell=1}^L$, with $\rho_\ell = \rho t_\ell$. According to Proposition 2 we would now have $M_n = R^{2n}$ for any ρ and R with $2\rho < R$, and this particular choice of data. The representation formula in Proposition 2 would imply that the solution, $v_{0,\rho}$, to

$$\nabla \cdot (a_0 \nabla v_{0,\rho}) = 0 \quad \text{in } B_R \setminus B_\rho, \quad v_{0,\rho} = \cos(n\theta) \quad \text{on } \partial B_R, \quad a_0 \frac{\partial v_{0,\rho}}{\partial \nu} = 0 \quad \text{on } \partial B_\rho,$$

is given by $v_{0,\rho}(r, \theta) = R^n r^{-n} \cos(n\theta)$ in $B_R \setminus B_{2\rho}$, independently of ρ , for this particular choice of data: $n, L, 1 = t_L < t_{L-1} < \dots < t_1 < t_0 = 2$ and $\{\sigma_\ell\}_{\ell=1}^L$, with $\rho_\ell = \rho t_\ell$. However this contradicts Lemma 2, which asserts that $v_{0,\rho}$ converges to $R^{-n} r^n \cos(n\theta)$, as $\rho \rightarrow 0$, on any fixed compact subset of $B_R \setminus B_{2\rho}$. \square

We are now in a position to estimate the constants M_n in the representation formula for $u_{0,\rho}$.

Proposition 3. *Suppose $\rho = \rho_L < \rho_{L-1} < \dots < \rho_1 < \rho_0 = 2\rho$, with $\rho_\ell/\rho =$ a fixed constant t_ℓ , $1 \leq t_\ell \leq 2$, $0 \leq \ell \leq L$, and suppose R is fixed with $R > 2\rho$. Then there exist positive constants C_* and δ , independent of ρ and n (but dependent on L, R , and the constants $\{\sigma_\ell\}, \{t_\ell\}$) such that*

$$|M_n| = \left| \frac{\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} R^{-2n} \\ -1 \end{pmatrix}} \right| \leq C_* \rho^{2n},$$

for $\rho \leq \delta$, and all $n \geq 1$.

Proof. Using Lemma 3, and the alternating nature of $A(I)$, we find

$$\begin{aligned}
& \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right| \\
&= \sigma_L \prod_{\ell=0}^{L-1} \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \left| \rho^{n-1} \left(1 + \sum_{|I|=2,4,\dots,L} \Lambda_I \rho_I^{2nA(I)} \right) - \rho^{-n-1} \sum_{|I|=1,3,\dots,L} \Lambda_I \rho_I^{2nA(I)} \right| \\
&= \sigma_L \prod_{\ell=0}^{L-1} \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \rho^{n-1} \left| \left(1 + \sum_{|I|=2,4,\dots,L} \Lambda_I t_I^{2nA(I)} \right) - \sum_{|I|=1,3,\dots,L} \Lambda_I t_I^{2nA(I)} \right| \\
&\leq \sigma_L \prod_{\ell=0}^{L-1} \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} C^n \rho^{n-1} \quad , \tag{28}
\end{aligned}$$

and

$$\begin{aligned}
& \left| \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} R^{-2n} \\ -1 \end{pmatrix} \right| \\
&= \sigma_L \prod_{\ell=0}^{L-1} \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \left| R^{-2n} \left(\rho^{n-1} \left(1 + \sum_{|I|=2,4,\dots,L} \Lambda_I \rho_I^{2nA(I)} \right) - \rho^{-n-1} \sum_{|I|=1,3,\dots,L} \Lambda_I \rho_I^{2nA(I)} \right) \right. \\
&\quad \left. - \left(\rho^{n-1} \sum_{|I|=1,3,\dots,L} \Lambda_I \rho_I^{-2nA(I)} - \rho^{-n-1} \left(1 + \sum_{|I|=2,4,\dots,L} \Lambda_I \rho_I^{-2nA(I)} \right) \right) \right| \\
&= \sigma_L \prod_{\ell=0}^{L-1} \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \rho^{-n-1} \left| \rho^{2n} R^{-2n} \left(1 + \sum_{|I|=2,4,\dots,L} \Lambda_I t_I^{2nA(I)} - \sum_{|I|=1,3,\dots,L} \Lambda_I t_I^{2nA(I)} \right) \right. \\
&\quad \left. - \sum_{|I|=1,3,\dots,L} \Lambda_I t_I^{-2nA(I)} + 1 + \sum_{|I|=2,4,\dots,L} \Lambda_I t_I^{-2nA(I)} \right| \\
&\geq \sigma_L \prod_{\ell=0}^{L-1} \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \rho^{-n-1} \left(\frac{1}{2} - C^n \rho^{2n} R^{-2n} \right) \\
&\geq \frac{1}{4} \sigma_L \prod_{\ell=0}^{L-1} \frac{\sigma_{\ell+1} + \sigma_\ell}{2\sigma_{\ell+1}} \rho^{-n-1} \quad , \tag{29}
\end{aligned}$$

for $\rho < \frac{R}{2\sqrt{C}}$, and $n \geq N_0$. In the next to last inequality we have used that $t_I^{A(I)} > 1$ and $|\Lambda_I| < 1$ for any $|I| \geq 1$, to conclude that

$$- \sum_{|I|=1,3,\dots,L} \Lambda_I t_I^{-2nA(I)} + 1 + \sum_{|I|=2,4,\dots,L} \Lambda_I t_I^{-2nA(I)} \geq \frac{1}{2}$$

for $n \geq N_0$, where N_0 is independent of ρ and the $\{\sigma_\ell\}$, but depends on L and the $\{t_\ell\}$. According to Lemma 4 we have that

$$- \sum_{|I|=1,3,\dots,L} \Lambda_I t_I^{-2nA(I)} + 1 + \sum_{|I|=2,4,\dots,L} \Lambda_I t_I^{-2nA(I)}$$

does not vanish for $1 \leq n \leq N_0$, and any choice of t_ℓ and σ_ℓ . It now follows from the second identity in (29) that there exists positive constants c and δ' , dependent on the σ_ℓ , the t_ℓ , L and R , but independent of ρ , such that

$$\left| \begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} R^{-2n} \\ -1 \end{pmatrix} \right| \geq c \rho^{-n-1} \quad , \quad (30)$$

for $\rho < \delta'$ and $1 \leq n < N_0$. By a combination of the estimates (28) and (29), (30) we now obtain

$$\left| \frac{\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}^T \mathbf{R}_{n,L}(\rho_L) \mathbf{R}_{n,L}^{-1}(\rho_{L-1}) \dots \mathbf{R}_{n,1}(\rho_1) \mathbf{R}_{n,1}^{-1}(\rho_0) \mathbf{R}_{n,0}(\rho_0) \begin{pmatrix} R^{-2n} \\ -1 \end{pmatrix}} \right| \leq C_*^m \rho^{2n} \quad ,$$

for $\rho < \delta = \min\{\frac{R}{2\sqrt{C}}, \delta'\}$, and all $n \geq 1$. This completes the proof of the proposition. \square

Given the formula for M_n and the second identity in (28), it follows that $M_n = 0$ if and only if

$$\left(1 + \sum_{|I|=2,4,\dots,L} \Lambda_I t_I^{2nA(I)} \right) - \sum_{|I|=1,3,\dots,L} \Lambda_I t_I^{2nA(I)} = 0 \quad . \quad (31)$$

In principle there are $2L - 1$ free parameters in the equation, however, we think of the relative layer position variables $1 = t_L < t_{L-1} < \dots < t_1 < t_0 = 2$ as fixed and consider (31) an equation in the L conductivities $\{\sigma_\ell\}_1^L$. The following result, a version of which is originally found in [3], generalizes the estimate in Lemma 2.

Proposition 4. *Let L and $1 = t_L < t_{L-1} < \dots < t_1 < t_0 = 2$ be given. Set $\rho_\ell = \rho t_\ell$ for ρ sufficiently small that $B_{2\rho} \subset \subset \Omega$. Let R be fixed, with $B_{2\rho} \subset B_R \subset \Omega$. Suppose $\{\sigma_\ell\}_{\ell=1}^L$ solve the equations (31) for $n = 1, \dots, N$. Let $u_{0,\rho}$ be the solution to (10) with a_0 given by (19), with this choice of $\{\rho_\ell\}_{\ell=0}^L$ and $\{\sigma_\ell\}_{\ell=1}^L$. Let U be the (background) solution to (21). There exists a constant C , independent of ρ and ϕ , such that*

$$\|U - u_{0,\rho}\|_{H^1(\Omega \setminus B_R)} \leq C \rho^{2N+2} \|\phi\|_{H^{1/2}(\partial\Omega)} \quad (32)$$

Proof. Let $\varphi = u_{0,\rho}|_{\partial B_R} = \sum_{n=0}^{\infty} R^n g_n^c \cos n\theta + \sum_{n=1}^{\infty} R^n g_n^s \sin n\theta$. From comments at the beginning of this section it follows that

$$\|\varphi\|_{H^{1/2}(\partial B_R)} \leq C \|\phi\|_{H^{1/2}(\partial\Omega)} \quad .$$

A simple calculation gives that

$$\|\varphi\|_{H^{1/2}(\partial B_R)}^2 \text{ is equivalent to } \sum_{n=1}^{\infty} n (|R^n g_n^c|^2 + |R^n g_n^s|^2) + |g_0^c|^2 \text{ ,}$$

with constants depending on R . Let U_ρ denote the solution to

$$\Delta U_\rho = 0 \text{ in } B_\rho \text{ , } U_\rho = u_{0,\rho} \text{ on } \partial B_R \text{ .} \quad (33)$$

From the representation formula in Proposition 2, and the fact that $U_\rho = \sum_{n=0}^{\infty} r^n g_n^c \cos n\theta + \sum_{n=1}^{\infty} r^n g_n^s \sin n\theta$, we get

$$U_\rho - u_{0,\rho} = \sum_{n=0}^{\infty} (R^{-2n} r^n - r^{-n}) M_n g_n^c \cos n\theta + \sum_{n=1}^{\infty} (R^{-2n} r^n - r^{-n}) M_n g_n^s \sin n\theta \text{ ,}$$

in $B_R \setminus B_{2\rho}$, and so

$$\begin{aligned} \left\| \frac{\partial(U_\rho - u_{0,\rho})^-}{\partial \nu} \right\|_{H^{-1/2}(\partial B_R)}^2 \text{ is equivalent to} \\ \sum_{n=1}^{\infty} n^{-1} (|nR^{-n-1} M_n g_n^c|^2 + |nR^{-n-1} M_n g_n^s|^2) \text{ ,} \end{aligned}$$

with constants depending on R . Since we suppose $M_1 = \dots = M_N = 0$, it follows that

$$\left\| \frac{\partial(U_\rho - u_{0,\rho})^-}{\partial \nu} \right\|_{H^{-1/2}(\partial B_R)}^2 \leq C \sum_{n=N+1}^{\infty} n^{-1} (|nR^{-n-1} M_n g_n^c|^2 + |nR^{-n-1} M_n g_n^s|^2) \text{ .}$$

Using the fact that $\sum_n a_n b_n \leq \sum_n a_n \sum_n b_n$ for sums of positive numbers, we conclude

$$\begin{aligned} \left\| \frac{\partial(U_\rho - u_{0,\rho})^-}{\partial \nu} \right\|_{H^{-1/2}(\partial B_R)}^2 \\ \leq C \|\varphi\|_{H^{\frac{1}{2}}(\partial B_R)}^2 \sum_{n=N+1}^{\infty} |R^{-2n-1} M_n|^2 \text{ .} \end{aligned}$$

The bound $|M_n| \leq C_* \rho^{2n}$ from Proposition 3 now implies that

$$\left\| \frac{\partial(U_\rho - u_{0,\rho})^-}{\partial \nu} \right\|_{H^{-1/2}(\partial B_R)} \leq C \rho^{2N+2} \|\varphi\|_{H^{1/2}(\partial B_R)} \leq C \rho^{2N+2} \|\phi\|_{H^{1/2}(\partial \Omega)} \text{ ,} \quad (34)$$

for $\rho \leq \delta$, where δ and C depend on R and C_* (and thus on R , L and the constants $\{\sigma_\ell\}$, $\{t_\ell\}$). Let $w_\rho \in H^1(\Omega)$ denote the function

$$w_\rho = \begin{cases} u_{0,\rho} & \text{in } \Omega \setminus B_R \\ U_\rho & \text{in } B_R \text{ .} \end{cases}$$

It follows immediately from (10), (19) and (33) that

$$\begin{aligned} \int_{\Omega} \nabla w_{\rho} \nabla v \, dx &= \int_{\partial B_R} \left(\frac{\partial U_{\rho}}{\partial \nu} \right)^{-} v \, ds - \int_{\partial B_R} \left(\frac{\partial u_{0,\rho}}{\partial \nu} \right)^{+} v \, ds \\ &= \int_{\partial B_R} \frac{\partial(U_{\rho} - u_{0,\rho})^{-}}{\partial \nu} v \, ds \, , \end{aligned}$$

for any $v \in H_0^1(\Omega)$. As a consequence of this and (21) we therefore get

$$\int_{\Omega} \nabla(U - w_{\rho}) \nabla v \, dx = - \int_{\partial B_R} \frac{\partial(U_{\rho} - u_{0,\rho})^{-}}{\partial \nu} v \, ds \quad \forall v \in H_0^1(\Omega),$$

which by insertion of $v = U - w_{\rho}$ (remember: $U = w_{\rho} = \phi$ on $\partial\Omega$), and use of (34) yields

$$\begin{aligned} \int_{\Omega} |\nabla(U - w_{\rho})|^2 \, dx &= - \int_{\partial B_R} \frac{\partial(U_{\rho} - u_{0,\rho})^{-}}{\partial \nu} (U - w_{\rho}) \, ds \\ &\leq \left\| \frac{\partial(U_{\rho} - u_{0,\rho})^{-}}{\partial \nu} \right\|_{H^{-1/2}(\partial B_R)} \|U - w_{\rho}\|_{H^{1/2}(\partial B_R)} \\ &\leq C \rho^{2N+2} \|\phi\|_{H^{1/2}(\partial\Omega)} \|U - w_{\rho}\|_{H^1(\Omega)} \, . \end{aligned}$$

An application of Poincaré's inequality now gives

$$\|U - w_{\rho}\|_{H^1(\Omega)} \leq C \rho^{2N+2} \|\phi\|_{H^{1/2}(\partial\Omega)} \, ,$$

and since $w_{\rho} = u_{0,\rho}$ in $\Omega \setminus B_R$

$$\|U - u_{0,\rho}\|_{H^1(\Omega \setminus B_R)} \leq C \rho^{2N+2} \|\phi\|_{H^{1/2}(\partial\Omega)} \, ,$$

as desired. \square

The simultaneous solvability of the algebraic equations (31), $1 \leq n \leq N$, for any given integer N , has not been established. Some evidence of this solvability has already been presented in [3]. In the next section we add to this evidence of solvability, and the emergence of asymptotic shapes.

3.1 Numerical Results

When employing L layers of fixed thickness in the enhanced cloak construction described in the previous section, one is left with L "free" variables, the conductivities $\{\sigma_{\ell}\}_{\ell=1}^L$, and so it is quite natural to hope to be able to solve the equations (31) simultaneously for $n = 1, 2, \dots, L$. In this section we shall present some evidence of the feasibility of this. In doing so we display the conductivity values $\sigma_1 \dots \sigma_L$ of numerous enhanced cloaks, as well as the ratios $\lambda_0 \dots \lambda_{L-1}$, with $\lambda_{\ell} = \frac{\sigma_{\ell+1} - \sigma_{\ell}}{\sigma_{\ell+1} + \sigma_{\ell}}$, $0 \leq \ell \leq L-1$, $\sigma_0 = 1$ (which more directly emerge from solving the system of algebraic equations (31) with $n = 1, \dots, L$).

For the cases $L \leq 4$ we are able to obtain analytical solutions (using the symbolic calculation package MATHEMATICA). Note that for $L = 3, 4$ the precise expressions are quite lengthy, and we present only rounded numerical values. In the case of equidistant layers, *i.e.*, $\rho_{\ell} = \frac{2L-\ell}{L} \rho$, our results are as follows:

- $L = 1$:

$$(\sigma_1, \lambda_0) = \left(\frac{5}{3}, \frac{1}{4}\right)$$

is the solution to

$$1 - 2^{2n}\lambda_0 = 0 \quad \text{with } n = 1.$$

- $L = 2$:

$$\begin{pmatrix} \sigma_1 & \lambda_0 \\ \sigma_2 & \lambda_1 \end{pmatrix} = \begin{pmatrix} \frac{-4825+4\sqrt{1613257}}{357} & \frac{931-\sqrt{1613257}}{2048} \\ \frac{43072-25\sqrt{1613257}}{3315} & \frac{-5551+5\sqrt{1613257}}{1224} \end{pmatrix} \\ \approx \begin{pmatrix} 0.71586 & -0.165596 \\ 3.41432 & 0.653352 \end{pmatrix}$$

is a solution to

$$1 + \left(\frac{4}{3}\right)^{2n} \lambda_0 \lambda_1 - \left(\frac{4}{2}\right)^{2n} \lambda_0 - \left(\frac{3}{2}\right)^{2n} \lambda_1 = 0, \quad \text{with } n = 1, 2 \text{ and } |\lambda_0|, |\lambda_1| < 1.$$

- $L = 3$:

$$\begin{pmatrix} \sigma_1 & \lambda_0 \\ \sigma_2 & \lambda_1 \\ \sigma_3 & \lambda_2 \end{pmatrix} \approx \begin{pmatrix} 1.22827 & 0.102444 \\ 0.42636 & -0.484645 \\ 5.51582 & 0.856496 \end{pmatrix}$$

is a solution to

$$1 + \left(\frac{6}{5}\right)^{2n} \lambda_0 \lambda_1 + \left(\frac{6}{4}\right)^{2n} \lambda_0 \lambda_2 + \left(\frac{5}{4}\right)^{2n} \lambda_1 \lambda_2 - \left(\frac{6}{3}\right)^{2n} \lambda_0 \\ - \left(\frac{5}{3}\right)^{2n} \lambda_1 - \left(\frac{4}{3}\right)^{2n} \lambda_2 - \left(\frac{24}{15}\right)^{2n} \lambda_0 \lambda_1 \lambda_2 = 0,$$

with $n = 1, 2, 3$ and $|\lambda_0|, |\lambda_1|, |\lambda_2| < 1$

- $L = 4$:

$$\begin{pmatrix} \sigma_1 & \lambda_0 \\ \sigma_2 & \lambda_1 \\ \sigma_3 & \lambda_2 \\ \sigma_4 & \lambda_3 \end{pmatrix} \approx \begin{pmatrix} 0.883265 & -0.0619857 \\ 1.832611 & 0.349555 \\ 0.281192 & -0.733947 \\ 7.602646 & 0.928666 \end{pmatrix}$$

is a solution to

$$1 + \left(\frac{8}{7}\right)^{2n} \lambda_0 \lambda_1 + \left(\frac{8}{6}\right)^{2n} \lambda_0 \lambda_2 + \left(\frac{8}{5}\right)^{2n} \lambda_0 \lambda_3 + \left(\frac{7}{6}\right)^{2n} \lambda_1 \lambda_2 + \left(\frac{7}{5}\right)^{2n} \lambda_1 \lambda_3 \\ + \left(\frac{6}{5}\right)^{2n} \lambda_2 \lambda_3 + \left(\frac{48}{35}\right)^{2n} \lambda_0 \lambda_1 \lambda_2 \lambda_3 - \left(\frac{8}{4}\right)^{2n} \lambda_0 - \left(\frac{7}{4}\right)^{2n} \lambda_1 - \left(\frac{6}{4}\right)^{2n} \lambda_2 \\ - \left(\frac{5}{4}\right)^{2n} \lambda_3 - \left(\frac{48}{28}\right)^{2n} \lambda_0 \lambda_1 \lambda_2 - \left(\frac{40}{24}\right)^{2n} \lambda_0 \lambda_2 \lambda_3 - \left(\frac{40}{28}\right)^{2n} \lambda_0 \lambda_1 \lambda_3 \\ - \left(\frac{35}{24}\right)^{2n} \lambda_1 \lambda_2 \lambda_3 = 0,$$

with $n = 1, 2, 3, 4$ and $|\lambda_0|, |\lambda_1|, |\lambda_2|, |\lambda_3| < 1$

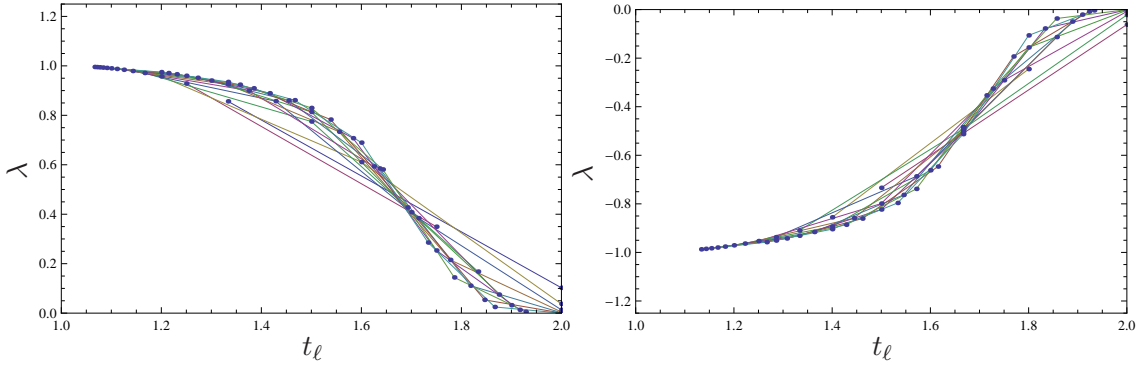


Figure 1: The solutions $(\lambda_{L-1}, \lambda_{L-3} \dots)$ (left) and $(\lambda_{L-2}, \lambda_{L-4} \dots)$ (right) for $L = 3, \dots, 14$ for the algebraic system (31) as functions of the equidistant rescaled layer interfaces $t_\ell = \frac{2L-\ell}{L}$. For visualization purposes we use linear interpolation between the discrete values.

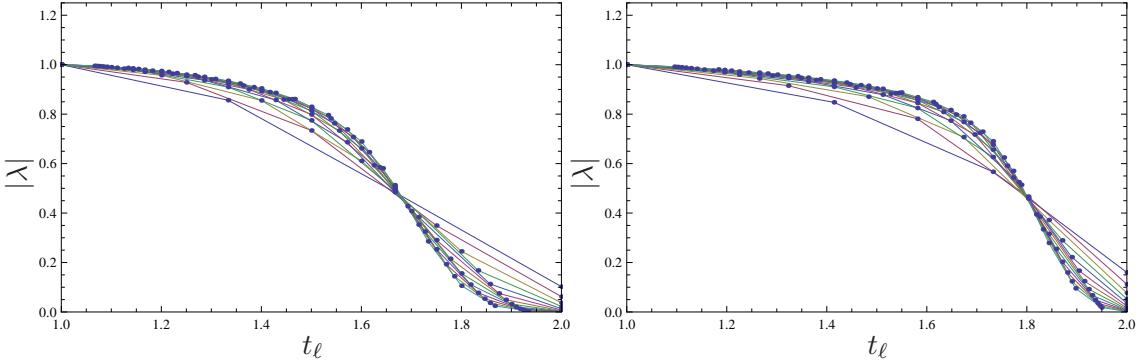


Figure 2: The modulus of the solutions $(|\lambda_{L-1}, |\lambda_{L-2} \dots)$ for $L = 3, \dots, 15$ for the algebraic system (31) as functions of the equidistant rescaled layer interfaces $t_\ell = \frac{2L-\ell}{L}$ (left) and non-equidistant rescaled layer interfaces (right).

In the latter three cases it appears very likely that these solutions are indeed the unique solutions with moduli smaller than 1. These first numbers seem to suggest the general existence of solutions for which $(\lambda_{L-1}, \lambda_{L-3} \dots)$ are positive and decreasing, and for which $\lambda_{L-2}, \lambda_{L-4} \dots$ are negative and increasing. This observation is confirmed (see Figure 1), if we use numerical methods to determine approximate solutions $\lambda_0, \dots, \lambda_{L-1}$ for $L > 4$. Moreover, we observe that $(|\lambda_\ell|)_{\ell=0}^L$, with $\lambda_L = 1$ converges to a sigmoidal curve (see Figure 2 left). The shape of this curve changes, if we choose a different grading for the layers (see Figure 2 right).

Finally, in Figure 3 we show the numerical approximations of the conductivity coefficients for 6, 9, 12, 15 and 18 enhancement layers. MATHEMATICA allows the use of arbitrarily high order precision for numerical functions. We use this feature to push the size of the coefficients M_n , that are supposed to vanish, below 10^{-50} .

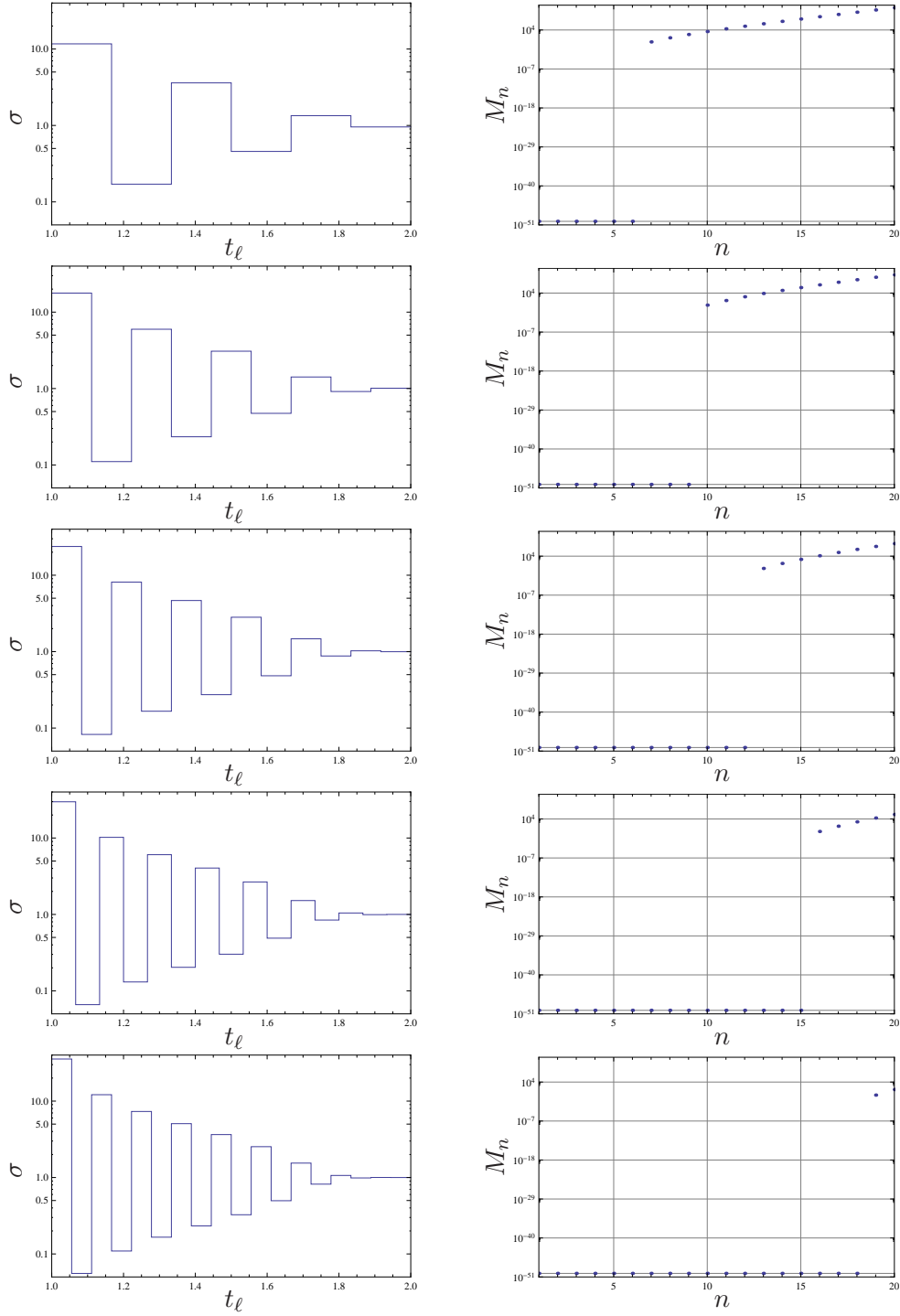


Figure 3: Numerical approximations of conductivity coefficients for 6, 9, 12, 15 and 18 enhancement layers (top to bottom, left), and the corresponding values of M_1, \dots, M_{20} from (23), with $R = 2$.

4 Main result and conclusions

Let $a_{0,\rho}$ be a family L^∞ -functions that satisfy

$$\begin{aligned} 0 < c_0 \leq a_{0,\rho}(x) \leq C_0 < \infty \quad \text{for a.e. } x \in \Omega \setminus B_\rho, \\ \text{with } a_{0,\rho}(x) = 1 \text{ for a.e. } x \in \Omega \setminus B_2, \end{aligned} \quad (35)$$

for some fixed constants c_0, C_0 . Define $A_{0,\rho} = (F_{2\rho})_* a_{0,\rho}$ on $\Omega \setminus B_{1/2}$, and let $U_{0,\rho}$ ($= u_{0,\rho} \circ F_{2\rho}^{-1}$) denote the solution to

$$\begin{aligned} \nabla \cdot (A_{0,\rho} \nabla U_{0,\rho}) &= 0 \text{ in } \Omega \setminus B_{1/2}, U_{0,\rho} = \phi \text{ on } \partial\Omega, \\ A_{0,\rho} \frac{\partial U_{0,\rho}}{\partial \nu} &= (A_{0,\rho} \nabla U_{0,\rho}) \cdot \nu = 0 \text{ on } \partial B_{1/2}. \end{aligned} \quad (36)$$

As $a_{0,\rho}$ we may for example take the piecewise constant conductivity distributions constructed in Section 3. $A_{0,\rho} = (F_{2\rho})_* a_{0,\rho}$ in $B_2 \setminus B_{1/2}$ (in physical space) thus represents one of the enhanced approximate cloaks, that have been designed to cloak the perfectly insulated ball $B_{1/2}$. Let $a_{\epsilon,\rho}$ denote the conductivity distribution

$$a_{\epsilon,\rho} = \begin{cases} a_{0,\rho} & \text{in } \Omega \setminus B_\rho, \\ \epsilon & \text{in } B_\rho \setminus B_{\rho/2}, \\ a_{obj} & \text{in } B_{\rho/2}, \end{cases}$$

where ϵ is a positive constant, and a_{obj} is an arbitrary, strictly positive L^∞ function in $B_{\rho/2}$. $A_{\epsilon,\rho} := (F_{2\rho})_* a_{\epsilon,\rho}$ in $B_2 \setminus B_{1/4}$ (of physical space) thus represents one of our enhanced approximate cloaks, that have been designed to cloak any conducting object $a_{obj}^* := (F_{2\rho})_* a_{obj}$, placed inside $B_{1/4}$. Indeed, let $U_{\epsilon,\rho}$ ($= u_{\epsilon,\rho} \circ F_{2\rho}^{-1}$) denote the solution to

$$\nabla \cdot (A_{\epsilon,\rho} \nabla U_{\epsilon,\rho}) = 0 \text{ in } \Omega, U_{\epsilon,\rho} = \phi \text{ on } \partial\Omega. \quad (37)$$

The extent to which we have been able to achieve the enhanced approximate cloaking of a_{obj}^* is measured by the closeness of $U_{\epsilon,\rho}$ to U , the solution of

$$\Delta U = 0 \text{ in } \Omega, U = \phi \text{ on } \partial\Omega, \quad (38)$$

strictly outside B_2 . An estimate of this closeness is the contents of our main theorem

Theorem 1. *Let $U_{0,\rho}, U_{\epsilon,\rho}$, and U be the solutions to (36), (37), and (38), respectively, with coefficients as described above. Let K be any compact subdomain of $\Omega \setminus B_2$. There exists a constant C , independent of ρ, ϵ, ϕ and a_{obj}^* such that*

$$\|U - U_{\epsilon,\rho}\|_{H^1(K)} \leq C\epsilon \|\phi\|_{H^{1/2}(\partial\Omega)} + \|U - U_{0,\rho}\|_{H^1(K)}. \quad (39)$$

C depends on c_0 and C_0 of (35), but is otherwise also independent of $a_{0,\rho}$, and thus also of the physical cloak $A_{\epsilon,\rho}|_{B_2 \setminus B_{1/4}}$.

Proof. This theorem follows directly by a combination of the triangle inequality and Proposition 1. Here we use that $U_{0,\rho} = u_{0,\rho}$, and $U_{\epsilon,\rho} = u_{\epsilon,\rho}$ in $\Omega \setminus B_2$, since $F_{2\rho}$ equals the identity there. \square

We note that since the coefficients of the three PDEs, involved in (36), (37), and (38), are all constantly equal to 1 in $\Omega \setminus B_2$, the functions $U_{\epsilon,\rho}$, $U_{0,\rho}$ and U are all harmonic there; consequently the H^1 norm on the left side of (39) may be replaced by any H^k norm $k > 1$, and the H^1 norm on the right hand side may be replaced by any H^k norm $k < 1$ (at the cost of replacing K in the right hand side by K' , with $K \subset\subset K' \subset \Omega \setminus B_2$).

As we saw in Section 3 (and [3]), it is in two dimension almost certainly possible to design $a_{0,\rho}$, so that

$$\|U - U_{0,\rho}\|_{H^1(K)} \leq C_N \rho^{2N+2} \|\phi\|_{H^{1/2}(\partial\Omega)} \quad ,$$

for any $N \geq 1$. This is rigorously verified for $1 \leq N \leq 4$ (due to the demonstrated presence of analytic solution to (31)), and it is very strongly indicated by the numerics for any N . The estimate (39) thus leads to

$$\|U_{\epsilon,\rho} - U\|_{H^1(K)} \leq C_N (\epsilon + \rho^{2N+2}) \|\phi\|_{H^{1/2}(\partial\Omega)} \quad ,$$

which suggests that a good choice for ϵ would be $\epsilon = \rho^{2N+2}$. With this choice of ϵ , the resulting approximate cloak will have anisotropy measure

$$\chi_{an}^* = O(\rho^{-2}) \quad ,$$

and degeneracy measures

$$\Lambda_{min}^* = O(\rho^{2N+2}) \quad , \quad \text{and} \quad \Lambda_{max}^* = O(\rho^{-1}) \quad .$$

Two of these measures, the measure of anisotropy χ_{an}^* , and the degeneracy measure Λ_{max}^* , are much more favorable than those associated with the approximate scheme using ρ^{N+1} in place of ρ (which also has a visibility of the order ρ^{2N+2} in two dimension). However, the degeneracy measure Λ_{min}^* is worse than that obtained by replacing ρ by ρ^{N+1} .

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