Approximate cloaking for the full wave equation via change of variables

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Abstract

We study, in the context of the full wave equation, an approximate cloaking scheme, that was previously considered for the Helmholtz equation [8], [17]. This cloaking scheme consists in a combination of an absorbing layer with an anisotropic layer, obtained by so-called transformation optics. We give optimal bounds for the visibility that tend to zero as a certain regularization parameter approaches 0. Our bounds are based on recent estimates for the Helmholtz equation [17], some low frequency improvements of these estimates, and the use of Fourier Transformation in time.

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1 Introduction and statement of main results

Cloaking via change of variables (sometimes referred to as "cloaking by mapping") has received quite a bit of attention since the 2006 papers by Pendry, Schurig, and Smith [18] and Leonhardt [11]. Pendry, Schurig, and Smith approached the problem in terms of the Helmholtz equation (describing monochromatic light) whereas Leonhardt took the "ray-optics" approach. In both cases the fundamental idea was to use a singular change of variables to create a cloaked region from a single point. This idea had already in 2003 been used by Greenleaf, Lassas, Uhlmann to generate extreme examples of non-uniqueness for the zero frequency Helmholtz inverse coefficient problem, the so-called Calderon Problem [5].

In the case of perfect cloaking the objective is to construct a region (the cloaked region), in which the fields trivialize in such a way that they are completely insensitive to changes in the coefficients inside this region. Furthermore, the presence of the cloak (the "shield" that surrounds the cloaked region) should not perturbe the fields outside the cloak. The need for very singular (and anisotropic) materials is not only the primary practical difficulty, it is also at the very heart of the theoretical difficulties of this cloaking problem. The main theoretical task is to define and analyze the properties of the appropriate notion of weak (and physical) solution [2], [22]. To avoid the use of singular materials, regularized schemes have been proposed in [1], [3], [9], [19], [20], [24]. The trade-off is that one no longer attains perfect cloaking, but only approximate cloaking. The approach originated in [9] appears particularly well-suited for rigorous estimation of the degree of approximate cloaking (near-invisibility). In this approach the regularization parameter $\varepsilon > 0$ represents the diameter of a small ball that is mapped to the (approximately) cloaked region by a change of variables. When ε approaches zero, one reaches the singular situation of a point being mapped to the (perfectly) cloaked region. The reader may find more information and references related to cloaking in the works mentioned above, and in the review articles [4] and [23].

Let us briefly review some facts about approximate cloaking for the Helmholtz equation for a finite range, and for the full range of frequencies. In order to succesfully achieve approximate cloaking it is often advantageous to introduce a lossy layer in addition to the standard (mapped) cloak. Using an appropriate lossy layer, it is proven in [8] that approximate cloaking works well on a bounded domain regardless of the contents of the cloaked region. In [12], the author proved that approximate cloaking works well for exterior problems with a zero Dirichlet boundary condition. In [15], the author established that approximate cloaking works well in the whole space regardless of the contents of the cloaked region (using a fixed lossy-layer). The result in [12] is very related to the results in [15], since a highly conducting media (as in a lossy layer) enforces the zero Dirichlet boundary condition approximately (see e.g. [6]). In both [8] and [15] the authors demonstrated the necessity of the lossy layer, in order to obtain a degree of approximate cloaking (near-invisibility) that is independent of the contents of the cloaked region. The paper [17] establishes precise estimates for the degree of near-invisibility at all frequencies, where the dependence on frequency is explicit. These estimates are sharp and independent of the contents of the cloaked region. To be a little more specific: in the high frequency case, our "lossy" approximate cloaking scheme works as well as in the finite frequency case. However, the estimates degenerate as frequency tends to 0. This follows from (or can be explained by) the fact that the effect of the lossy layer becomes weaker and weaker, as frequency tends to 0. Without a lossy layer the situation becomes quite complicated, as explored in [16]. For example, in the $3d$ non-resonant case, i.e., when k^2 is not an eigenvalue of the Neumann problem inside the cloaked region (here k denotes the wave number), the approximate scheme works well: cloaking is achieved (as the parameter of regularization goes to zero) and the limiting field inside the cloaked region is the corresponding solution to the Neumann problem. In the $3d$ resonant case, the situation changes completely. Sometimes cloaking is achieved; nevertheless, the limiting field inside the cloaked region depends on the solution in the free space. Sometimes cloaking is not achieved, and the energy inside the cloaked region tends to infinity as the parameter of regularization tends to 0. In the $2d$ non-resonant case, the limiting field inside the cloaked region inherits a non-local structure. In the $2d$ resonant case, cloaking sometimes is not achieved, and the energy inside the cloaked region can go to infinity. These "lossless" facts are somewhat different from what is frequently asserted in the literature, namely that (a) in $3d$, cloaking is always achieved, the limiting field inside and outside the cloaked region completely separate, and the energy of the field inside the cloaked region remains bounded, and (b) in $2d$, the limiting field inside the cloaked region satisfies the corresponding Neumann problem.

The goal of this paper is to study approximate cloaking for the wave equation via change of variables. Although approximate cloaking has been extensively investigated for the Helmholtz equation, this is, to the best of our knowledge, the first work for the full wave equation. In our approximate cloaking scheme, we use again two layers. One comes from the standard scheme introduced in the work of Kohn et al. in [9]. The other is an appropriate lossy layer, similar to what has been used for the Helmholtz equation in [8], [17]. We estimate the degree of approximate cloaking (near-invisibility) in $2d$ and $3d$. Our results assert that the visibility is of order ε in $3d$ (Theorem 1) and of the order $1/|\ln \varepsilon|$ in $2d$ (Theorem 2). We emphasize that our estimates hold for an arbitrary finite range of material parameters inside the cloaked region, but that the constants depend on this range (and only on this range). We also note that this dependence of the constants on the range is real, and totally consistent with the fact that the uniformly valid estimate of the degree of near-invisibility for the Helmholtz equation degenerates as frequency goes to 0.

To obtain our wave equation estimate of the degree of near-invisibility we, briefly described, proceed as follows. We first transform the wave equation into a family of Helmholtz equations by taking the Fourier Transform with respect to time. After obtaining the appropriate degree of near-invisibility estimates for the Helmholtz equation, where the dependence on frequency is explicit, we simply invert the Fourier Transform. For the high frequency regime we can directly use the estimate of the degree of near-invisibility established in [17], but for the low frequency Helmholtz equation we have to establish new estimates (in Section 2.2) which improve the ones in [17] under the (additional) finite range assumption. The proof of these new estimates are among the central results in this paper. We emphasize here that, the estimates for the Helmholtz equation blow up as frequency goes to 0. However, they blow up in an integrable way thanks to the new estimates in Section 2.2. Another important (albeit technical) point is, that we need to establish that the Fourier Transform of solutions to the wave equation (with respect to time) satisfy an outgoing radiation condition. The proof of this fact is contained in Appendix A.

Our analysis differs significantly between $2d$ and $3d$. In three dimension, we rely on Huyghens' principle to pass from the wave equation to a family of Helmholtz equations and to obtain appropriate estimates for the data of these equations. In two dimension, we have apriori to establish sufficient decay of solutions to the wave equation at infinity, in order to carry out a similar analysis. The required decay estimate is found in Appendix B. Furthermore, the two dimensional analysis of the low frequency Helmholtz equation is considerably more delicate than the $3d$ analogue, due to the non-uniqueness of "finite energy" solutions to the homogeneous Laplace equation (the zero frequency limit) in all of space (see Lemma 4).

We now state our main results precisely. For simplicity suppose the cloak occupies the annular region $\{1/2 < |x| < 2\}$, and that the cloaked region is the ball $B_{1/2} = \{|x| < 1/2\}$ of $\mathbb{R}^d$ $(d = 2, 3)$. Let F_ε denote the radial Lipschitz map, which transforms the ball B_ε into B_1, maps B_2 onto itself, and which is given by

$$F_\varepsilon(x) = \begin{cases} x & \text{if } x \in \mathbb{R}^d \setminus B_2 \ , \\ \Big(\dfrac{2-2\varepsilon}{2-\varepsilon} + \dfrac{|x|}{2-\varepsilon}\Big)\dfrac{x}{|x|} & \text{if } x \in B_2 \setminus B_\varepsilon \ , \\ \dfrac{x}{\varepsilon} & \text{if } x \in B_\varepsilon \ . \end{cases} \tag{1.1}$$

We shall use the standard notation

$$F_* A(y) = \frac{DF(x)A(x)DF^T(x)}{\det DF(x)} \quad \text{and} \quad F_*\Sigma(y) = \frac{\Sigma(x)}{\det DF(x)}, \quad \text{with } x = F^{-1}(y) \ , \tag{1.2}$$

for any real, symmetric matrix-valued function A, and any complex function Σ. Let u and u_c be the unique solution to the wave equation

$$\begin{cases} \partial_{tt}^2 u - \Delta u = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ u(t=0) = u_0 & \text{in } \mathbb{R}^d, \\ \partial_t u(t=0) = u_1 & \text{in } \mathbb{R}^d, \end{cases} \tag{1.3}$$

and to the "damped" wave equation

$$\begin{cases} \Sigma_{1,c}\partial_{tt}^2 u_c - \operatorname{div}(A_c \nabla u_c) + \Sigma_{2,c}\partial_t u_c = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ u_c(t=0) = u_0 & \text{in } \mathbb{R}^d, \\ \partial_t u_c(t=0) = u_1 & \text{in } \mathbb{R}^d, \end{cases} \tag{1.4}$$

respectively. Here A_c, $\Sigma_{1,c}$, and $\Sigma_{2,c}$ are time independent, and defined as follows

$$A_c, \Sigma_{1,c}, \Sigma_{2,c} = \begin{cases} I, 1, 0 & \text{in } \mathbb{R}^d \setminus B_2 \ , \\ F_{\varepsilon *}I, F_{\varepsilon *}1, 0 & \text{in } B_2 \setminus B_1 \ , \\ F_{\varepsilon *}I, F_{\varepsilon *}1, F_{\varepsilon *}\Big(1/\varepsilon^{2+\gamma}\Big) & \text{in } B_1 \setminus B_{1/2} \ , \\ a, \sigma, 0 & \text{in } B_{1/2} \ , \end{cases}$$

for some positive constant γ (a parameter of our scheme) [1]. Note that damping is only present in the annulus $B_1 \setminus B_{1/2}$. We assume that the real, symmetric matrix-valued function a, and the real-valued function σ each has a "finite range", in the sense that

$$\frac{1}{\Lambda}|\xi|^2 \leq \langle a\xi, \xi\rangle \leq \Lambda|\xi|^2 \ , \quad \frac{1}{\Lambda} \leq \sigma \leq \Lambda \ , \tag{1.5}$$

for some positive constant Λ.

Remark 1. *Notice that to cloak the region $B_{1/2}$, we use two layers in the region $B_2 \setminus B_{1/2}$. The first layer, in the region $B_2 \setminus B_1$, is the standard "mapped cloak" introduced in [9]. The second layer, in the region $B_1 \setminus B_{1/2}$, is a lossy (damping) layer used in [8], [17].*

The main results of this paper are the following two theorems.

Theorem 1. *Suppose $d = 3$ and $\gamma > 0$. Suppose f is a smooth function defined on $\mathbb{R}_+ \times \mathbb{R}^3$ and suppose u_0, u_1 are smooth functions defined on $\mathbb{R}^3$ with $\operatorname{supp} f \subset [0,1] \times (B_4 \setminus B_2)$ and $\operatorname{supp} u_0, \operatorname{supp} u_1 \subset B_4 \setminus B_2$. Let u and u_c denote the unique solutions to (1.3) and (1.4), respectively, where a and σ satisfy (1.5). Given any $R > 2$, there exists a positive constant C, depending only on the range-constant Λ, the constant γ, and R, such that*

$$\sup_{t>0} \|u_c(t,\cdot) - u(t,\cdot)\|_{L^2(B_R \setminus B_2)} \leq C\Big(\|f\| + \|u_0\| + \|u_1\|\Big)\varepsilon \ .$$

Here

$$\|f\| = \|f\|_{C^m}, \quad \|u_0\| = \|u_0\|_{C^m}, \quad \text{and} \quad \|u_1\| = \|u_1\|_{C^m} \ ,$$

for some $m > 0$.

Theorem 2. *Suppose $d = 2$ and $\gamma > 0$. Suppose f is a smooth function defined on $\mathbb{R}_+ \times \mathbb{R}^2$ and suppose u_0, u_1 are smooth functions defined on $\mathbb{R}^2$ with $\operatorname{supp} f \subset [0,1] \times (B_4 \setminus B_2)$ and $\operatorname{supp} u_0, \operatorname{supp} u_1 \subset B_4 \setminus B_2$. Let u and u_c denote the unique solutions to (1.3) and (1.4), respectively, where a and σ satisfy (1.5). Given any $R > 2$, there exists a positive constant C, depending only on the range-constant Λ, the constant γ, and R, such that*

$$\sup_{t>0} \|u_c(t,\cdot) - u(t,\cdot)\|_{L^2(B_R \setminus B_2)} \leq C\Big(\|f\| + \|u_0\| + \|u_1\|\Big)\frac{1}{|\ln \varepsilon|} \ .$$

Here

$$\|f\| = \|f\|_{C^m}, \quad \|u_0\| = \|u_0\|_{C^m}, \quad \text{and} \quad \|u_1\| = \|u_1\|_{C^m} \ ,$$

for some $m > 0$.

Remark 2. *The estimates in theorems 1 and 2 are sharp (in the regularization parameter ε) since the minimal visibility for the Helmholtz equation is of order ε in 3d, and of order $1/|\ln \varepsilon|$ in 2d, in the finite frequency regime.*

[1] Our analysis immediately extends to the case where the triplet $(a, \sigma, 0)$ is replaced by (a, σ_1, σ_2) with (a, σ_1) satisfying the same condition as (a, σ), and $0 \leq \sigma_2 \leq \Lambda$.

2 Preliminaries

As mentioned in the introduction, to obtain the degree of near invisibility estimates for the wave equation, we first transform the wave equations into a family of Helmholtz equations and establish the appropriate degree of near invisibility estimates for these Helmholtz equations, where the dependence on frequency is explicit. To this end, we recall some known results from the work of Morawetz-Ludwig [13], Nguyen [15, 16] and Nguyen-Vogelius [17], and then we establish new results, which will be used in the proof of theorems 1 and 2.

2.1 Some known results

We first recall two results concerning exterior problems. The first one, dealing with the high frequency regime, is very related to results of Morawetz-Ludwig [13], and can be proved in the same fashion as [17, Proposition 1]. In the following, whenever we talk about outgoing solutions to an exterior Helmholtz problem at frequency k, we mean solutions that satisfy

$$\partial_r v - ikv = o(r^{-\frac{d-1}{2}}) \quad \text{as } r \text{ goes to infinity}.$$

Proposition 1. *Let $d = 2, 3$ and $k > k_0$, for some $k_0 > 0$. Let $g \in H^1(\partial B_1)$ and let $v \in H^1_{loc}(\mathbb{R}^d \setminus B_1)$ be the unique outgoing solution of*

$$\begin{cases} \Delta v + k^2 v = 0 \ \text{ in } \mathbb{R}^d \setminus B_1 \ , \\ v = g \ \text{ on } \partial B_1 \ . \end{cases}$$

Then

$$\frac{1}{\beta} \int_{B_\beta \setminus B_1} \left(|\nabla v|^2 + k^2 |v|^2 \right) \leq C \|g\|^2_{H^1(\partial B_1)} \ ,$$

for some constant C depending only on k_0.

Remark 3. *Proposition 1 also holds if the unit ball B_1 is replaced by a smooth, bounded convex domain of $\mathbb{R}^d$.*

The second result, concerning the low frequency regime, is from [16] ([16, Lemmas 1 and 4]).

Proposition 2. *Let $d = 2, 3$, and $0 < \varepsilon < 1$. Let $D \subset B_1$ be a smooth, nonempty open subset of $\mathbb{R}^d$, and $g_\varepsilon \in H^{\frac{1}{2}}(\partial D)$. Assume $\mathbb{R}^d \setminus \overline{D}$ is connected and $v_\varepsilon \in H^1_{loc}(\mathbb{R}^d \setminus D)$ is the unique outgoing solution of*

$$\begin{cases} \Delta v_\varepsilon + \varepsilon^2 v_\varepsilon = 0 & \text{in } \mathbb{R}^d \setminus \bar{D} \ , \\ v_\varepsilon = g_\varepsilon & \text{on } \partial D \ . \end{cases}$$

i) We have

$$\|v_\varepsilon\|_{H^1(B_R \setminus D)} \leq C_R \|g_\varepsilon\|_{H^{\frac{1}{2}}(\partial D)} \quad \forall R > 1 \ , \tag{2.1}$$

and for all $\beta > 1$,

$$\begin{cases} \|v_\varepsilon\|_{L^2(B_{2\beta} \setminus B_\beta)} \leq C \beta^{\frac{1}{2}} \|g_\varepsilon\|_{H^{\frac{1}{2}}(\partial D)} & \text{if } d = 3, \\ \|v_\varepsilon\|_{L^2(B_{2\beta} \setminus B_\beta)} \leq C \beta \dfrac{|H_0^{(1)}(\varepsilon\beta)|}{|H_0^{(1)}(\varepsilon)|} \|g_\varepsilon\|_{H^{\frac{1}{2}}(\partial D)} & \text{if } d = 2, \end{cases} \tag{2.2}$$

for some positive constants $C_R = C(R, D)$ *and* $C = C(D)$.

ii) Assume in addition that g_ε *converges to* g *weakly in* $H^{\frac{1}{2}}(\partial D)$, *as* $\varepsilon \to 0$. *Then* v_ε *converges to* v *weakly in* $H^1_{loc}(\mathbb{R}^d \setminus D)$, *where* $v \in W^1(\mathbb{R}^d \setminus D)$ *is the unique solution of*

$$\begin{cases} \Delta v = 0 & \text{in } \mathbb{R}^d \setminus \overline{D} \ , \\ \\ v = g & \text{on } \partial D \ . \end{cases} \tag{2.3}$$

Here and in the following, $H_0^{(1)}$ denotes the Hankel function of the first kind of order 0. For a connected, smooth open region U of $\mathbb{R}^d$ with a bounded complement (this includes $U = \mathbb{R}^d$), $W^1(U)$ is defined as follows:

$$W^1(U) = \Big\{ \psi \in L^2_{loc}(U) \ ; \ \ \frac{\psi(x)}{\sqrt{1+|x|^2}} \in L^2(U) \text{ and } \nabla\psi \in L^2(U) \Big\} \quad \text{for } d = 3 \ ,$$

and

$$W^1(U) = \Big\{ \psi \in L^2_{loc}(U) : \frac{\psi(x)}{\ln(2+|x|)\sqrt{1+|x|^2}} \in L^2(U) \text{ and } \nabla\psi \in L^2(U) \Big\} \quad \text{for } d = 2 \ .$$

Remark 4. *The estimates in* (2.2) *are not stated explicitly in [16, Lemmas 1 and 4]. However, their proofs follow immediately from the ones of [16, Lemmas 1 and 4]. We can also view* (2.2) *as a limiting case of Lemma 1 below (which is [17, Theorem 1]) as* $\lambda \to 0$. *In fact* (2.1) *and* (2.2), *with* $\beta = 1/\varepsilon$, *already appeared in [15].*

The following lemma, which will be used in the proof of Proposition 3, was established in [17, Theorem 1].

Lemma 1. *Let* $d = 2, 3$, $0 < \lambda < 1$ *and* $k > 0$. *Let* a *be a real, symmetric matrix valued function, and let* σ *be a complex function, both defined on* $B_{1/2}$. *Suppose* a *is bounded and uniformly elliptic, and suppose* σ *satisfies* $0 \le \text{ess inf } \Im(\sigma) \le \text{ess sup } \Im(\sigma) < +\infty$, *and* $0 < \text{ess inf } \Re(\sigma) \le \text{ess sup } \Re(\sigma) < +\infty$ [2]. *Let* $f \in L^2(\mathbb{R}^d)$ *with* $\text{supp } f \subset B_4 \setminus B_1$, *and let* $v \in H^1_{loc}(\mathbb{R}^d)$ *be the unique outgoing solution of*

$$\text{div}(A\nabla v) + k^2 \Sigma v = f \qquad \text{in } \mathbb{R}^d \ ,$$

with

$$A, \Sigma = \begin{cases} I, 1 & \text{in } \mathbb{R}^d \setminus B_1 \ , \\ \\ I, 1 + i/(k\lambda) & \text{in } B_1 \setminus B_{1/2} \ , \\ \\ a, \sigma & \text{in } B_{1/2} \ . \end{cases}$$

Then for any $k_0 > 0$, *there exists* $C > 0$ *such that*

a) For $k > k_0$, *and* $\beta > 1$

$$\frac{1}{\beta} \int_{B_\beta \setminus B_1} \left(|\nabla v|^2 + k^2 |v|^2 \right) \le C \int_{\mathbb{R}^d} |f|^2 \ .$$

[2] In this paper, we use the notation $\Im(\xi)$ and $\Re(\xi)$ for the imaginary part and the real part of ξ, respectively.

b) For $0 < k < k_0$, and $\beta > 1$

$$\begin{cases} \|v\|_{L^2(B_\beta\setminus B_1)} \le C\beta^{\frac{1}{2}} \max\{1, \lambda/k\}\|f\|_{L^2} & \text{for } d = 3 \ , \\ \|v\|_{L^2(B_{2\beta}\setminus B_\beta)} \le C\beta \max\{1, \lambda/k\} \dfrac{|H_0^{(1)}(k\beta)|}{|H_0^{(1)}(k)|}\|f\|_{L^2} & \text{for } d = 2 \ . \end{cases}$$

The constant C depends on k_0, but is independent of a, σ, k, β, λ, and f.

The following proposition, which is an immediate consequence of Lemma 1, will be used in the proofs of theorems 1 and 2.

Proposition 3. *Let $d = 2, 3$, $0 < \lambda < 1$ and $k > 0$. Let a be a real, symmetric matrix valued function, and let σ be a complex function, both defined on $B_{1/2}$. Suppose a is bounded and uniformly elliptic, and suppose σ satisfies $0 \le \text{ess inf}\, \Im(\sigma) \le \text{ess sup}\, \Im(\sigma) < +\infty$, and $0 < \text{ess inf}\, \Re(\sigma) \le \text{ess sup}\, \Re(\sigma) < +\infty$. Let $V \in H^1(B_4\setminus \bar{B}_1)$ be such that $\Delta V + k^2 V = 0$ in $B_4 \setminus \bar{B}_1$, $V = 0$ on ∂B_1. Assume that $v \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution of the system* [3]

$$\begin{cases} \operatorname{div}(A\nabla v) + k^2\Sigma v = 0 & \text{in } \mathbb{R}^d \setminus \partial B_1 \ , \\ \big[\partial_r v\big] = \partial_r V & \text{on } \partial B_1 \ , \end{cases}$$

with

$$A, \Sigma = \begin{cases} I, 1 & \text{in } \mathbb{R}^d \setminus B_1 \ , \\ I, 1 + i/(k\lambda) & \text{in } B_1 \setminus B_{1/2} \ , \\ a, \sigma & \text{in } B_{1/2} \ . \end{cases}$$

Then for any $k_0 > 0$, there exists $C > 0$ such that

a) For $k > k_0$, and $\beta > 1$

$$\frac{1}{\beta}\int_{B_{2\beta}\setminus B_\beta} k^2|v|^2 \le C\|V\|^2_{H^1(B_4\setminus B_1)} + \frac{2}{\beta}\int_{B_4\setminus B_\beta} k^2|V|^2.$$

b) For $0 < k < k_0$, and $\beta > 1$,

$$\|v\|_{L^2(B_{2\beta}\setminus B_\beta)} \le C\beta^{\frac{1}{2}} \max\Big\{1, \lambda/k\Big\}\|V\|_{H^1(B_4\setminus B_1)},$$

for $d = 3$, and

$$\|v\|_{L^2(B_{2\beta}\setminus B_\beta)} \le C\beta \max\Big\{1, \lambda/k\Big\}\frac{|H_0^{(1)}(k\beta)|}{|H_0^{(1)}(k)|}\|V\|_{H^1(B_4\setminus B_1)},$$

for $d = 2$.

The constant C depends on k_0, but is independent of a, σ, k, β, λ, and V.

[3] In this paper, we use the notation $[v]$ to denote $v\Big|_{\text{ext}} - v\Big|_{\text{int}}$ on ∂D, for any $D \subset \mathbb{R}^d$.

Proof of Proposition 3. Let $\phi \in C^2(\mathbb{R}^d)$ be such that $\operatorname{supp}\phi \subset B_4$, $0 \le \phi \le 1$, and $\phi = 1$ in B_3. Define

$$v_1 = \begin{cases} v - \phi V & \text{if } x \in \mathbb{R}^d \setminus B_1 \ , \\ \\ v & \text{if } x \in B_1 \ . \end{cases}$$

Since $V = 0$ on ∂B_1 and $\big[\langle A\nabla v, x/|x|\rangle\big] = \partial_r V$, it follows that $v_1 \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to the equation

$$\operatorname{div}(A\nabla v_1) + k^2 \Sigma v_1 = f \quad \text{ in } \mathbb{R}^d \ ,$$

where

$$f = \begin{cases} -(\Delta V + k^2 V)\phi - 2\nabla V \nabla\phi - V\Delta\phi & \text{in } \mathbb{R}^d \setminus B_1 \ , \\ \\ 0 & \text{in } B_1 \ . \end{cases}$$

It is clear that $\operatorname{supp} f \subset B_4 \setminus B_1$ and since $\Delta V + k^2 V = 0$ in $B_4 \setminus B_1$,

$$\|f\|_{L^2} \le C\|V\|_{H^1(B_4 \setminus B_1)} \ .$$

The estimates of this proposition now follow from Lemma 1 and the fact that

$$\frac{\beta |H_0^{(1)}(k\beta)|}{|H_0^{(1)}(k)|} \ge c > 0 \text{ for } 0 < k < k_0 \text{ and } \beta > 1.$$

□

2.2 New estimates for Helmholtz problems in the low frequency regime

In this section we improve the low frequency results from the previous section, under the additional assumption that a and σ are in finite ranges. The notion of G-convergence plays a role in the proof of these improved results. We therefore first recall the definition of G-convergence and state one of the fundamental properties associated with this notion. We emphasize that if we considered cloaking for fixed a and σ, then our approach would work without the use of G-convergence.

2.2.1 G-convergence

We recall here the definition of a particular version of G-convergence (also frequently referred to as H-convergence) and state a central theorem involving this notion.

Suppose $0 < \alpha < \beta < \infty$ and let Ω be a connected, bounded, smooth open subset of $\mathbb{R}^d$. $M(\alpha, \beta, \Omega)$ denotes the set of real symmetric matrix valued functions a defined on Ω such that

$$\alpha|\xi|^2 \le \langle a\xi, \xi\rangle \le \beta|\xi|^2,$$

and $H^1_\sharp(\Omega)$ denotes the Sobolev space

$$H^1_\sharp(\Omega) = \left\{\psi \in H^1(\Omega); \int_\Omega \psi = 0\right\} \ .$$

Definition 1. *A sequence of matrices* (a_n) *in* $M(\alpha, \beta, \Omega)$ *G-converges to a matrix* $a \in M(\alpha', \beta', \Omega)$ *iff for all* $g \in \left[H^1(\Omega)\right]^*$ *(the dual of* $H^1(\Omega)$*) with* $\langle g, 1\rangle_{[H^1]^*, H^1} = 0$*, the solution* $u_n \in H^1_\sharp(\Omega)$ *of the equation*

$$\operatorname{div}(a_n \nabla u_n) = g, \quad i.e., \int_\Omega a_n \nabla u_n \nabla \psi = \langle g, \psi\rangle_{\left[H^1\right]^*, H^1} \quad \forall\, \psi \in H^1_\sharp(\Omega),$$

has the property that

$$u_n \rightharpoonup u \text{ weakly in } H^1(\Omega) \quad \text{and} \quad a_n \nabla u_n \rightharpoonup a\nabla u \text{ weakly in } L^2(\Omega),$$

where $u \in H^1_\sharp(\Omega)$ *is the unique solution of the problem*

$$\operatorname{div}(a \nabla u) = g, \quad i.e., \int_\Omega a \nabla u \nabla \psi = \langle g, \psi\rangle_{\left[H^1\right]^*, H^1} \quad \forall\, \psi \in H^1_\sharp(\Omega).$$

Concerning G-convergence, one has

Proposition 4. *Let* $0 < \alpha < \beta < +\infty$*, let* Ω *be a connected, bounded, and smooth subset of* $\mathbb{R}^d$*, and suppose* $(a_n) \subset M(\alpha, \beta, \Omega)$*, then*

i) there exist a subsequence (a_{n_k}) *of* (a_n) *and* $a \in M(\alpha', \beta', \Omega)$ *for some* $0 < \alpha' < \beta' < +\infty$ *such that* (a_{n_k}) *G-converges to* a*.*

ii) suppose $(a_n), (b_n) \subset M(\alpha, \beta, \Omega)$ *with* $a_n = b_n$ *on an open subset* Ω' *of* Ω*, and suppose* a_n *G-converges to* $a \in M(\alpha', \beta', \Omega)$*,* b_n *G-converges to* $b \in M(\alpha', \beta', \Omega)$ *for some* $0 < \alpha' < \beta' < +\infty$*. Then* $a = b$ *on* Ω'*.*

Proof of Proposition 4. The result analogous to Proposition 4 for the zero-Dirichlet boundary condition is well-known, see e.g. [7], and [14]. The proof of Proposition 4 follows by a slight variation of the proof for the zero-Dirichlet boundary condition. The details are left to the reader. □

2.2.2 New estimates in the $3d$ low frequency regime

This section is devoted to new estimates for the Helmholtz equation in the $3d$ low frequency regime. The main result is the following proposition:

Proposition 5. *Let* a *be a real, symmetric matrix valued function, and let* σ *be a complex function, both defined on* $B_{1/2}$*. Assume that*

$$\frac{1}{\Lambda}|\xi|^2 \le \langle a\xi, \xi\rangle \le \Lambda|\xi|^2 \ , \quad \frac{1}{\Lambda} \le \Re(\sigma) \le \Lambda \ , \quad \text{and} \quad 0 \le \Im(\sigma) \le \Lambda \ , \tag{2.4}$$

for some positive constant Λ*. Given* $0 < k < 1$ *and* $0 < \varepsilon < 1$*, let* $V_\varepsilon \in H^1(B_4 \setminus \bar{B}_1)$ *be such that* $\Delta V_\varepsilon + \varepsilon^2 k^2 V_\varepsilon = 0$ *in* $B_4 \setminus B_1$*,* $V_\varepsilon = 0$ *on* ∂B_1*, and let* $v_\varepsilon \in H^1_{_{loc}}(\mathbb{R}^3)$ *be the unique outgoing solution of the system*

$$\begin{cases} \operatorname{div}(A_\varepsilon \nabla v_\varepsilon) + \varepsilon^2 k^2 \Sigma_\varepsilon v_\varepsilon = 0 & \text{in } \mathbb{R}^3 \setminus \partial B_1 \ , \\ [\partial_r v_\varepsilon] = \partial_r V_\varepsilon & \text{on } \partial B_1 \ , \end{cases}$$

with

$$A_\varepsilon, \Sigma_\varepsilon = \begin{cases} I, 1 & \text{in } \mathbb{R}^3 \setminus B_1 \ , \\ I, 1 + i/(\varepsilon^{2+\gamma} k) & \text{in } B_1 \setminus B_{1/2} \ , \\ \dfrac{1}{\varepsilon} a, \dfrac{1}{\varepsilon^3}\sigma & \text{in } B_{1/2} \ , \end{cases} \tag{2.5}$$

for some constant $\gamma \geq 0$. Then there exist two positive constants c and C such that $k < c\min\{\varepsilon^{1/2}, \varepsilon^{\gamma}\}$ implies

$$\|v_\varepsilon\|_{L^2(B_{2\beta}\setminus B_\beta)} \leq C\beta^{\frac{1}{2}}\|V_\varepsilon\|_{H^1(B_4\setminus B_1)} \quad \textit{for all } \beta > 1.$$

The constants c and C depend on the "range" constant Λ and on γ, but are independent of V_ε, k, ε, β, a and σ.

Proposition 5 is a consequence of Lemma 2 below. The proof of Proposition 5 is similar to the one of Proposition 3, where instead of using Lemma 1, one uses Lemma 2. The details are left to the reader.

Lemma 2. *Let a be a real, symmetric matrix valued function, and let σ be a complex function, both defined on $B_{1/2}$. Assume that*

$$\frac{1}{\Lambda}|\xi|^2 \leq \langle a\xi, \xi\rangle \leq \Lambda|\xi|^2, \quad \frac{1}{\Lambda} \leq \Re(\sigma) \leq \Lambda, \quad \textit{and} \quad 0 \leq \Im(\sigma) \leq \Lambda\ , \tag{2.6}$$

for some positive constant Λ. Given $0 < k < 1$, $0 < \varepsilon < 1$, and $f \in L^2(\mathbb{R}^3)$ with $\operatorname{supp} f \subset B_4 \setminus B_1$, *let $v_\varepsilon \in H^1_{loc}(\mathbb{R}^3)$ be the unique outgoing solution of*

$$\operatorname{div}(A_\varepsilon \nabla v_\varepsilon) + \varepsilon^2 k^2 \Sigma_\varepsilon v_\varepsilon = f \quad \textit{in } \mathbb{R}^3\ ,$$

with

$$A_\varepsilon, \Sigma_\varepsilon = \begin{cases} I, 1 & \textit{in } \mathbb{R}^3 \setminus B_1\ , \\ I, 1 + i/(\varepsilon^{2+\gamma}k) & \textit{in } B_1 \setminus B_{1/2}\ , \\ \dfrac{1}{\varepsilon}a, \dfrac{1}{\varepsilon^3}\sigma & \textit{in } B_{1/2}\ . \end{cases} \tag{2.7}$$

for some constant $\gamma \geq 0$. Then there exist two positive constants c and C such that $k < c\min\{\varepsilon^{1/2}, \varepsilon^{\gamma}\}$ implies

$$\|v_\varepsilon\|_{L^2(B_5)} \leq C\|f\|_{L^2} \tag{2.8}$$

and

$$\|v_\varepsilon\|_{L^2(B_{2\beta}\setminus B_\beta)} \leq C\beta^{1/2}\|f\|_{L^2}\ . \tag{2.9}$$

The constants c and C depend on the "range" constant Λ and γ, but are independent of f, k, ε, β, a and σ.

Proof of Lemma 2. We only establish (2.8). The estimate (2.9), for $\beta > 4.5$, follows immediately from (2.8) and Proposition 2, since

$$\|v_\varepsilon\|_{H^{1/2}(\partial B_{4.5})} \leq C\|v_\varepsilon\|_{H^1(B_{4.8}\setminus B_{4.2})} \leq C\|v_\varepsilon\|_{L^2(B_5\setminus B_4)},$$

by standard regularity theory for elliptic equations. The extension of (2.9) to all $\beta > 1$ is a simple consequence of (2.8).

The proof of (2.8) uses ideas from [16, Lemma 3] and the theory of G-convergence (Proposition 4) and proceeds by contradiction. Suppose (2.8) is not true, then there exist $k_n \to 0$, $0 < \varepsilon_n < 1$, a_n, σ_n that satisfy (2.6), and $\{f_n\}$ with $\operatorname{supp} f_n \subset B_4 \setminus B_1$, such that

$$\lim_{n\to\infty} k_n/\min\{\varepsilon_n^{1/2}, \varepsilon_n^{\gamma}\} = 0, \quad \|f_n\|_{L^2} \to 0, \quad \text{and} \quad \|v_n\|_{L^2(B_5)} = 1\ . \tag{2.10}$$

Here $v_n \in H^1_{loc}(\mathbb{R}^3)$ is the unique outgoing solution to the equation

$$\operatorname{div}(A_n \nabla v_n) + \varepsilon_n^2 k_n^2 \Sigma_n v_n = f_n \quad \text{in } \mathbb{R}^3 , \tag{2.11}$$

where A_n and Σ_n are defined similarly to A_ε and Σ_ε of (2.7), with a, σ, ε, and k replaced by a_n, σ_n, ε_n, and k_n, respectively. Applying elliptic estimates and Proposition 2, we obtain

$$\|v_n\|_{H^1(K)} \le C_K, \tag{2.12}$$

for any compact subset K of $\mathbb{R}^3 \setminus B_{9/2}$. Multiplication of (2.11) by $\bar{v}_n$ (the conjugate of v_n) and integration of the obtained expression on B_5 yields

$$\int_{B_5 \setminus B_{1/2}} |\nabla v_n|^2 + \frac{1}{\varepsilon_n} \int_{B_{1/2}} \langle a_n \nabla v_n, \nabla v_n \rangle - \varepsilon_n^2 k_n^2 \int_{B_5 \setminus B_{1/2}} |v_n|^2$$

$$- \frac{i k_n}{\varepsilon_n^\gamma} \int_{B_1 \setminus B_{1/2}} |v_n|^2 - \frac{k_n^2}{\varepsilon_n} \int_{B_{1/2}} \sigma_n |v_n|^2 = - \int_{B_5} f_n \bar{v}_n + \int_{\partial B_5} \partial_r v_n \bar{v}_n . \tag{2.13}$$

From (2.10), (2.12), (2.13), and the fact that a_n and σ_n satisfy (2.6), we have

$$\|v_n\|_{H^1(B_5)} \le C.$$

Hence it follows from (2.12) that

$$\|v_n\|_{H^1(K)} \le C_K \quad \forall\, K \subset\subset \mathbb{R}^3. \tag{2.14}$$

Set

$$v_{1,n} = \fint_{B_{1/2}} v_n, \tag{2.15}$$

and define $v_{2,n}$, w_n on $B_{1/2}$ as follows

$$v_{2,n} = v_n - v_{1,n} , \quad \text{ and } \quad w_n = \frac{1}{\varepsilon_n} v_{2,n} .$$

From the equation for v_n, we have

$$\operatorname{div}(a_n \nabla v_n) + k_n^2 \sigma_n v_n = 0 \text{ in } B_{1/2} \quad \text{and} \quad a_n \nabla v_n \cdot \frac{x}{|x|}\Big|_{\text{int}} = \varepsilon_n \partial_r v_n \Big|_{\text{ext}} \text{ on } \partial B_{1/2}. \tag{2.16}$$

This implies

$$\operatorname{div}(a_n \nabla w_n) = - \frac{k_n^2 \sigma_n}{\varepsilon_n} v_n \text{ in } B_{1/2} , \;\; a_n \nabla w_n \cdot \frac{x}{|x|} = \partial_r v_n \Big|_{\text{ext}} \text{ on } \partial B_{1/2} , \tag{2.17}$$

and $\displaystyle\int_{B_{1/2}} w_n = 0$. Since $(k_n^2 \sigma_n / \varepsilon_n) v_n \to 0$ in $L^2(B_{1/2})$, it follows from (2.14) that w_n is bounded in $H^1(B_{1/2})$. From (2.14) and (2.16), it is clear that

$$a_n \nabla v_n \cdot \frac{x}{|x|}\Big|_{\text{int}} \to 0 \text{ in } H^{-1/2}(\partial \Omega), \tag{2.18}$$

and so

$$\int_{B_{1/2}} \langle a_n \nabla v_n, \nabla v_n \rangle = k_n^2 \int_{B_{1/2}} \sigma_n |v_n|^2 + \int_{\partial B_{1/2}} a_n \nabla v_n \cdot \frac{x}{|x|}\Big|_{\text{int}} \bar{v}_n \to 0. \tag{2.19}$$

Due to (2.6), (2.14) and (2.19), we may (after extraction of a subsequence) assume that $\big(v_n\big)$ converges to a constant in $H^1(B_{1/2})$. On the other hand, from (2.14) and the fact that $\Delta v_n + \varepsilon_n^2 k_n^2 v_n = 0$ for $|x| > 4$, we may (after extraction of a subsequence and a diagonalization argument) assume that $\big(v_n\big)$ converges in $H^1(K)$ for any $K \subset\subset \mathbb{R}^3 \setminus B_{9/2}$. Hence $\big(v_n\big)$ converges in $H^{1/2}(\partial B_5)$ and $H^{1/2}(\partial B_{1/2})$. Using the equation for v_n in $B_5 \setminus \bar{B}_{1/2}$ and the theory of elliptic equations, we obtain that $\big(v_n\big)$ converges in $H^1(B_5 \setminus \bar{B}_{1/2})$. Here we also used (2.14) and the fact that $k_n/\varepsilon_n^\gamma \to 0$. In summary, we have that (v_n) converges in $H^1_{_{loc}}(\mathbb{R}^3)$. Let v be the limit of v_n in $H^1_{_{loc}}(\mathbb{R}^3)$. By Proposition 4, we may (after extraction of a subsequence) assume that $\big(a_n\big)$ G-converges to a and that $\big(w_n\big)$ converges weakly to w in $H^1(B_{1/2})$. It now follows from (2.16) and (2.17) that

$$\begin{cases} \operatorname{div}(a\nabla w) = 0 \text{ in } B_{1/2} & \operatorname{div}(a\nabla v) = 0 \text{ in } B_{1/2}, \\ \langle a\nabla w, \dfrac{x}{|x|}\rangle = \partial_r v\Big|_{\text{ext}} \text{ on } \partial B_{1/2} & \langle a\nabla v\Big|_{\text{int}}, \dfrac{x}{|x|}\rangle = 0 \text{ on } \partial B_{1/2}. \end{cases}$$

This is consistent with the fact that v is constant on $B_{1/2}$. It is not difficult to see that $\Delta v = 0$ in $\mathbb{R}^3 \setminus B_{1/2}$ and that $v \in W^1(\mathbb{R}^3)$ (by Proposition 2). Since $\displaystyle\int_{B_{1/2}} w_n = 0$, it follows that and $\displaystyle\int_{B_{1/2}} w = 0$. Lemma 3 (stated below) now implies that $v = 0$ and $w = 0$. However, this contradicts the fact that $\|v\|_{L^2(B_5)} = \lim_{n\to\infty} \|v_n\|_{L^2(B_5)} = 1$. $\square$

In the proof of Lemma 2 we used the following result, which was established in [16, Lemma 2].

Lemma 3. *Let a be a real, symmetric, positive definite matrix valued function defined on a connected bounded smooth domain D of $\mathbb{R}^3$. There exists no nonzero solution (v, w) in $W^1(\mathbb{R}^3) \times H^1_\sharp(D)$ of the system*

$$\begin{cases} \Delta v = 0 & \textit{in } \mathbb{R}^3 \setminus \overline{D}\ , \\ \operatorname{div}(a\nabla v) = 0 & \textit{in } D\ , \\ \operatorname{div}(a\nabla w) = 0 & \textit{in } D\ , \end{cases} \quad \textit{and} \quad \begin{cases} a\nabla v \cdot \nu\Big|_{\text{int}} = 0 & \textit{on } \partial D, \\ \dfrac{\partial v}{\partial \nu}\Big|_{\text{ext}} - a\nabla w \cdot \nu = 0 & \textit{on } \partial D. \end{cases} \tag{2.20}$$

Here ν denotes the outward normal unit vector on ∂D.

The proof of Lemma 3 goes as follows:

Proof. Since $v \in W^1(\mathbb{R}^3 \setminus \overline{D})$, it follows from (2.20) that

$$\int_{\mathbb{R}^3\setminus D} |\nabla v|^2 = -\int_{\partial D} \frac{\partial v}{\partial \nu}\Big|_{\text{ext}} \bar{v} = -\int_{\partial D} (a\nabla w \cdot \nu)\bar{v}. \tag{2.21}$$

We also deduce from (2.20) that

$$\int_{\partial D} (a\nabla w \cdot \nu)\bar{v} = \int_D a\nabla w \nabla \bar{v} = 0. \tag{2.22}$$

For the latter, we have used that v is constant in D. A combination of (2.21) and (2.22) yields $v = 0$ in $\mathbb{R}^3 \setminus D$. It follows from the continuity of v across ∂D that

$$v = 0 \quad \text{in } \mathbb{R}^3 \quad \text{and} \quad w = 0 \text{ in } D.$$

$\square$

2.2.3 New estimates in the $2d$ low frequency regime

In this section, we establish new estimates for the Helmholtz equation in the $2d$ low frequency regime. These estimates will play an important role for the analysis of cloaking for the full wave equation in $2d$. The analogue of Proposition 5 is

Proposition 6. *Let a be a real, symmetric matrix valued function, and let σ be a complex function, both defined on $B_{1/2}$. Assume that*

$$\frac{1}{\Lambda}|\xi|^2 \le \langle a\xi, \xi\rangle \le \Lambda|\xi|^2, \quad \frac{1}{\Lambda} \le \Re(\sigma) \le \Lambda, \quad \textit{and} \quad 0 \le \Im(\sigma) \le \Lambda \ ,$$

for some positive constant Λ. Given $0 < k < 1$, $0 < \lambda < 1$, $0 < \varepsilon < 1$, let $V_\varepsilon \in H^1(B_4 \setminus \bar{B}_1)$ be such that $\Delta V_\varepsilon + \varepsilon^2 k^2 V_\varepsilon = 0$ in $B_4 \setminus B_1$, $V_\varepsilon = 0$ on ∂B_1, and let $v_\varepsilon \in H^1_{loc}(\mathbb{R}^2)$ be the unique outgoing solution of the system

$$\begin{cases} \operatorname{div}(A\nabla v_\varepsilon) + \varepsilon^2 k^2 \Sigma_\varepsilon v_\varepsilon = 0 & \textit{in } \mathbb{R}^2 \setminus \partial B_1 \ , \\[2mm] \big[\partial_r v_\varepsilon\big] = \partial_r V_\varepsilon & \textit{on } \partial B_1 \ , \end{cases}$$

with

$$A \ , \Sigma_\varepsilon = \begin{cases} I, 1 & \textit{in } \mathbb{R}^2 \setminus B_1 \ , \\[2mm] I, 1 + i/(\varepsilon k \lambda) & \textit{in } B_1 \setminus B_{1/2} \ , \\[2mm] a, \dfrac{1}{\varepsilon^2}\sigma & \textit{in } B_{1/2} \ . \end{cases}$$

There exist two positive constants c and C, depending only on Λ, such that if $\varepsilon k < c$, $\varepsilon k |\ln(\varepsilon k)|^2 < c\lambda$, and $k^2 |\ln(\varepsilon k)|^2 < c$, then

$$\|v_\varepsilon\|_{L^2(B_5)} \le C|\ln(\varepsilon k)|^2 \|V_\varepsilon\|_{H^1(B_4\setminus B_1)}$$

and

$$\|v_\varepsilon\|_{L^2(B_{2\beta}\setminus B_\beta)} \le C\beta |H_0^{(1)}(\varepsilon k \beta)| |\ln(\varepsilon k)| \|V_\varepsilon\|_{H^1(B_4\setminus B_1)} \quad \textit{for all } \beta > 1.$$

The constants c and C depend on the "range" constant Λ, but are independent of V_ε, k, ε, λ, a, σ and β.

Proposition 6 is a consequence of Lemma 5 (and Lemma 4) below. The proof of Proposition 6 is similar to the one of Proposition 5, where instead of Lemma 2, we use Lemma 5. The details of this proof are left for the reader.

The following result plays an important role in $2d$ low frequency regime.

Lemma 4. *Let a be a real, symmetric matrix valued function defined on $B_{1/2}$. Assume that*

$$\frac{1}{\Lambda}|\xi|^2 \le \langle a\xi, \xi\rangle \le \Lambda|\xi|^2,$$

for some positive constant Λ. Given $f \in L^2(\mathbb{R}^2)$ with $\operatorname{supp} f \subset B_4$, let $v_\varepsilon \in H^1_{loc}(\mathbb{R}^2)$ be the outgoing solution of the Helmholtz equation

$$\operatorname{div}(A\nabla v_\varepsilon) + \varepsilon^2 v_\varepsilon = f$$

with $A = I$ for $x \in \mathbb{R}^2 \setminus B_{1/2}$ and $A = a$ in $B_{1/2}$. There exist constants c and C depending only on Λ such that $0 < \varepsilon < c$ implies

$$\|v_\varepsilon\|_{L^2(B_5)} \le C|\ln \varepsilon|^2 \|f\|_{L^2} \ . \tag{2.23}$$

Remark 5. *Lemma 4 is obvious if $a = I$; the proof follows immediately from the behavior of the fundamental solution of the Helmholtz equation in 2d. In fact, in this case we have a better estimate:*

$$\|v_\varepsilon\|_{L^2(B_5)} \le C|\ln \varepsilon| \|f\|_{L^2} \ .$$

We believe that this estimate also holds in the setting of Lemma 4, however, we do not know how to prove it. The weaker estimate in Lemma 4 is sufficient to obtain our desired cloaking estimate in the 2d case.

We are ready to give

Proof of Lemma 4. We proceed by contradiction. Suppose there exist a sequence $\varepsilon_n \to 0$, a sequence $(f_n) \subset L^2(\mathbb{R}^2)$, and a sequence (a_n) of symmetric matrices such that $\operatorname{supp} f_n \subset B_4$, $(1/\Lambda)|\xi|^2 \le \langle a_n \xi, \xi \rangle \le \Lambda |\xi|^2$,

$$\lim_{n\to\infty} |\ln \varepsilon_n|^2 \|f_n\|_{L^2} = 0, \quad \text{and} \quad \|v_n\|_{L^2(B_5)} = 1. \tag{2.24}$$

Here $v_n \in H^1_{loc}(\mathbb{R}^2)$ is the unique outgoing solution to the equation

$$\operatorname{div}(A_n \nabla v_n) + \varepsilon_n^2 v_n = f_n,$$

with $A_n = I$ for $x \in \mathbb{R}^2 \setminus B_{1/2}$ and $A_n = a_n$ otherwise. Multiplying this equation by $\bar{v}_n$ and integrating the obtained expression on B_5, we have

$$\int_{B_5} \langle A_n \nabla v_n, \nabla v_n \rangle - \varepsilon_n^2 \int_{B_5} |v_n|^2 = -\int_{B_5} f \bar{v}_n + \int_{\partial B_5} \frac{\partial v_n}{\partial r} \bar{v}_n. \tag{2.25}$$

Using interior elliptic estimates in combination with (2.24), and applying Proposition 2, we obtain

$$\|v_n\|_{H^1(B_R \setminus B_{9/2})} \le C_R \quad \text{for } R > 9/2. \tag{2.26}$$

Thus it follows from (2.24) and (2.25) that

$$\int_{B_5} |\nabla v_n|^2 \le C. \tag{2.27}$$

A combination of (2.26) and (2.27) yields

$$\|v_n\|_{H^1(B_R)} \le C_R \quad \text{for all } R > 0.$$

From the second part of Proposition 2 and Proposition 4, it now follows that there exists a symmetric matrix $a \in M(\alpha, \beta, B_{1/2})$, for some $0 < \alpha < \beta < +\infty$, such that (after extraction of a subsequence) $v_n \to v$ in $L^2_{loc}(\mathbb{R}^2)$ where $v \in W^1(\mathbb{R}^2)$ is a solution to the equation

$$\operatorname{div}(A \nabla v) = 0,$$

Here $A = I$ for $x \in \mathbb{R}^2 \setminus B_{1/2}$ and $A = a$ otherwise. It is clear that $v = \alpha$ for some (complex) constant α. Since $\Delta v_n + \varepsilon_n^2 v_n = 0$ in $\mathbb{R}^2 \setminus B_4$, and v_n satisfies the outgoing radiation condition, v_n can be represented as

$$v_n(x) = \sum_{l=-\infty}^{\infty} a_{l,n} H_l^{(1)}(\varepsilon_n |x|) e^{il\theta} \quad |x| > 4,$$

where $H_l^{(1)}$ is the Hankel function of the first kind of order l. This implies

$$v_n = v_{0,n} + v_{1,n} \quad |x| > 4. \tag{2.28}$$

where

$$v_{0,n} = a_{0,n} H_0^{(1)}(\varepsilon_n |x|), \quad \text{and} \quad v_{1,n} = \sum_{l \neq 0} a_{l,n} H_l^{(1)}(\varepsilon_n |x|) e^{il\theta}, \quad |x| > 4. \tag{2.29}$$

We recall that, see e.g. [21],

$$\lim_{r \to 0} \frac{1}{|\ln r|} H_0^{(1)}(r) = \frac{2}{i\pi}, \quad |H_0^{(1)}| \searrow \text{ on } \mathbb{R}_+, \quad \lim_{r \to 0} r \frac{dH_0^{(1)}(r)}{dr} = -\frac{2}{i\pi}, \tag{2.30}$$

$$\lim_{r \to \infty} \sqrt{\frac{\pi r}{2}} e^{-i(r - \pi/4)} H_0^{(1)}(r) = 1, \tag{2.31}$$

and

$$\int_{\partial B_t} |H_l^{(1)}|^2 \leq \int_{\partial B_s} |H_l^{(1)}|^2, \quad 0 < s < t, \tag{2.32}$$

for all $l \neq 0$. By orthogonality, it is clear that for any $R > 4$,

$$\|v_{0,n}\|_{H^1(B_R \setminus B_4)} + \|v_{1,n}\|_{H^1(B_R \setminus B_4)} \leq C \|v_n\|_{H^1(B_R \setminus B_4)}.$$

Hence, after extraction of a subsequence, we may assume that $v_{0,n} \to \alpha_0$ in $L^2_{loc}(\mathbb{R}^2)$ and $v_{1,n} \to v_1$ in $L^2_{loc}(\mathbb{R}^2)$ for some (complex) constant α_0 and some $v_1 \in L^2_{loc}(\mathbb{R}^2)$. Therefore,

$$\alpha = v = \alpha_0 + v_1 \quad |x| > 4.$$

This implies that v_1 is constant on $\{|x| > 4\}$. From (2.32) we deduce that

$$\int_{B_{t+1} \setminus B_t} |v_1|^2 \leq \int_{B_{s+1} \setminus B_s} |v_1|^2, \quad 4 < s < t,$$

and so v_1 must be equal to 0. Thus $v_{0,n} \to v = \alpha$ in $L^2_{loc}(\mathbb{R}^2 \setminus B_4)$. From (2.29) and (2.30), we now have

$$\lim_{n \to \infty} |a_{0,n}| |\ln \varepsilon_n| = \frac{\pi}{2} |\alpha|. \tag{2.33}$$

On the other hand, the outgoing radiation condition implies

$$\lim_{R \to \infty} \varepsilon_n \int_{\partial B_R} |v_n|^2 \leq \int_{\mathbb{R}^2} |f_n| |v_n|. \tag{2.34}$$

By orthogonality and (2.31),

$$\lim_{R \to \infty} \varepsilon_n \int_{\partial B_R} |v_n|^2 \geq \lim_{R \to \infty} \varepsilon_n \int_{\partial B_R} |v_{0,n}|^2 = \lim_{R \to \infty} 2\pi \varepsilon_n R |a_{0,n} H_0^{(1)}(\varepsilon_n R)|^2 = 4|a_{0,n}|^2. \tag{2.35}$$

Since $\|v_n\|_{L^2(B_5)} = 1$ and $\operatorname{supp} f \subset B_4$,

$$\int_{\mathbb{R}^2} |f_n||v_n| \le C\|f_n\|_{L^2}. \tag{2.36}$$

A combination of (2.24), (2.33), (2.34), (2.35), and (2.36) yields that $v = \alpha = 0$, which in turn contradicts the fact that $\|v_n\|_{L^2(B_5)} = 1$. This completes the proof of (2.23). □

Based on Lemma 4 we now state and prove the $2d$ analogue of Lemma 2.

Lemma 5. *Assume that*

$$\frac{1}{\Lambda}|\xi|^2 \le \langle a\xi, \xi\rangle \le \Lambda|\xi|^2, \quad \frac{1}{\Lambda} \le \Re(\sigma) \le \Lambda, \quad \textit{and} \quad 0 \le \Im(\sigma) \le \Lambda \ ,$$

for some positive constant Λ. Let $f \in L^2(\mathbb{R}^2)$ with $\operatorname{supp} f \subset B_4 \setminus B_1$, and let $v_\varepsilon \in H^1_{loc}(\mathbb{R}^2)$ be the unique outgoing solution of

$$\operatorname{div}(A\nabla v_\varepsilon) + \varepsilon^2 k^2 \Sigma_\varepsilon v_\varepsilon = f \quad \textit{in } \mathbb{R}^2 \ ,$$

where A and Σ_ε are as in Proposition 6. There exist two positive constants c and C, depending only on Λ, such that if $0 < \varepsilon k < c$, $\varepsilon k|\ln(\varepsilon k)|^2 < c\lambda$, and $k^2|\ln(\varepsilon k)|^2 < c$ then

$$\|v_\varepsilon\|_{L^2(B_5)} \le C|\ln(\varepsilon k)|^2\|f\|_{L^2} \tag{2.37}$$

and

$$\|v_\varepsilon\|_{L^2(B_{2\beta}\setminus B_\beta)} \le C\beta|H_0^{(1)}(\varepsilon k\beta)||\ln(\varepsilon k)|\|V_\varepsilon\|_{H^1(B_4\setminus B_1)} \quad \textit{for all } \beta > 1. \tag{2.38}$$

The constants c and C depend on the "range" constant Λ but are otherwise independent of a, σ, f, k, ε, λ, and β.

Proof. We have

$$\operatorname{div}(A\nabla v_\varepsilon) + \varepsilon^2 k^2 v_\varepsilon = f + \varepsilon^2 k^2 v_\varepsilon - \varepsilon^2 k^2 \Sigma_\varepsilon v_\varepsilon \ .$$

From Lemma 4, we deduce that

$$\|v_\varepsilon\|_{L^2(B_5)} \le C|\ln(k\varepsilon)|^2\Big(\|f\|_{L^2} + \varepsilon^2 k^2\|v_\varepsilon\|_{L^2(B_5)} + \frac{k\varepsilon}{\lambda}\|v_\varepsilon\|_{L^2(B_5)} + k^2\|v_\varepsilon\|_{L^2(B_5)}\Big) \ .$$

By selecting c sufficiently small, and using the facts that $\varepsilon^2 k^2|\ln(\varepsilon k)|^2 \le c$, $\varepsilon k|\ln(\varepsilon k)|^2 \le c\lambda$, and $k^2|\ln(\varepsilon k)|^2 \le c$, the last three terms on the right hand side may be absorbed by (half) the left hand side, and we arrive at the first estimate (2.37).

The second estimate (2.38) follows from a combination of (2.37) and Proposition 2. □

3 Proof of the main results

Let u_ε be the unique solution of the wave equation

$$\begin{cases} \Sigma_{1,\varepsilon}\partial_{tt}^2 u_\varepsilon - \operatorname{div}(A_\varepsilon\nabla u_\varepsilon) + \Sigma_{2,\varepsilon}\partial_t u_\varepsilon = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^d, \\ u_\varepsilon(t=0) = u_0 & \text{in } \mathbb{R}^d, \\ \partial_t u_\varepsilon(t=0) = u_1 & \text{in } \mathbb{R}^d, \end{cases} \tag{3.1}$$

where $A_\varepsilon, \Sigma_{1,\varepsilon}, \Sigma_{2,\varepsilon}$, $0 < \varepsilon < 1$, are time independent and defined as follows,

$$A_\varepsilon, \Sigma_{1,\varepsilon}, \Sigma_{2,\varepsilon} = \begin{cases} I, 1, 0 & \text{in } \mathbb{R}^3 \setminus B_\varepsilon, \\ \\ I, 1, 1/\varepsilon^{2+\gamma} & \text{in } B_\varepsilon \setminus B_{\varepsilon/2}, \\ \\ F_\varepsilon^{-1}{}_* a, F_\varepsilon^{-1}{}_* \sigma, 0 & \text{in } B_{\varepsilon/2}, \end{cases}$$

with a, σ satisfying (1.5). By direct computation, $A_c, \Sigma_{1,c}, \Sigma_{2,c} = F_{\varepsilon *} A_\varepsilon, F_{\varepsilon *} \Sigma_{1,\varepsilon}, F_{\varepsilon *} \Sigma_{2,\varepsilon}$. Hence, using the invariance of the wave equation under change of variables, and the fact that u_0, u_1, and f vanish for $|x| < 2$, we have

$$u_c(t, x) = u_\varepsilon(t, F_\varepsilon^{-1}(x)),$$

and so

$$u_c(t, x) = u_\varepsilon(t, x) \quad \text{ for } x \in \mathbb{R}^d \setminus B_2.$$

The main theorems, Theorem 1 and Theorem 2, are now consequences of the following results.

Theorem 3. *Let $d = 3$ and $\gamma > 0$. Given $R > 2$, there exists a positive constant C depending only on Λ, γ, and R, such that*

$$\sup_{t>0} \|u_\varepsilon(t, \cdot) - u(t, \cdot)\|_{L^2(B_R \setminus B_2)} \le C\, \varepsilon \Big(\|f\| + \|u_0\| + \|u_1\| \Big),$$

where u is the solution to (1.3).

Theorem 4. *Let $d = 2$ and $\gamma > 0$. Given $R > 2$, there exists a positive constant C depending only on Λ, γ, and R, such that*

$$\sup_{t>0} \|u_\varepsilon(t, \cdot) - u(t, \cdot)\|_{L^2(B_R \setminus B_2)} \le \frac{C}{|\ln \varepsilon|} \Big(\|f\| + \|u_0\| + \|u_1\| \Big),$$

where u is the solution to (1.3).

3.1 Proof of Theorem 3

We split $u_\varepsilon - u$ into two parts. To this end, consider $\tilde{u}_\varepsilon \in H^1_{loc}(\mathbb{R}_+; H^1(\mathbb{R}^3))$, uniquely determined by

$$\begin{cases} \partial^2_{tt} \tilde{u}_\varepsilon - \Delta \tilde{u}_\varepsilon = f & \text{in } \mathbb{R}_+ \times \big(\mathbb{R}^3 \setminus \bar{B}_\varepsilon\big), \\ \\ \tilde{u}_\varepsilon = 0 & \text{in } \mathbb{R}_+ \times B_\varepsilon, \\ \\ \tilde{u}_\varepsilon(t = 0) = u_0 & \text{in } \mathbb{R}^3, \\ \\ \partial_t \tilde{u}_\varepsilon(t = 0) = u_1 & \text{in } \mathbb{R}^3, \end{cases}$$

and set

$$\begin{cases} v_\varepsilon = \tilde{u}_\varepsilon - u, \\ \\ w_\varepsilon = u_\varepsilon - \tilde{u}_\varepsilon. \end{cases} \tag{3.2}$$

We proceed to show that

$$\sup_{t>0} \|v_\varepsilon(t,\cdot)\|_{L^2(B_R\setminus B_2)} \le C\varepsilon Data, \tag{3.3}$$

and

$$\sup_{t>0} \|w_\varepsilon(t,\cdot)\|_{L^2(B_R\setminus B_2)} \le C\varepsilon Data, \tag{3.4}$$

where

$$Data = \|f\| + \|u_0\| + \|u_1\|.$$

Here and in the following, C denotes a positive constant depending only on Λ, γ, and R. Since $u_\varepsilon - u = v_\varepsilon + w_\varepsilon$, the inequalities (3.3) and (3.4) are sufficient to obtain the estimate of Theorem 3.

Step 1: Proof of (3.3). From the definition of v_ε in (3.2), we have

$$\begin{cases} \partial^2_{tt} v_\varepsilon - \Delta v_\varepsilon = 0 & \text{in } \mathbb{R}_+ \times \left(\mathbb{R}^3 \setminus \bar{B}_\varepsilon\right), \\ v_\varepsilon = -u & \text{on } \mathbb{R}_+ \times \partial B_\varepsilon, \\ v_\varepsilon(t=0) = \partial_t v_\varepsilon(t=0) = 0\ . \end{cases}$$

Let $\hat{v}_\varepsilon(k,x)$ denote the Fourier Transform (in time) of $v_\varepsilon(t,x)$. In this paper, by the Fourier Transform of a function defined on $[0,\infty)$, we mean the Fourier Transform of the extension by 0 for negative time. We claim that $\hat{v}_\varepsilon(k,\cdot)$ satisfies the outgoing radiation condition. To see this, let $\phi \in C^\infty(\mathbb{R}^3)$ be such that $\phi(x) = 0$ on B_1 and $\phi(x) = 1$ for $|x| > 2$. Set $\xi = \tilde{u}_\varepsilon - u\phi$. We have

$$\begin{cases} \partial^2_{tt} \xi - \Delta \xi = u\Delta\phi + 2\nabla u \nabla \phi & \text{in } \mathbb{R}_+ \times \left(\mathbb{R}^3 \setminus \bar{B}_\varepsilon\right), \\ \xi = 0 & \text{on } \mathbb{R}_+ \times \partial B_\varepsilon, \\ \xi(t=0) = \partial_t \xi(t=0) = 0\,, \end{cases}$$

for $\varepsilon < 1$. Applying Huyghen's principle and Theorem A2, we conclude that $\hat{\xi}(k,\cdot)$ satisfies the outgoing radiation condition. Since $\hat{v}_\varepsilon(k,x) = \hat{\xi}(k,x)$ for $|x| > 2$, $\hat{v}_\varepsilon(k,\cdot)$ also satisfies the outgoing radiation condition. Thus $\hat{v}_\varepsilon(k,\cdot) \in H^1_{loc}(\mathbb{R}^3 \setminus B_\varepsilon)$ is the unique outgoing solution to

$$\begin{cases} \Delta \hat{v}_\varepsilon(k,\cdot) + k^2 \hat{v}_\varepsilon(k,\cdot) = 0 & \text{in } \mathbb{R}^3 \setminus \bar{B}_\varepsilon, \\ \hat{v}_\varepsilon(k,\cdot) = -\hat{u}(k,\cdot) & \text{on } \partial B_\varepsilon, \end{cases}$$

for almost all $k > 0$. Set

$$\hat{V}_\varepsilon(k,x) = \hat{v}_\varepsilon(k,\varepsilon x).$$

It follows that

$$\begin{cases} \Delta \hat{V}_\varepsilon(k,\cdot) + k^2\varepsilon^2 \hat{V}_\varepsilon(k,\cdot) = 0 & \text{in } \mathbb{R}^3 \setminus \bar{B}_1, \\ \hat{V}_\varepsilon(k,\cdot) = -\hat{u}(k,\varepsilon\cdot) & \text{on } \partial B_1, \end{cases}$$

for almost all $k > 0$. Applying propositions 1 and 2, we have

$$\varepsilon \int_{B_{R/\varepsilon}\setminus B_{2/\varepsilon}} |\hat{V}_\varepsilon(k,\cdot)|^2 \le C \|\hat{u}(k,\varepsilon\cdot)\|^2_{H^1(\partial B_1)} \quad \text{for } k > 0, \tag{3.5}$$

and

$$\int_{B_R\setminus B_1}\Big(|\nabla\hat{V}_\varepsilon(k,x)|^2+|\hat{V}_\varepsilon(k,x)|^2\Big)\,dx\le C\|\hat{u}(k,\varepsilon\cdot)\|^2_{H^1(\partial B_1)}\quad\text{for } k>0. \tag{3.6}$$

Let $\hat{u}(k,\cdot)$ be the Fourier Transform (in time) of u. Since $u(\cdot,x)\in L^1(\mathbb{R}_+)$ (due to Huyghen's principle), we have

$$\hat{u}(k,x)=\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{itk}u(t,x)\,dt. \tag{3.7}$$

As a consequence of (3.7) and the fact that $u(t=0,x)=\partial_t u(t=0,x)=0$ for $x\in B_1$,

$$(1+k^2)\Big(|\hat{u}(k,x)|+|\nabla\hat{u}(k,x)|\Big)$$
$$\le\int_0^\infty\Big(|u(t,x)|+|\partial_{tt}u(t,x)|+|\nabla u(t,x)|+|\partial_{tt}\nabla u(t,x)|\Big)\,dt\quad\text{for } x\in B_1.$$

This implies

$$(1+k^2)\Big(|\hat{u}(k,x)|+|\nabla\hat{u}(k,x)|\Big)\le C Data\quad\text{for } x\in B_1. \tag{3.8}$$

It follows from (3.5), (3.6), and (3.8) that

$$\varepsilon\int_{B_{R/\varepsilon}\setminus B_{2/\varepsilon}}|\hat{V}_\varepsilon(k,\cdot)|^2\le\frac{C}{1+k^4}Data^2\quad\text{for } k>0, \tag{3.9}$$

and

$$\int_{B_R\setminus B_1}\Big(|\nabla\hat{V}_\varepsilon(k,x)|^2+|\hat{V}_\varepsilon(k,x)|^2\Big)\,dx\le\frac{C}{1+k^4}Data^2. \tag{3.10}$$

This implies

$$\sqrt{\varepsilon}\int_0^\infty\|\hat{V}_\varepsilon(k,\cdot)\|_{L^2(B_{R/\varepsilon}\setminus B_{2/\varepsilon})}\,dk\le C Data,$$

and by a change of variable,

$$\int_0^\infty\|\hat{v}_\varepsilon(k,\cdot)\|_{L^1(B_R\setminus B_2)}\,dk\le C\int_0^\infty\|\hat{v}_\varepsilon(k,\cdot)\|_{L^2(B_R\setminus B_2)}\,dk\le C\varepsilon Data. \tag{3.11}$$

Hence $\hat{v}(\cdot,x)\in L^1(\mathbb{R}_+)$ for almost all $x\in\mathbb{R}^3$, and by the inversion formula

$$v_\varepsilon(t,x)=2\Re\Big\{\frac{1}{\sqrt{2\pi}}\int_0^\infty\hat{v}_\varepsilon(k,x)e^{-ikt}\,dk\Big\},$$

we obtain [4]

$$\sup_{t>0}\|v_\varepsilon(t,\cdot)\|_{L^2(B_R\setminus B_2)}\le C\varepsilon Data.$$

Step 2: Proof of (3.4). From the definition of w_ε in (3.2), we have

$$\begin{cases}\Sigma_{1,\varepsilon}\partial^2_{tt}w_\varepsilon-\operatorname{div}(A_\varepsilon\nabla w_\varepsilon)+\Sigma_{2,\varepsilon}\partial_t w_\varepsilon=0 & \text{in } \mathbb{R}_+\times(\mathbb{R}^3\setminus\partial B_\varepsilon),\\[2mm] [w_\varepsilon]=0 \text{ and } \big[\partial_r w_\varepsilon\big]=-\partial_r\tilde{u}_\varepsilon\Big|_{\text{ext}}=-\partial_r(v_\varepsilon+u)\Big|_{\text{ext}} & \text{on } \mathbb{R}_+\times\partial B_\varepsilon,\\[2mm] w_\varepsilon(t=0)=\partial_t w_\varepsilon(t=0)=0.\end{cases}$$

[4]Using the fact that $\|\int_0^\infty\phi(k)\,dk\|_{L^2}\le\int_0^\infty\|\phi(k)\|_{L^2}\,dk$.

Let $\hat{w}_\varepsilon(\cdot, x)$ denote the Fourier Transform of $w_\varepsilon(\cdot, x)$. We have $\hat{w}_\varepsilon = (\hat{u}_\varepsilon - \hat{u}) - \hat{v}_\varepsilon$. By Theorem A2, $\hat{v}_\varepsilon(k, \cdot)$ satisfies the outgoing radiation condition. We claim that $(\hat{u}_\varepsilon - \hat{u})(k, \cdot)$ also satisfies the outgoing radiation condition. To see this, let $\phi \in C^\infty(\mathbb{R}^3)$ be such that $\phi(x) = 0$ on B_1 and $\phi(x) = 1$ for $|x| > 2$. Set $\xi = u_\varepsilon - u\phi$. We have

$$\begin{cases} \Sigma_{1,\varepsilon}\partial^2_{tt}\xi - \operatorname{div}\big(A_\varepsilon \nabla \xi\big) + \Sigma_{2,\varepsilon}\partial_t \xi = u\Delta\phi + 2\nabla u \nabla \phi & \text{in } \mathbb{R}_+ \times \mathbb{R}^3, \\ \xi(t=0) = \partial_t \xi(t=0) = 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{3.12}$$

for $\varepsilon < 1$. Using Huyghen's principle and Theorem A1, we obtain that $\hat{\xi}(k, \cdot)$ satisfies the outgoing radiation condition. Since $\hat{\xi}(k, x) = \hat{u}_\varepsilon(k, x) - \hat{u}(k, x)$ for $|x| > 2$, the claim is proved. As a consequence $\hat{w}_\varepsilon(k, \cdot)$ satisfies the outgoing radiation condition. It follows that $\hat{w}_\varepsilon(k, \cdot) \in H^1_{loc}(\mathbb{R}^3)$ is the unique outgoing solution to

$$\begin{cases} \operatorname{div}(A_\varepsilon \nabla \hat{w}_\varepsilon(k, \cdot)) + \Sigma_\varepsilon k^2 \hat{w}_\varepsilon(k, \cdot) = 0 & \text{in } \mathbb{R}^3 \setminus \partial B_\varepsilon, \\ \big[\partial_r \hat{w}_\varepsilon(k, \cdot)\big] = -\partial_r(\hat{v}_\varepsilon + \hat{u})(k, \cdot)\Big|_{\text{ext}} & \text{on } \partial B_\varepsilon, \end{cases}$$

for almost all $k > 0$. Here

$$\Sigma_\varepsilon = \Sigma_{1,\varepsilon} + \frac{i}{k}\Sigma_{2,\varepsilon}.$$

Define $\hat{W}_\varepsilon(k, x) = \hat{w}_\varepsilon(k, \varepsilon x)$ and $\hat{U}_\varepsilon(k, x) = \hat{u}(k, \varepsilon x)$. It follows that $\hat{W}_\varepsilon(k, \cdot) \in H^1_{loc}(\mathbb{R}^3)$ is the unique outgoing solution of

$$\begin{cases} \operatorname{div}(A_\varepsilon(\varepsilon x) \nabla \hat{W}_\varepsilon(k, \cdot)) + \varepsilon^2 k^2 \Sigma_\varepsilon(\varepsilon x) \hat{W}_\varepsilon(k, \cdot) = 0 & \text{in } \mathbb{R}^3 \setminus \partial B_1, \\ \big[\partial_r \hat{W}_\varepsilon(k, \cdot)\big] = -\partial_r \hat{V}_\varepsilon(k, \cdot) - \partial_r \hat{U}_\varepsilon(k, \cdot)\Big|_{\text{ext}} & \text{on } \partial B_1, \end{cases}$$

for almost all $k > 0$. From (3.8), we have

$$\int_{B_4} \Big(|\nabla \hat{U}_\varepsilon(k, x)|^2 + |\hat{U}_\varepsilon(k, x)|^2\Big)\, dx \le \frac{C}{1 + k^4} Data^2, \tag{3.13}$$

for $\varepsilon < 1/4$. Using (3.10) and (3.13) in combination with Proposition 3, we obtain

$$\varepsilon \int_{B_{R/\varepsilon} \setminus B_{2/\varepsilon}} |\hat{W}_\varepsilon(k, \cdot)|^2 \le \frac{C}{1 + k^4} Data^2 \quad \text{for } k \ge \varepsilon^\gamma,$$

and

$$\varepsilon \int_{B_{R/\varepsilon} \setminus B_{2/\varepsilon}} |\hat{W}_\varepsilon(k, \cdot)|^2 \le \frac{C\varepsilon^{2\gamma}}{k^2} Data^2 \quad \text{for } c \min\{\varepsilon^{1/2}, \varepsilon^\gamma\} \le k \le \varepsilon^\gamma.$$

Using (3.10) and (3.13) in combination with Proposition 5, we obtain

$$\varepsilon \int_{B_{R/\varepsilon} \setminus B_{2/\varepsilon}} |\hat{W}_\varepsilon(k, \cdot)|^2 \le C Data^2 \quad \text{for } 0 < k \le c \min\{\varepsilon^{1/2}, \varepsilon^\gamma\}.$$

We therefore have

$$
\begin{aligned}
&\sqrt{\varepsilon}\int_0^\infty \|\hat{W}_\varepsilon(k,\cdot)\|_{L^2(B_{R/\varepsilon}\setminus B_{2/\varepsilon})}\,dk\\
&\quad=\Big(\int_{\varepsilon^\gamma}^\infty+\int_{c\min\{\varepsilon^{1/2},\varepsilon^\gamma\}}^{\varepsilon^\gamma}+\int_0^{c\min\{\varepsilon^{1/2},\varepsilon^\gamma\}}\Big)\sqrt{\varepsilon}\|\hat{W}_\varepsilon(k,\cdot)\|_{L^2(B_{R/\varepsilon}\setminus B_{2/\varepsilon})}\,dk\\
&\quad\le C\Big(Data+Data\,\varepsilon^\gamma\int_{c\min\{\varepsilon^{1/2},\varepsilon^\gamma\}}^{\varepsilon^\gamma}\frac{1}{k}+Data\min\{\varepsilon^{1/2},\varepsilon^\gamma\}\Big).
\end{aligned}
$$

Since $\gamma>0$, this implies

$$
\sqrt{\varepsilon}\int_0^\infty \|\hat{W}_\varepsilon(k,\cdot)\|_{L^2(B_{R/\varepsilon}\setminus B_{2/\varepsilon})}\,dk\le Data,
$$

and by a change of variables,

$$
\int_0^\infty \|\hat{w}_\varepsilon(k,\cdot)\|_{L^2(B_R\setminus B_2)}\,dk\le C\varepsilon Data.
$$

Hence $\hat{w}_\varepsilon(\cdot,x)\in L^1(\mathbb{R}_+)$ for almost all x, and due to the inversion formula

$$
w_\varepsilon(t,x)=2\Re\Big\{\frac{1}{\sqrt{2\pi}}\int_0^\infty \hat{w}_\varepsilon(k,x)e^{-ikt}\,dk\Big\},
$$

we conclude that

$$
\sup_{t>0}\|w_\varepsilon(t,\cdot)\|_{L^2(B_R\setminus B_2)}\le C\varepsilon Data.
$$

□

3.2 Proof of Theorem 4

We follow the strategy used in the proof of Theorem 3. We split $u_\varepsilon-u$ into two parts. To this end, consider $\tilde{u}_\varepsilon\in H^1_{loc}(\mathbb{R}_+,H^1(\mathbb{R}^2))$ uniquely determined by

$$
\begin{cases}
\partial^2_{tt}\tilde{u}_\varepsilon-\Delta\tilde{u}_\varepsilon=f & \text{in } \mathbb{R}_+\times(\mathbb{R}^2\setminus\bar{B}_\varepsilon),\\
\tilde{u}_\varepsilon=0 & \text{in } \mathbb{R}_+\times B_\varepsilon,\\
\tilde{u}_\varepsilon(t=0)=u_0 & \text{in } \mathbb{R}^2,\\
\partial_t\tilde{u}_\varepsilon(t=0)=u_1 & \text{in } \mathbb{R}^2,
\end{cases}
$$

and set

$$
\begin{cases}
v_\varepsilon=\tilde{u}_\varepsilon-u,\\
w_\varepsilon=u_\varepsilon-\tilde{u}_\varepsilon.
\end{cases}
\tag{3.14}
$$

We proceed to show that

$$
\sup_{t>0}\|v_\varepsilon(t,\cdot)\|_{L^2(B_R\setminus B_2)}\le\frac{C}{|\ln\varepsilon|}Data
\tag{3.15}
$$

and

$$\sup_{t>0} \|w_\varepsilon(t,\cdot)\|_{L^2(B_R\setminus B_2)} \leq \frac{C}{|\ln\varepsilon|} Data, \tag{3.16}$$

where

$$Data = \|f\| + \|u_0\| + \|u_1\|.$$

Here and in the following, C denotes a positive constant depending only on Λ, γ, and R. Since $u_\varepsilon - u = v_\varepsilon + w_\varepsilon$, the inequalities (3.15) and (3.16) are sufficient to obtain the estimate of Theorem 4.

Step 1: Proof of (3.15). From the definition of v_ε in (3.14), we have

$$\begin{cases} \partial^2_{tt} v_\varepsilon - \Delta v_\varepsilon = 0 & \text{in } \mathbb{R}_+ \times (\mathbb{R}^2 \setminus \bar{B}_\varepsilon), \\ v_\varepsilon = -u & \text{on } \mathbb{R}_+ \times \partial B_\varepsilon, \\ v_\varepsilon(t=0) = \partial_t v_\varepsilon(t=0) = 0\,. \end{cases}$$

Let $\hat{v}_\varepsilon(k,x)$ be the Fourier Transform of $v_\varepsilon(\cdot,x)$ with respect to time. We claim that $\hat{v}_\varepsilon(k,\cdot)$ satisfies the outgoing radiation condition. To see this, let $\phi \in C^\infty(\mathbb{R}^2)$ be such that $\phi(x) = 0$ on B_1 and $\phi(x) = 1$ for $|x| > 2$. Set $\xi = \tilde{u}_\varepsilon - u\phi$. We have

$$\begin{cases} \partial^2_{tt} \xi - \Delta \xi = u\Delta\phi + 2\nabla u \nabla \phi & \text{in } \mathbb{R}_+ \times (\mathbb{R}^2 \setminus \bar{B}_\varepsilon), \\ \xi = 0 & \text{on } \mathbb{R}_+ \times \partial B_\varepsilon, \\ \xi(t=0) = \partial_t \xi(t=0) = 0\,, \end{cases}$$

for $\varepsilon < 1$. Applying theorems A2 and B1, we conclude that $\hat{\xi}(k,\cdot)$ satisfies the outgoing radiation condition. Since $\hat{v}_\varepsilon(k,x) = \hat{\xi}(k,x)$ for $|x| > 2$, $\hat{v}_\varepsilon(k,\cdot)$ also satisfies the outgoing radiation condition. Thus $\hat{v}_\varepsilon(k,\cdot) \in H^1_{loc}(\mathbb{R}^2 \setminus B_\varepsilon)$ is the unique outgoing solution to

$$\begin{cases} \Delta \hat{v}_\varepsilon(k,\cdot) + k^2 \hat{v}_\varepsilon(k,\cdot) = 0 & \text{in } \mathbb{R}^2 \setminus \bar{B}_\varepsilon, \\ \hat{v}_\varepsilon(k,\cdot) = -\hat{u}(k,\cdot) & \text{on } \partial B_\varepsilon, \end{cases}$$

for almost all $k > 0$. Here $\hat{u}$ is the Fourier Transform of u with respect to time. Set

$$\hat{V}_\varepsilon(k,x) = \hat{v}_\varepsilon(k,\varepsilon x).$$

It follows that $\hat{V}_\varepsilon(k,\cdot) \in H^1_{loc}(\mathbb{R}^2 \setminus B_1)$ is the unique outgoing solution to

$$\begin{cases} \Delta \hat{V}_\varepsilon(k,\cdot) + \varepsilon^2 k^2 \hat{V}_\varepsilon(k,\cdot) = 0 & \text{in } \mathbb{R}^2 \setminus \bar{B}_1, \\ \hat{V}_\varepsilon(k,\cdot) = -\hat{u}(k,\varepsilon\cdot) & \text{on } \partial B_1, \end{cases}$$

for almost all $k > 0$. Applying propositions 1 and 2, we have

$$\varepsilon^2 \int_{B_{R/\varepsilon}\setminus B_{2/\varepsilon}} |\hat{V}_\varepsilon(k,\cdot)|^2 \leq C\varepsilon \|\hat{u}(k,\varepsilon\cdot)\|^2_{H^1(\partial B_1)} \quad \text{for } k \geq 1/(2\varepsilon), \tag{3.17}$$

$$\varepsilon^2 \int_{B_{R/\varepsilon}\setminus B_{2/\varepsilon}} |\hat{V}_\varepsilon(k,\cdot)|^2 \leq \frac{C|H_1^{(0)}(k)|^2}{|\ln(\varepsilon k)|^2} \|\hat{u}(k,\varepsilon\cdot)\|^2_{H^{1/2}(\partial B_1)} \quad \text{for } 0 < k \leq 1/(2\varepsilon), \tag{3.18}$$

and

$$\int_{B_R\setminus B_1}\Big(|\nabla \hat V_\varepsilon(k,x)|^2+|\hat V_\varepsilon(k,x)|^2\Big)\,dx \le C\|\hat u(k,\varepsilon\cdot)\|^2_{H^1(\partial B_1)}, \tag{3.19}$$

for $k>0$. By Placherel's theorem, we have

$$\int_0^\infty (1+k^2)\|\hat u(k,\cdot)\|^2_{H^3(B_1)}\,dk \le \int_0^\infty \Big(\|u(t,\cdot)\|^2_{H^3(B_1)}+\|\partial_t u(t,\cdot)\|^2_{H^3(B_1)}\Big)\,dt,$$

since $u(t=0,x)=0$ for $x\in B_1$. Applying Theorem B1 (in the appendix), we obtain

$$\int_0^\infty (1+k^2)\|\hat u(k,\cdot)\|^2_{W^{1,\infty}(B_1)} \le C Data^2. \tag{3.20}$$

A combination of (3.17), (3.18), and (3.20) yields

$$\varepsilon\int_0^\infty \|\hat V_\varepsilon(k,\cdot)\|_{L^2(B_{R/\varepsilon}\setminus B_{2/\varepsilon})}\,dk \le \frac{C Data}{|\ln\varepsilon|}. \tag{3.21}$$

Here we use the fact that

$$\frac{|H_0^{(1)}(k)|}{|\ln(\varepsilon k)|} \le \frac{C}{|\ln\varepsilon|}(|\ln k|+1)\,, \quad 0<k<1/(2\varepsilon), \tag{3.22}$$

which follows from

$$\frac{|H_0^{(1)}(k)|}{|\ln(\varepsilon k)|} \le C\begin{cases} \dfrac{|\ln k|+1}{|\ln\varepsilon|} & \text{if } 0<k<\dfrac{1}{2\sqrt{\varepsilon}} \\ \varepsilon^{1/4} & \text{if } \dfrac{1}{2\sqrt{\varepsilon}}<k<\dfrac{1}{2\varepsilon}. \end{cases}$$

After a change of variables, (3.21) yields

$$\int_0^\infty \|\hat v_\varepsilon(k,\cdot)\|_{L^2(B_R\setminus B_2)}\,dk \le \frac{C Data}{|\ln\varepsilon|}.$$

Hence $\hat v(\cdot,x)\in L^1(\mathbb{R}_+)$ for almost all $x\in\mathbb{R}^2$, and by the inversion formula

$$v_\varepsilon(t,x) = 2\Re\Big\{\frac{1}{\sqrt{2\pi}}\int_0^\infty \hat v_\varepsilon(k,x)e^{-ikt}\,dk\Big\},$$

we obtain (3.15).

Step 2: Proof of (3.16). From the definition of w_ε in (3.14), we have

$$\begin{cases} \Sigma_{1,\varepsilon}\partial^2_{tt}w_\varepsilon - \operatorname{div}(A_\varepsilon\nabla w_\varepsilon)+\Sigma_{2,\varepsilon}\partial_t w_\varepsilon = 0 & \text{in } \mathbb{R}_+\times(\mathbb{R}^2\setminus\partial B_\varepsilon), \\ [w_\varepsilon]=0 \text{ and } \big[\partial_r w_\varepsilon\big] = -\partial_r\tilde u_\varepsilon\Big|_{\text{ext}} = -\partial_r(v_\varepsilon+u)\Big|_{\text{ext}} & \text{on } \mathbb{R}_+\times\partial B_\varepsilon, \\ w_\varepsilon(t=0)=\partial_t w_\varepsilon(t=0)=0. \end{cases}$$

Let $\hat w_\varepsilon(\cdot,x)$ denote the Fourier Transform of $w_\varepsilon(\cdot,x)$. We have $\hat w_\varepsilon = (\hat u_\varepsilon-\hat u)-\hat v_\varepsilon$. By Theorem A2, $\hat v_\varepsilon$ satisfies the outgoing radiation condition. We claim that $\hat u_\varepsilon-\hat u$ also satisfies the outgoing radiation condition. To see this, let $\phi \in C^\infty(\mathbb{R}^2)$ be such that $\phi(x) = 0$ on B_1 and $\phi(x) = 1$ for $|x| > 2$. Set $\xi = u_\varepsilon - u\phi$. We have

$$\begin{cases} \Sigma_{1,\varepsilon}\partial^2_{tt}\xi - \operatorname{div}\left(A_\varepsilon \nabla \xi\right) + \Sigma_{2,\varepsilon}\partial_t \xi = u\Delta\phi + 2\nabla u \nabla \phi & \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ \xi(t=0) = \partial_t \xi(t=0) = 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{3.23}$$

for $\varepsilon < 1$. Applying theorems A1 and B1, we conclude that $\hat{\xi}(k, \cdot)$ satisfies the outgoing radiation condition. Hence the claim is proved. As a consequence $\hat{w}_\varepsilon(k, \cdot)$ satisfies the outgoing radiation condition. It follows that $\hat{w}_\varepsilon(k, \cdot) \in H^1_{_{loc}}(\mathbb{R}^2)$ is the unique outgoing solution to

$$\begin{cases} \operatorname{div}(A_\varepsilon \nabla \hat{w}_\varepsilon(k, \cdot)) + k^2 \Sigma_\varepsilon \hat{w}_\varepsilon(k, \cdot) = 0 & \text{in } \mathbb{R}^2 \setminus \partial B_\varepsilon, \\ \left[\partial_r \hat{w}_\varepsilon(k, \cdot)\right] = -\partial_r(\hat{v}_\varepsilon + \hat{u})(k, \cdot)\Big|_{\text{ext}} & \text{on } \partial B_\varepsilon, \end{cases}$$

for almost all $k > 0$. Here

$$\Sigma_\varepsilon = \Sigma_{1,\varepsilon} + \frac{i}{k}\Sigma_{2,\varepsilon}.$$

Define $\hat{W}_\varepsilon(k, x) = \hat{w}_\varepsilon(k, \varepsilon x)$ and $\hat{U}_\varepsilon(k, x) = \hat{u}(k, \varepsilon x)$. It follows that $\hat{W}_\varepsilon(k, \cdot) \in H^1_{_{loc}}(\mathbb{R}^2)$ is the unique outgoing solution to

$$\begin{cases} \operatorname{div}(A_\varepsilon(\varepsilon x) \nabla \hat{W}_\varepsilon(k, \cdot)) + \varepsilon^2 k^2 \Sigma_\varepsilon(\varepsilon x) \hat{W}_\varepsilon(k, \cdot) = 0 & \text{in } \mathbb{R}^2 \setminus \partial B_1, \\ \left[\partial_r \hat{W}_\varepsilon(k, \cdot)\right] = -\partial_r \hat{V}_\varepsilon(k, \cdot) - \partial_r \hat{U}_\varepsilon(k, \cdot)\Big|_{\text{ext}} & \text{on } \partial B_1. \end{cases}$$

We have (for $\varepsilon < 1/4$)

$$\int_{B_4} \left(|\nabla \hat{U}_\varepsilon(k, x)|^2 + |\hat{U}_\varepsilon(k, x)|^2\right) dx \le C \|\hat{u}(k, \cdot)\|^2_{W^{1,\infty}(B_1)} \tag{3.24}$$

Using (3.19),(3.24), and Proposition 3, we obtain

$$\varepsilon^2 \int_{B_{R/\varepsilon} \setminus B_{2/\varepsilon}} |\hat{W}_\varepsilon(k, \cdot)|^2 \le \varepsilon \|\hat{u}(k, \cdot)\|^2_{W^{1,\infty}(B_1)} \quad \text{for } k \ge 1/(2\varepsilon),$$

$$\varepsilon^2 \int_{B_{R/\varepsilon} \setminus B_{2/\varepsilon}} |\hat{W}_\varepsilon(k, \cdot)|^2 \le \frac{C|H_0^{(1)}(k)|^2}{|\ln(\varepsilon k)|^2} \|\hat{u}(k, \cdot)\|^2_{W^{1,\infty}(B_1)} \quad \text{for } \varepsilon^\gamma \le k \le 1/(2\varepsilon),$$

and

$$\varepsilon^2 \int_{B_{R/\varepsilon} \setminus B_{2/\varepsilon}} |\hat{W}_\varepsilon(k, \cdot)|^2 \le \frac{C\varepsilon^{2\gamma}}{k^2} \frac{|H_0^{(1)}(k)|^2}{|\ln(\varepsilon k)|^2} \|\hat{u}(k, \cdot)\|^2_{W^{1,\infty}(B_1)} \quad \text{for } \varepsilon^{5\gamma/4} \le k \le \varepsilon^\gamma.$$

Using (3.19), (3.24), and Proposition 6 [5], we have

$$\varepsilon^2 \int_{B_{R/\varepsilon} \setminus B_{2/\varepsilon}} |\hat{W}_\varepsilon(k, \cdot)|^2 \le C|H_0^{(1)}(k)|^2 |\ln(\varepsilon k)|^2 \|\hat{u}(k, \cdot)\|^2_{W^{1,\infty}(B_1)} \quad \text{for } 0 < k \le \varepsilon^{5\gamma/4}.$$

[5]Note that $k < \varepsilon^{5\gamma/4}$ ensures that $\varepsilon k < c$, $\varepsilon k |\ln(\varepsilon k)|^2 < c\varepsilon^{1+\gamma}$, and $k^2 |\ln(\varepsilon k)|^2 < c$ for ε sufficiently small, as required in Proposition 6.

In combination with (3.20) and (3.22), this now yields

$$\varepsilon\int_0^{\infty}\|\hat{W}_\varepsilon(k,\cdot)\|_{L^2(B_{R/\varepsilon}\setminus B_{2/\varepsilon})}\,dk$$

$$\begin{aligned}
&= \Big(\int_{1/(2\varepsilon)}^{\infty}+\int_{\varepsilon^{\gamma}}^{1/(2\varepsilon)}+\int_{\varepsilon^{5\gamma/4}}^{\varepsilon^{\gamma}}+\int_0^{\varepsilon^{5\gamma/4}}\Big)\varepsilon\|\hat{W}_\varepsilon(k,\cdot)\|_{L^2(B_{R/\varepsilon}\setminus B_{2/\varepsilon})}\,dk\\
&\le C\Big(\sqrt{\varepsilon}Data+\frac{1}{|\ln\varepsilon|}Data+\frac{\varepsilon^{\gamma}}{|\ln\varepsilon|}\int_{\varepsilon^{5\gamma/4}}^{\varepsilon^{\gamma}}\frac{|\ln k|}{k}\|\hat{u}(k,\cdot)\|_{W^{1,\infty}}\,dk\Big)\\
&\quad + C\int_0^{\varepsilon^{5\gamma/4}}|H_0^{(1)}(k)||\ln(\varepsilon k)|\|\hat{u}(k,\cdot)\|_{W^{1,\infty}}\,dk. \qquad (3.25)
\end{aligned}$$

Since

$$\begin{aligned}
\int_{\varepsilon^{5\gamma/4}}^{\varepsilon^{\gamma}}\frac{|\ln k|}{k}\|\hat{u}(k,\cdot)\|_{W^{1,\infty}} &\le \Big(\int_{\varepsilon^{5\gamma/4}}^{\varepsilon^{\gamma}}\frac{|\ln k|^2}{k^2}\Big)^{1/2}\Big(\int_{\varepsilon^{5\gamma/4}}^{\varepsilon^{\gamma}}\|\hat{u}(k,\cdot)\|_{W^{1,\infty}}^2\Big)^{1/2}\\
&\le C\varepsilon^{-3\gamma/4}Data,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{\varepsilon^{5\gamma/4}}|H_0^{(1)}(k)||\ln(\varepsilon k)|\|\hat{u}(k,\cdot)\|_{W^{1,\infty}} &\le C\Big(\int_0^{\varepsilon^{5\gamma/4}}|\ln k|^4\Big)^{1/2}\Big(\int_0^{\varepsilon^{5\gamma/4}}\|\hat{u}(k,\cdot)\|_{W^{1,\infty}}^2\Big)^{1/2}\\
&\le C\varepsilon^{\gamma/2}Data,
\end{aligned}$$

it follows from (3.25) that

$$\varepsilon\int_0^{\infty}\|\hat{W}_\varepsilon(k,\cdot)\|_{L^2(B_{R/\varepsilon}\setminus B_{2/\varepsilon})}\,dk\le CData\Big(\sqrt{\varepsilon}+\frac{1}{|\ln\varepsilon|}+\frac{\varepsilon^{\gamma/4}}{|\ln\varepsilon|}+\varepsilon^{\gamma/2}\Big).$$

The fact that $\gamma>0$ now implies

$$\int_0^{\infty}\varepsilon\|\hat{W}_\varepsilon(k,\cdot)\|_{L^2(B_{R/\varepsilon}\setminus B_{2/\varepsilon})}\,dk\le\frac{C}{|\ln\varepsilon|}Data,$$

which by a change of variables becomes

$$\int_0^{\infty}\|\hat{w}_\varepsilon(k,\cdot)\|_{L^2(B_R\setminus B_2)}\,dk\le\frac{C}{|\ln\varepsilon|}Data.$$

It follows that $\hat{w}_\varepsilon(\cdot,x)\in L^1(\mathbb{R}_+)$ for almost all x, and by the inversion formula

$$w_\varepsilon(t,x)=2\Re\Big\{\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\hat{w}_\varepsilon(k,x)e^{-ikt}\,dk\Big\},$$

we obtain (3.16). □

A Appendix: The outgoing radiation condition

Suppose $A(x)$ is a real, symmetric matrix valued function, and Σ_1, Σ_2 are two real functions defined on $\mathbb{R}^d$ such that

$$|\xi|^2/\Lambda \leq \langle A(x)\xi, \xi\rangle \leq \Lambda|\xi|^2, \quad 1/\Lambda < \Sigma_1(x) < \Lambda \quad \text{and} \quad 0 \leq \Sigma_2(x) < \Lambda, \tag{A1}$$

for some positive number Λ. Furthermore suppose $A, \Sigma_1, \Sigma_2 = I, 1, 0$ outside a bounded domain. Given any $f \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ such that $\operatorname{supp} f \subset \mathbb{R}_+ \times K$ for some compact set K of $\mathbb{R}^d$, let v be the unique solution of

$$\Sigma_1(x)\partial^2_{tt}v(t,x) - \operatorname{div}\big(A(x)\nabla v(t,x)\big) + \Sigma_2(x)\partial_t v(t,x) = f(t,x), \tag{A2}$$

with

$$v(0,x) = \partial_t v(0,x) = 0. \tag{A3}$$

We first recall the following classic result, which is a direct consequence of an energy estimate.

Lemma A1. *Let $A(x)$ be a real, symmetric matrix valued function, and let Σ_1, Σ_2 be two real functions defined on $\mathbb{R}^d$ such that* (A1) *holds for some $\Lambda > 0$. Given any $f \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$, let v be the unique solution of* (A2) *and* (A3). *We have*

$$\|\partial_t v(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla v(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq Ct\int_0^t\int_{\mathbb{R}^d}|f(s,x)|^2\,dx\,ds \quad t > 0\ ,$$

and

$$\|v(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq Ct^3\int_0^t\int_{\mathbb{R}^d}|f(s,x)|^2\,dx\,ds\ , \quad t > 0\ ,$$

Here C denotes a positive constant depending only on Λ.

Proof. Multiplying the equation (A2) by $\partial_t v$ and integrating the obtained expression over $\mathbb{R}^d$, we have

$$\frac{1}{2}\frac{d}{dt}\Big(\int_{\mathbb{R}^d}\Sigma_1|\partial_t v|^2 + \langle A\nabla v, \nabla v\rangle\Big) + \int_{\mathbb{R}^d}\Sigma_2|\partial_t v|^2 = \int_{\mathbb{R}^d} f\partial_t v. \tag{A4}$$

It follows from (A1) and (A3) that

$$\|\partial_t v(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla v(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq C\Big(\int_0^t\Big(\int_{\mathbb{R}^d}|f(x,s)|^2\,dx\Big)^{1/2}ds\Big)^2 \quad t > 0\ , \tag{A5}$$

which implies

$$\|\partial_t v(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla v(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq Ct\|f\|^2_{L^2([0,t]\times\mathbb{R}^d)} \quad t > 0\ , \tag{A6}$$

for some positive constant C depending only on Λ. This completes the proof of the first inequality. The second inequality follows immediately from the first one and (A3).
$\Box$

We extend v by zero for $t < 0$. As a consequence of the preceeding lemma, v is a tempered distribution for a.e. $x \in \mathbb{R}^d$. Hence we can as usual define the Fourier Transform $\hat{v}$ of v (with respect to t) by the formula

$$\int_{-\infty}^{\infty}\hat{v}(k,x)\overline{\phi}(k)\ dk = \int_0^{\infty}v(t,x)\overline{\check{\phi}}(t)\ dt \qquad \text{for any } \phi \in \mathcal{S}(\mathbb{R})\ .$$

Here

$$\check{\phi}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{-ikt} \, dk$$

denotes the inverse of the classical Fourier Transform

$$\hat{\phi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(t) e^{ikt} \, dt \ .$$

In this appendix we show that the Fourier Transform $\hat{v}$ is indeed a function, and that the corresponding functions $\hat{v}(k, \cdot)$ for almost all $k > 0$ are outgoing solutions to the Helmholtz equation; in other words: they are solutions to the Helmholtz equation and they satisfy the outgoing radiation condition

$$\frac{\partial}{\partial r} \hat{v}(k, \cdot) - ik\hat{v}(k, \cdot) = o(r^{-\frac{d-1}{2}}) \ .$$

Theorem A1. *Let $A(x)$ be a real, symmetric matrix valued function, and let Σ_1, Σ_2 be two real functions defined on $\mathbb{R}^d$, such that* (A1) *holds for some $\Lambda > 0$, and $A, \Sigma_1, \Sigma_2 = I, 1, 0$ outside a bounded domain. Given any $f \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ such that $\operatorname{supp} f \subset \mathbb{R}_+ \times K$, for some compact subset K of $\mathbb{R}^d$, let v be the unique solution of* (A2) *and* (A3). *For almost all $k > 0$, $\hat{v}(k, \cdot) \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to the equation*

$$\operatorname{div}\big(A(x)\nabla \hat{v}(k, x)\big) + k^2 \Sigma_1(x) \hat{v}(k, x) + ik\Sigma_2(x)\hat{v}(k, x) = -\hat{f}(k, x) \ ,$$

with $\hat{f}(k, x)$ denoting the Fourier Transform of $f(t, x)$ (extended by zero for negative time).

Proof. Let v_ε be the unique solution to

$$\Sigma_1(x)\partial^2_{tt} v_\varepsilon(t, x) - \operatorname{div}(A(x)\nabla v_\varepsilon(t, x)) + (\Sigma_2(x) + \varepsilon)\partial_t v_\varepsilon(t, x) = f(t, x) \ ,$$

with $v_\varepsilon(0, x) = \partial_t v_\varepsilon(0, x) = 0$. From the analogue of (A4), we conclude that

$$\varepsilon^2 \int_0^t \|\partial_s v_\varepsilon(s, \cdot)\|^2_{L^2(\mathbb{R}^d)} \, ds \le C, \tag{A7}$$

where C denotes a positive constant depending only on $\|f\|_{L^2(\mathbb{R}_+ \times \mathbb{R}^d)}$ and Λ. Set

$$w_\varepsilon = v_\varepsilon - v.$$

We have

$$\Sigma_1(x)\partial^2_{tt} w_\varepsilon(t, x) - \operatorname{div}(A(x)\nabla w_\varepsilon(t, x)) + (\Sigma_2(x) + \varepsilon)\partial_t w_\varepsilon(t, x) = -\varepsilon\partial_t v \ ,$$

with $w_\varepsilon(0, x) = \partial_t w_\varepsilon(0, x) = 0$. It follows immediately from Lemma A1 that

$$v_\varepsilon \to v \quad \text{in} \quad L^2\big((0, T) \times \mathbb{R}^d\big) \quad , \quad \text{as} \quad \varepsilon \to 0 \ , \quad \text{for any } T > 0. \tag{A8}$$

Let $\hat{v}_\varepsilon$ denote the Fourier Transform of v_ε. From Plancherel's Theorem and the energy estimate (A7) we have

$$\int_{\mathbb{R}} k^2 \|\hat{v}_\varepsilon(k, \cdot)\|^2_{L^2(\mathbb{R}^d)} \, dk < \infty \ , \tag{A9}$$

in other words $\hat{v}_\varepsilon(k,\cdot) \in L^2(\mathbb{R}^d)$ for almost all k. It is straightforward to check that $\hat{v}_\varepsilon(k,\cdot)$ satisfies the equation

$$\operatorname{div}\big(A(x)\nabla\hat{v}_\varepsilon(k,x)\big) + k^2\Sigma_1(x)\hat{v}_\varepsilon(k,x) + ik\Sigma_2(x)\hat{v}_\varepsilon(k,x) + ik\varepsilon\hat{v}_\varepsilon(k,x) = -\hat{f}(k,x)\ . \tag{A10}$$

As a consequence $\hat{v}_\varepsilon(k,\cdot)$ lies in $H^1(\mathbb{R}^d)$ for almost all k, and it is the unique solution to the equation (A10) in this space. By the limiting absorption principle (see e.g [10, Section 4.6]) we have, for $k > 0$,

$$\hat{v}_\varepsilon(k,\cdot) \to \hat{V}(k,\cdot) \text{ weakly in } H^1_{loc}(\mathbb{R}^d)\ , \tag{A11}$$

where $\hat{V}(k,x) \in H^1_{loc}(\mathbb{R}^d)$ is the unique outgoing solution to

$$\operatorname{div}(A(x)\nabla\hat{V}(k,x)) + k^2\Sigma_1(x)\hat{V}(k,x) + ik\Sigma_2(x)\hat{V}(k,x) = -\hat{f}(k,x)\ .$$

From (A9) and (A11) it follows that

$$\hat{v}_\varepsilon(k,x) \text{ converges to } \hat{V}(k,x) \text{ in the distributional sense on } \mathbb{R}_+ \times \mathbb{R}^d\ . \tag{A12}$$

On the other hand, let $\phi(k,x)$ be a C^∞ test function (in k and x) with compact support, then

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{\mathbb{R}^d}(\hat{v}_\varepsilon(k,x) - \hat{v}(k,x))\overline{\phi}(k,x)\,dxdk \qquad\qquad (A13)\\
&\qquad = \int_0^\infty\int_{\mathbb{R}^d}(v_\varepsilon(t,x) - v(t,x))\overline{\check{\phi}}(t,x)\,dxdt\\
&\qquad = \int_0^T\int_{\mathbb{R}^d}(v_\varepsilon(t,x) - v(t,x))\overline{\check{\phi}}(t,x)\,dxdt\\
&\qquad\quad + \int_T^\infty\int_{\mathbb{R}^d}(v_\varepsilon(t,x) - v(t,x))\overline{\check{\phi}}(t,x)\,dxdt\ .
\end{aligned}$$

Since $\operatorname{supp}\check{\phi}(t,x) \subset \mathbb{R}\times B_R$ for some $R > 0$, it follows from Lemma A1 that

$$\int_T^\infty\int_{\mathbb{R}^d}(v_\varepsilon(t,x) - v(t,x))\overline{\check{\phi}}(t,x)\,dxdt \le C\int_T^\infty t^{3/2}\|\check{\phi}(t,\cdot)\|_{L^2(\mathbb{R}^d)}\,dt\ .$$

Since $\|\check{\phi}(t,\cdot)\|_{L^2(\mathbb{R}^d)}$ decreases faster than any negative power of t, it follows that, given any $\delta > 0$, we may choose T such that

$$\left|\int_T^\infty\int_{\mathbb{R}^d}(v_\varepsilon(t,x) - v(t,x))\overline{\check{\phi}}(t,x)\,dxdt\right| < \delta/2 \qquad \text{for all } 0 < \varepsilon < 1.$$

Since v_ε converges to v in $L^2((0,T)\times\mathbb{R}^d)$, according to (A8), we may now choose ε sufficiently small that

$$\left|\int_0^T\int_{\mathbb{R}^d}(v_\varepsilon(t,x) - v(t,x))\check{\phi}(t,x)\,dxdt\right| < \delta/2\ .$$

A combination of these two estimates with (A13) yields that

$$\hat{v}_\varepsilon \text{ converges to } \hat{v} \text{ in the distributional sense (with respect to } k \text{ and } x)\ .$$

As a consequence of this and (A12) we conclude that $\hat{v}(k,x) = \hat{V}(k,x)$ for almost all $k > 0$, which completes the proof of Theorem A1. $\square$

Using the same technique, we can also prove

Theorem A2. *Let D be a smooth, bounded, open subset of $\mathbb{R}^d$. Let $f \in L^2\big(\mathbb{R}_+ \times (\mathbb{R}^d \setminus D)\big)$ be such that* $\operatorname{supp} f \in \mathbb{R}_+ \times K$ *for some compact subset K of $\mathbb{R}^d \setminus D$. Suppose $A(x)$ is a real, symmetric matrix valued function, and Σ_1, Σ_2 are two real functions defined on $\mathbb{R}^d \setminus D$ such that* (A1) *holds for some $\Lambda > 0$. Suppose also $A, \Sigma_1, \Sigma_2 = I, 1, 0$ outside a bounded domain and let v be the unique solution to the equation*

$$\begin{cases} \Sigma_1(x)\partial^2_{tt} v(t,x) - \operatorname{div}\big(A(x)\nabla v(t,x)\big) + \Sigma_2(x)\partial_t v(t,x) = f(t,x) & \text{in } \mathbb{R}_+ \times (\mathbb{R}^d \setminus \bar{D}) \ , \\ v = 0 \quad \text{on } \mathbb{R}_+ \times \partial D \ , & \\ v(t=0) = \partial_t v(t=0) = 0 \ . & \end{cases} \tag{A14}$$

Let $\hat{v}(k,x)$ denote the Fourier Transform of $v(t,x)$ with respect to t. Then $\hat{v}(k,\cdot) \in H^1_{loc}(\mathbb{R}^d \setminus D)$ is the unique outgoing solution to

$$\operatorname{div}\big(A\nabla \hat{v}(k,x)\big) + k^2 \Sigma_1 \hat{v}(k,x) + ik\Sigma_2 \hat{v}(k,x) = -\hat{f}(k,x) \quad \text{in } \mathbb{R}^d \setminus \bar{D} \ ,$$

for almost all $k > 0$.

B Appendix: Decay of solutions of the $2d$ wave equation

In this section, we establish the decay of solutions of the $2d$ wave equation which is an ingredient in the proof of Theorem 4.

Theorem B1. *Let $f(t,x)$, $u_0(x)$, and $u_1(x)$ be smooth functions such that* $\operatorname{supp} f \subset [0,1] \times (B_4 \setminus B_2)$, *and* $\operatorname{supp} u_0$, $\operatorname{supp} u_1 \subset B_4 \setminus B_2$. *Let u be the unique solution of the system*

$$\begin{cases} \partial^2_{tt} u - \Delta u = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ u(t=0) = u_0 & \text{in } \mathbb{R}^2, \\ \partial_t u(t=0) = u_1 & \text{in } \mathbb{R}^2. \end{cases}$$

There exist a positive constant C and an integer $m > 0$ such that

$$\int_0^\infty \|u(t,\cdot)\|^2_{L^2(B_1)}\, dt \le C Data^2,$$

where $Data = \|f\| + \|u_0\| + \|u_1\|$, and $\|\cdot\|$ denotes the C^m-norm.

Proof. The theorem follows immediately from the explicit formula for solutions of the wave equation in $2d$. For the convenience of the reader, we present the proof. Let v and w be the unique solutions to the systems

$$\begin{cases} \partial^2_{tt} v - \Delta v = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ v(t=0) = u_0 & \text{in } \mathbb{R}^2, \\ \partial_t v(t=0) = u_1 & \text{in } \mathbb{R}^2. \end{cases}$$

and

$$\begin{cases} \partial_{tt}^2 w - \Delta w = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \\ w(t=0) = 0 & \text{in } \mathbb{R}^2, \\ \partial_t w(t=0) = 0 & \text{in } \mathbb{R}^2, \end{cases}$$

respectively. It is clear that

$$u = v + w. \tag{B1}$$

We have

$$v(t,x) = \frac{1}{2} \fint_{B(x,t)} \frac{t u_0(y) + t^2 u_1(y) + t\langle \nabla u_0(y), y - x\rangle}{\big(t^2 - |y-x|^2\big)^{1/2}}\, dy,$$

which implies

$$|v(t,x)| \le C\Big(\frac{1}{t^2}\|u_0\| + \frac{1}{t}\|u_1\|\Big) \quad \text{for all } x \in B_1,\ t > 6.$$

Direct integration therefore yields

$$\int_6^\infty \|v(t,\cdot)\|^2_{L^2(B_1)}\, dt \le C Data^2. \tag{B2}$$

By a standard energy estimate

$$\int_0^6 \|v(t,\cdot)\|^2_{L^2(B_1)}\, dt \le C Data^2,$$

and a combination with (B2) now gives

$$\int_0^\infty \|v(t,\cdot)\|^2_{L^2(B_1)}\, dt \le C Data^2. \tag{B3}$$

On the other hand, we have

$$w(t,x) = \int_0^t w(t,x;s)\, ds, \tag{B4}$$

where $w(t,x;s)$ is the unique solution to the equation

$$\begin{cases} \partial_{tt}^2 w(t,x;s) - \Delta w(t,x;s) = 0 & \text{in } (s,+\infty) \times \mathbb{R}^2, \\ w(t=s,x;s) = 0 & \text{in } \mathbb{R}^2, \\ \partial_t w(t=s,x;s) = f(s,x) & \text{in } \mathbb{R}^2. \end{cases}$$

The explicit formula for solutions to the wave equation gives

$$w(t,x;s) = \frac{1}{2} \fint_{B(x,t-s)} \frac{(t-s)^2 f(s,y)}{\big((t-s)^2 - |y-x|^2\big)^{1/2}}\, dy \quad t > s > 0\ .$$

For $x \in B_1$ this implies

$$\begin{aligned} w(t,x;s) &= 0 \quad \text{for } s > 1,\, t > s > 0 \quad \text{and} \\ |w(x,t,s)| &\le Ct^{-1} Data \quad \text{for } s < 1,\, t > 7. \end{aligned} \tag{B5}$$

Combining (B4), (B5) and the standard energy estimate for $w(t,x;s)$, we obtain

$$\int_0^\infty \|w(t,\cdot)\|^2_{L^2(B_1)}\, dt \le C Data^2. \tag{B6}$$

The estimate of Theorem B1 follows from (B1), (B3), and (B6). $\square$

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