DIFFUSION AND HOMOGENIZATION LIMITS WITH SEPARATE SCALES

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Abstract. We consider the simultaneous diffusion and homogenization limit of the linear Boltzmann equation in a periodic medium in the regime where the mean free path is much smaller than the lattice constant. The resulting equation is a diffusion equation, with an averaged diffusion matrix that is formally obtained by first performing the diffusion limit and then the homogenization one. The rigorous proof relies on the use of two-scale limits, in combination with an asymptotic expansion of the equilibrium profile in powers of the ratio between the mean free path and the lattice constant.

Key words. Diffusion approximation, two-scale limits, kinetic equations,

AMS subject classifications. 35B25 35B27 82B40 82B21

1. Introduction

1.1. Setting of the problem. The derivation of macroscopic equations like diffusion equations or hydrodynamic equations from kinetic theory is a topic which has received a lot of attention in the last decades (see for instance [16], [9], [17], [27], [28], [29],[32], [21], [22]). The starting point is the Boltzmann equation, the unknown of which is the particle distribution function f, a function of time, positition and velocity ($f = f(t, \mathbf{x}, \mathbf{v})$). A typical Boltzmann equation asserts that

$$\partial_t f + \mathbf{v} \cdot \nabla_{\mathbf{x}} f = Q(f) \ ,$$

where the left hand side is the total derivative taking into account the free streaming of particles, while the right hand side is the so-called collision operator, which describes in a statistical way the scattering of the particles by an exterior medium and/or other particles. A Boltzmann equation may contain other terms, describing the effects of driving forces, sources or absorption; it may also include other degrees of freedom than time, space and velocity.

Macroscopic effects prevail when the time scale between collisions is much smaller than the observation time scale, or equivalently when the mean free path (which is the average distance between two successive collisions) becomes much smaller than the specimen length scale. The rescaled Boltzmann equation reads

$$\partial_t f + \frac{1}{\varepsilon} \mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{Q(f)}{\varepsilon^2} \ ,$$

where ϵ is the small mean free path. As $\varepsilon \to 0$, the particles are driven towards a local equilibrium characterized by the equation $Q(f) = 0$.

One of the simplest collision operators is the linear BGK operator

$$Q(f)(\mathbf{v}) = \int (\sigma(\mathbf{v}' \to \mathbf{v}) f(\mathbf{v}') - \sigma(\mathbf{v} \to \mathbf{v}') f(\mathbf{v}))\, d\nu(\mathbf{v}') \ ,$$

whose kernel is generated by a positive function [16] (with an abuse of notation, we shall refer to this function as the Maxwellian). The local equilibrium is then a multiple of the Maxwellian, the multiplier being the limiting particle density, a function of space and time. This particle density can be shown to solve a diffusion equation (see also [15]).

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The cross section $\sigma(\mathbf{v}' \to \mathbf{v})$, which represents the probability per unit time that a collision changes the particle velocity from $\mathbf{v}'$ to $\mathbf{v}$, will depend on position in the case of a non homogeneous medium. We shall consider the case when this position dependence is periodic, with a small scale of periodicity (this scale is referred to as the lattice constant). In this situation one might expect some form of homogenization to occur, giving rise to a macroscopic equation with averaged coefficients (see [12], [2], [18], [19], [5], [13], [14]). The linear Boltzmann equation has been widely studied when the mean free path and the lattice constant are small but of the same order of magnitude, see for instance Bal [6] , Allaire - Bal [3], Goudon-Poupaud [23], [24] , Goudon-Mellet [21] and Ben Abdallah-Tayeb [11]. In this case the local equilibrium profile is a solution of the cell problem $\mathbf{v} \cdot \nabla_{\mathbf{y}} f = Q(f)$. The equilibrium is in general a function of both the velocity and the scaled position variable y.

The aim of the present paper is to tackle the simultaneous diffusion and homogenization limit of the linear Boltzmann equation in the case when the mean free path is much smaller than the already small lattice constant. Namely, we consider the equation

$$\partial_t f + \frac{1}{\varepsilon}\mathbf{v} \cdot \nabla_{\mathbf{x}} f = \frac{Q_\alpha(f)}{\varepsilon^2} \ , \tag{1.1}$$

where

$$Q_\alpha(f)(\mathbf{v}) = \int \Big(\sigma(\frac{\mathbf{x}}{\alpha}, \mathbf{v}, \mathbf{v}')f(\mathbf{v}') - \sigma(\frac{\mathbf{x}}{\alpha}, \mathbf{v}', \mathbf{v})f(\mathbf{v})\Big)\, d\nu(\mathbf{v}') \ , \tag{1.2}$$

and the mean free path, ε, and the lattice constant, α, both tend to zero, subject to

$$\varepsilon \ll \alpha \ .$$

The cross section σ is 1-periodic with respect to the variable $\mathbf{y} = \mathbf{x}/\alpha$. Notice that the operator Q_α satisfies $\int_{\mathbf{v}} Q_\alpha(f)(\mathbf{v})\, d\nu(\mathbf{v}) = 0$. Thus, by integration of the equation (1.1) with respect to $\mathbf{x}$ and $\mathbf{v}$, we obtain that total mass is conserved, in the sense that the solution satisfies $\partial_t \int_{x,\mathbf{v}} f = 0$.

One expects that the diffusion limit is achieved relatively "faster" than the homogenization limit. And indeed, the simultaneous limiting equation we rigorously derive corresponds to first performing the diffusion limit (obtaining a drift-diffusion equation with oscillating parameters) and then averaging this equation by homogenization techniques. Such a convergence result has previously been proven by Sentis, [33] when $\alpha = \varepsilon^p$ for some constant $p < 1$, and provided the local equilibrium (the Maxwellian) does not depend on the "fast" variable $\mathbf{y}$. His proof relies on a formal expansion in powers of ε and α, which yields an estimate of the remainder when a sufficiently large number of terms is included (basically, the closer p is to 1, the more terms are needed in the expansion). The aim of our work is to remove these two hypotheses, thus allowing for instance $\alpha = \varepsilon|\log(\varepsilon)|$) and also permitting the local Maxwellian to depend on the periodic position variable. By removal of either of the hypotheses, the method proposed in [33] fails, and we need to find another route. It is convenient to introduce the small parameter $\eta = \dfrac{\varepsilon}{\alpha}$. With this notation our proof relies heavily on the expansion of solutions to the cell problem

$$\eta \mathbf{v} \cdot \nabla_{\mathbf{y}} f = Q(f)$$

with respect to the parameter η.

1.2. Notations, Hypotheses and Main Result. The position variable x lies in $I\!R^d$, where d is an integer, while the velocity variables $\mathbf{v}$ lies in a compact, symmetric set V of $I\!R^d$ equiped with a symmetric probability measure ν. We shall rely on the results of [21]; for that reason we shall throughout this paper also assume that the measure ν satisfies

- There exist constants $C, \gamma > 0$ such that $\nu(\{\mathbf{v} \in V,\ |\mathbf{v} \cdot \xi| \le h\}) \le Ch^\gamma, \quad$ for all $\xi \in S^{d-1},\ h > 0$.

This in particular implies that $\nu(\{\mathbf{v} \in V,\ |\mathbf{v}\cdot\xi| = 0\}) = 0$ for all $\xi \neq 0$, and so

$$\mathbf{v}\cdot\xi = 0 \quad \text{a.e. in } \mathbf{v} \text{ implies } \xi = 0 \ .$$

The time variable t lies in an arbitrarily large time interval $[0,T]$. The distribution function $f^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{v})$ is the unique weak solution to the equation

$$\partial_t f^{\varepsilon,\eta} + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}} f^{\varepsilon,\eta} = \frac{1}{\varepsilon^2} Q_{\varepsilon/\eta}(f^{\varepsilon,\eta}) \quad , \quad f^{\varepsilon,\eta}(0,\mathbf{x},\mathbf{v}) = f_{ini}(\mathbf{x},\mathbf{v}) \ , \tag{1.3}$$

where

$$Q_\alpha(f)(\mathbf{v}) = Q[\frac{\mathbf{x}}{\alpha}](f) \ , \quad \text{with} \quad Q[\mathbf{y}](f) = \int \left(\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')f(\mathbf{v}') - \sigma(\mathbf{y},\mathbf{v}',\mathbf{v})f(\mathbf{v})\right) d\nu(\mathbf{v}') \ . \tag{1.4}$$

Whenever one of the traditional normed function spaces is given a subscript *per* it signifies that the correponding elements are 1-periodic with respect to the variable $\mathbf{y}$; the norm is the one induced from the *unit cell* $Y = [-1/2, 1/2]^d$. The cross section σ is assumed to satisfy

- **(H0)** $\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')$ is in $L^\infty(V_{\mathbf{v}} \times V_{\mathbf{v}'}; C^\infty_{per}(I\!R^d_{\mathbf{y}}))$. It is bounded from below and above by positive constants and is 1-periodic with respect to the variable $\mathbf{y}$.
- **(H1)** σ has the following symmetries:

$$\sigma(-\mathbf{y},\mathbf{v},\mathbf{v}') = \sigma(\mathbf{y},\mathbf{v},\mathbf{v}'), \text{ and } \sigma(\mathbf{y},-\mathbf{v},-\mathbf{v}') = \sigma(\mathbf{y},\mathbf{v},\mathbf{v}').$$

- **(H2)** σ furthermore exhibits the detailed balance

$$\sigma(\mathbf{y},\mathbf{v},\mathbf{v}') = \tilde\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')M(\mathbf{y},\mathbf{v}) \quad \text{with} \quad M > 0 \quad \text{and} \quad \tilde\sigma(\mathbf{y},\mathbf{v},\mathbf{v}') = \tilde\sigma(\mathbf{y},\mathbf{v}',\mathbf{v}) \ .$$

The functions M and $\tilde\sigma$ lie in $L^\infty(V_{\mathbf{v}}; C^\infty(I\!R^d_{\mathbf{y}}))$ and $L^\infty(V_{\mathbf{v}} \times V_{\mathbf{v}'}; C^\infty(I\!R^d_{\mathbf{y}}))$, respectively, and they are 1-periodic with respect to the variable $\mathbf{y}$. We may without loss of generality suppose that $\int_V M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = 1$. The function M is referred to as the local Maxwellian; as we shall show later it is also the unique solution to

$$Q[\mathbf{y}](M) = 0, \quad \text{with} \quad \int_V M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = 1 \ .$$

THEOREM 1.1. *[31][16]. Let $f_{ini}(\mathbf{x},\mathbf{v})$ be a nonnegative function in $L^1(I\!R^d \times V)$. The equation (1.3) has a unique weak solution in $C^0(I\!R^+; L^1(I\!R^d \times V))$ satisfying:*

$$\int_{I\!R^d\times V} f_{ini}(\mathbf{x},\mathbf{v})\phi(0,\mathbf{x},\mathbf{v})\, d\mathbf{x}\, d\nu(\mathbf{v}) + \int_{I\!R^+}\int_{I\!R^d\times V} f^{\varepsilon,\eta}(\partial_t\phi + \frac{\mathbf{v}\cdot\nabla_{\mathbf{x}}\phi}{\varepsilon} + \frac{Q^*_{\varepsilon/\eta}(\phi)}{\varepsilon^2})\, d\mathbf{x} d\nu(\mathbf{v})\, dt = 0, \qquad \forall\phi\in\mathcal{T} \ . \tag{1.5}$$

The test functions (the elements of $\mathcal{T}$) are C^1 in the variables $(t,\mathbf{x})$, continuous in $\mathbf{v}$, and have compact support. The solution $f^{\varepsilon,\eta}$ is nonnegative and $\|f^{\varepsilon,\eta}(t,\cdot,\cdot)\|_{L^1} = \|f_{ini}\|_{L^1}$.

In our analysis we shall assume that the three parameters ε, η and ε/η (also referred to as α) all tend to zero. The main result established in this paper is the following convergence theorem.

THEOREM 1.2. *Assume $\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')$ satisfies* **(H0)**,**(H1)** *and* **(H2)** *above. Let $f^{\varepsilon,\eta}$ be the solution to (1.3) with nonnegative initial datum $f_{ini} \in L^1(I\!R^d, V)$. Let $\mathbf{X}^* = \mathbf{X}^*(\mathbf{y},\mathbf{v})$, be the unique solution to the cell problem*

$$Q^*[\mathbf{y}](\mathbf{X}^*) = -\mathbf{v} \ , \quad \int_V \mathbf{X}^* d\nu(\mathbf{v}) = 0 \ ,$$

where $Q^*[\mathbf{y}]$ *is the* $L^2(V)$ *adjoint of the operator* $Q[\mathbf{y}]$*. Let* M *be the local Maxwellian, and let* $\lambda = \lambda(\mathbf{y}, \mathbf{v})$ *be the unique solution to*

$$Q[\mathbf{y}](\lambda) = \mathbf{v} \cdot \nabla_{\mathbf{y}} M \ , \quad \int_V \lambda(\mathbf{y}, \mathbf{v}) d\nu(\mathbf{v}) = 0 \ .$$

There exists a positive, regular Borel measure $\mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\mathbf{v})$ *such that* $f^{\varepsilon,\eta}$ *two scale converges (at the scale 1 and* $\frac{\varepsilon}{\eta}$*) towards* $\mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\mathbf{v})$ *for any sequence* $\varepsilon \to 0$*,* $\eta \to 0$*, with* $\varepsilon/\eta \to 0$*.*[1] *The limiting measure has the form* $\mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\mathbf{v}) = N(dt, d\mathbf{x}, d\mathbf{y}) M(\mathbf{y}, \mathbf{v}) d\nu(\mathbf{v})$*, where the two-scale density* $N(t, \mathbf{x}, \mathbf{y})$ *decomposes as* $N(t, \mathbf{x}, \mathbf{y}) = \rho(\mathbf{y}) d\mathbf{y}\, n(dt, d\mathbf{x})$*. Here* $\rho(\mathbf{y})$ *is the positive 1-periodic function satisfying*

$$L(\rho(\mathbf{y})) = 0 \ , \quad \int_Y \rho(\mathbf{y}) = 1 \ ,$$

with

$$L(\rho(\mathbf{y})) = -div_{\mathbf{y}}(D(\mathbf{y}) \nabla_{\mathbf{y}} \rho(\mathbf{y})) + div_{\mathbf{y}}(U(\mathbf{y}) \rho(\mathbf{y})) \ ,$$

and

$$D(\mathbf{y}) = \int_V \mathbf{X}^*(\mathbf{y}, \mathbf{v}) \otimes \mathbf{v} M(\mathbf{y}, \mathbf{v}) d\nu(\mathbf{v}) \ , \quad U(\mathbf{y}) = \int_V \mathbf{v} \lambda(\mathbf{y}, \mathbf{v}) d\nu(\mathbf{v}) \ ,$$

while the macroscopic density $n(dt, d\mathbf{x})$ *is given by* $n(dt, d\mathbf{x}) = n(t, \mathbf{x}) dt\, d\mathbf{x}$*, where the function* $n(t, \mathbf{x})$ *is the solution to the diffusion initial value problem*

$$\partial_t n - div_{\mathbf{x}}(\mathbf{D} \nabla_{\mathbf{x}} n) = 0 \quad , \quad n(t=0, \mathbf{x}) = n_{ini}(\mathbf{x}) = \int_V f_{ini}(\mathbf{x}, \mathbf{v})\, d\nu(\mathbf{v}) \ . \tag{1.6}$$

The homogenized diffusion matrix is given by the formula

$$\mathbf{D} = \int_Y \rho(\mathbf{y}) D(\mathbf{y})\, d\mathbf{y} - \int_Y \mathbf{\Theta}^{[-1]}(\mathbf{y}) \otimes H(\mathbf{y})\, d\mathbf{y} \ , \tag{1.7}$$

where

$$H(\mathbf{y}) = div_{\mathbf{y}}(\rho(\mathbf{y}) D^\top(\mathbf{y})) + D(\mathbf{y}) \nabla_{\mathbf{y}} \rho - \rho(\mathbf{y}) U(\mathbf{y}) \ , \quad \mathbf{\Theta}^{[-1]}(\mathbf{y}) = L^{*-1}(\int M(\mathbf{y}, \mathbf{v})(\mathbf{v} \cdot \nabla_{\mathbf{y}})\, \mathbf{X}^*(\mathbf{y}, \mathbf{v}) d\mathbf{v})$$

and L^* *is the* (L^2-) *adjoint operator of* L *(acting on vectors).*

2. Heuristics and strategy of proof

In this section, we explain why the rigorous convergence result stated in the main theorem corresponds to performing the diffusion limit of the Boltzmann equation (while keeping the spatial length scale α fixed) and then homogenizing the so obtained drift-diffusion equation by letting α go to zero. We then explain the strategy of the proof which deals simultaneously with both limits.

[1] For the precise definition of this notion of convergence see section 4

2.1. Heuristics. We base our formal argument on a Hilbert expansion, first expanding f in powers of ε (for fixed α). In this context we use the notation $f^{\epsilon,\alpha}$ in place of $f^{\varepsilon,\eta}$ (remember, for fixed ε, there is a one-to-one correspondance between η and α, given by $\alpha = \varepsilon/\eta$).

$$f^{\varepsilon,\alpha}(t,\mathbf{x},\mathbf{v}) = f_0^\alpha(t,\mathbf{x},\mathbf{v}) + \varepsilon f_1^\alpha(t,\mathbf{x},\mathbf{v}) + \varepsilon^2 f_2^\alpha(t,\mathbf{x},\mathbf{v}) + \cdots \ .$$

Inserting this expansion in the Boltzmann equation and equating terms of same powers of ε leads to the three equations

$$Q_\alpha(f_0^\alpha) = 0\ , \qquad Q_\alpha(f_1^\alpha) = \mathbf{v}\cdot\nabla f_0^\alpha\ , \qquad Q_\alpha(f_2^\alpha) = \mathbf{v}\cdot\nabla f_1^\alpha + \partial_t f_0^\alpha\ .$$

The first equation implies that $f_0^\alpha(t,\mathbf{x},\mathbf{v}) = N^\alpha(t,\mathbf{x})M(\frac{\mathbf{x}}{\alpha},\mathbf{v})$ where $M(\mathbf{y},\cdot)$ is the Maxwellian profile, namely the unique solution of $Q[\mathbf{y}](M) = 0\ ,\ \int_V M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = 1 \quad \forall \mathbf{y}\in I\!R^d$. The formula for Q (ensuring mass conservation) implies that 1 belongs to the kernel of the adjoint Q^*. Furthermore, in Proposition 3.2, we show that 1 generates the kernel of the adjoint Q^*. The Fredholm alternative therefore implies that the equation for f_1^α has a solution if the Maxwellian profile satisfies the non drift-condition

$$\int_V \mathbf{v}M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = 0 \ \ \forall \mathbf{y}\in I\!R^d\ .$$

This condition is satisfied thanks to Hypothesis **(H1)**, which implies that M is even with respect to the velocity variable $\mathbf{v}$. Therefore

$$f_1^\alpha = \frac{1}{\alpha}N^\alpha(t,\mathbf{x})\lambda(\frac{\mathbf{x}}{\alpha},\mathbf{v}) - \mathbf{X}(\frac{\mathbf{x}}{\alpha},\mathbf{v})\cdot\nabla N^\alpha(t,\mathbf{x})$$

where λ is defined by $Q[\mathbf{y}](\lambda) = \mathbf{v}\cdot\nabla_{\mathbf{y}}M$, and $\mathbf{X}$ is defined by $Q[\mathbf{y}](\mathbf{X}) = -\mathbf{v}M$.

Finally, the Fredholm alternative asserts that the equation defining f_2^α can be solved if and only if

$$\int_V (\partial_t f_0^\alpha + \mathbf{v}\cdot\nabla_{\mathbf{x}} f_1^\alpha)\, d\nu(\mathbf{v}) = 0\ .$$

This last compatibility condition gives the equation that has to be satisfied by N^α, namely

$$\partial_t N^\alpha - \mathrm{div}(D(\frac{\mathbf{x}}{\alpha})\cdot\nabla N^\alpha) + \frac{1}{\alpha}\mathrm{div}(U(\frac{\mathbf{x}}{\alpha})N^\alpha) = 0\ , \tag{2.1}$$

where

$$D(\mathbf{y}) = \int_V \mathbf{v}\otimes\mathbf{X}(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = \int_V \mathbf{X}^*(\mathbf{y},\mathbf{v})\otimes\mathbf{v}M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})\ , \quad \text{and} \quad U(\mathbf{y}) = \int_V \mathbf{v}\lambda(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})\ . \tag{2.2}$$

The conclusion of this first step of the heuristics is that $f^\varepsilon(t,\mathbf{x},\mathbf{v}) \sim N^\alpha(t,\mathbf{x})M(\frac{\mathbf{x}}{\alpha},\mathbf{v}) + O(\varepsilon)$. But one has to be aware that $O(\varepsilon)$ is not necessarily uniform in α.

Now we continue the heuristics by passing to the limit $\alpha\to 0$ in the above drift-diffusion equation. Here we do not expect a simple Hilbert expansion as in the previous step, since the fast variable $\mathbf{y} = \frac{\mathbf{x}}{\alpha}$ must be taken into account. The usual approach consists in looking for a two-scale expansion involving both the space variable and the fast space variable

$$N^\alpha(t,\mathbf{x}) \sim N_0(t,\mathbf{x},\mathbf{y}) + \alpha N_1(t,\mathbf{x},\mathbf{y}) + \alpha^2 N_2(t,\mathbf{x},\mathbf{y})\ , \quad \mathbf{y} = \mathbf{x}/\alpha\ ,$$

N_i being 1-periodic in $\mathbf{y}$. Consequently, one can write $\nabla N^\alpha \sim \sum_{i=0}^{2} \alpha^i (\nabla_{\mathbf{x}} N_i + \frac{1}{\alpha}\nabla_{\mathbf{y}} N_i)$. Equating terms of the same power of α one is led to the following set of equations

$$L(N_0) = 0 \ , \tag{2.3}$$

$$L(N_1) = \mathrm{div}_{\mathbf{y}}(D(\mathbf{y})\nabla_{\mathbf{x}} N_0) + \mathrm{div}_{\mathbf{x}}(D(\mathbf{y})\nabla_{\mathbf{y}} N_0) - \mathrm{div}_{\mathbf{x}}(U(\mathbf{y})N_0) \ , \tag{2.4}$$

$$L(N_2) = -\partial_t N_0 + \mathrm{div}_{\mathbf{x}}(D(\mathbf{y})\nabla_{\mathbf{x}} N_0) + \mathrm{div}_{\mathbf{y}}(D(\mathbf{y})\nabla_{\mathbf{x}} N_1) + \mathrm{div}_{\mathbf{x}}(D(\mathbf{y})\nabla_{\mathbf{y}} N_1) - \mathrm{div}_{\mathbf{x}}(U(\mathbf{y})N_1) \ , \tag{2.5}$$

where the operator L is defined by

$$L(n) = -\mathrm{div}_{\mathbf{y}}(D(\mathbf{y})\nabla_{\mathbf{y}} n) + \mathrm{div}_{\mathbf{y}}(U(\mathbf{y})n) \quad \text{with periodic boundary conditions} \ . \tag{2.6}$$

As shall be proven in Proposition 3.1 (see the following section) the operator L has a one dimensional kernel spanned by a positive function ρ (that we normalize such that its integral over Y is equal to one). Equation (2.3) now implies that $N_0(t,\mathbf{x},\mathbf{y}) = n(t,\mathbf{x})\rho(\mathbf{y})$. Since L has divergence form (and a one dimensional kernel) its range is exactly the set of functions with zero average in $\mathbf{y}$. Therefore, (2.4) has a solution if and only if its right hand side has zero $\mathbf{y}$ average, *i.e.*,

$$\begin{aligned} 0 &= \int_Y \mathrm{div}_{\mathbf{x}}(D(\mathbf{y})\nabla_{\mathbf{y}} N_0) - \mathrm{div}_{\mathbf{x}}(U(\mathbf{y})N_0) \\ &= \left[\int_Y \int_V [\mathbf{X}^*(\mathbf{y},\mathbf{v}) \otimes \mathbf{v}]\, \nabla_{\mathbf{y}} \left[\rho(\mathbf{y}) M(\mathbf{y},\mathbf{v})\right] d\mathbf{y} d\nu(\mathbf{v})\right] \cdot \nabla_{\mathbf{x}} n(t,\mathbf{x}) \ . \end{aligned} \tag{2.7}$$

As will be shown later on, this non drift condition is true due to the symmetry hypothesis **(H1)**. Proceeding analogously we deduce that (2.5) has a solution if and only if

$$\int_Y [\partial_t N_0 - \mathrm{div}_{\mathbf{x}}(D(\mathbf{y})\nabla_{\mathbf{x}} N_0) - \mathrm{div}_{\mathbf{x}}(D(\mathbf{y})\nabla_{\mathbf{y}} N_1) + \mathrm{div}_{\mathbf{x}}(U(\mathbf{y})N_1)] = 0 \ ,$$

which can be rewritten

$$\partial_t n - \mathrm{div}_{\mathbf{x}}(\int_Y \rho(\mathbf{y}) D(\mathbf{y})\, d\mathbf{y} \nabla_{\mathbf{x}} n) + \mathrm{div}_{\mathbf{x}} \int_Y (\mathrm{div}_{\mathbf{y}} D(\mathbf{y}) + U(\mathbf{y})) N_1 \, d\mathbf{y} = 0 \ .$$

Here the divergence of a matrix is taken by row. Furthermore, since

$$L(N_1) = H(\mathbf{y}) \cdot \nabla_{\mathbf{x}} n, \quad \text{with} \quad H(\mathbf{y}) = \mathrm{div}_{\mathbf{y}}(\rho(\mathbf{y}) D^\top(\mathbf{y})) + D(\mathbf{y})\nabla_{\mathbf{y}}\rho(\mathbf{y}) - \rho(\mathbf{y})U(\mathbf{y}) \ ,$$

we can write $N_1 = L^{-1}(H(\mathbf{y})) \cdot \nabla_{\mathbf{x}} n$ (modulo the kernel of L), and so the solvability condition for N_2 can be rewritten

$$\partial_t n - \mathrm{div}_{\mathbf{x}}(\int_Y \rho(\mathbf{y}) D(\mathbf{y}) d\mathbf{y} \nabla_{\mathbf{x}} n) + \mathrm{div}_{\mathbf{x}}(\int_Y (\mathrm{div}_{\mathbf{y}} D(\mathbf{y}) + U(\mathbf{y}))(L^{-1} H(\mathbf{y}) \cdot \nabla_{\mathbf{x}} n)\, d\nu(\mathbf{v})\, d\mathbf{y}) = 0 \ .$$

Thanks to (2.7), this equation is independent of the (unkown) additive kernel element in the definition of L^{-1}. Now

$$\begin{aligned} \int_Y (\mathrm{div}_{\mathbf{y}} D(\mathbf{y}) + U(\mathbf{y}))(L^{-1} H(\mathbf{y}) \cdot \nabla_{\mathbf{x}} n)\, d\mathbf{y} &= \left[\int_Y (\mathrm{div}_{\mathbf{y}} D(\mathbf{y}) + U(\mathbf{y})) \otimes L^{-1} H(\mathbf{y})\, d\mathbf{y}\right] \nabla_{\mathbf{x}} n \\ &= \left[\int_Y L^{*-1}(\mathrm{div}_{\mathbf{y}} D(\mathbf{y}) + U(\mathbf{y})) \otimes H(\mathbf{y})\, d\mathbf{y}\right] \nabla_{\mathbf{x}} n \\ &= \left[\int_Y \boldsymbol{\Theta}^{[-1]}(\mathbf{y}) \otimes H(\mathbf{y})\, d\mathbf{y}\right] \nabla_{\mathbf{x}} n \ , \end{aligned}$$

where

$$\mathbf{\Theta}^{[-1]}(\mathbf{y}) = L^{*-1}(\mathrm{div}_{\mathbf{y}} D(\mathbf{y}) + U(\mathbf{y})) = L^{*-1} \int_V [(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^* M + (\mathbf{v}\cdot\nabla_{\mathbf{y}} M)\mathbf{X}^* + \mathbf{v}\lambda(\mathbf{y},\mathbf{v})]\, d\nu(\mathbf{v})$$
$$= L^{*-1} \int_V (\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^* M d\nu(\mathbf{v}) \ .$$

The diffusion equation for n can finally be written

$$\partial_t n - \mathrm{div}_{\mathbf{x}}(\mathbf{D}\nabla n) = 0 \ ,$$

where

$$\mathbf{D} = \int_Y \left(\rho(\mathbf{y})D(\mathbf{y}) - \mathbf{\Theta}^{[-1]}(\mathbf{y}) \otimes H(\mathbf{y})\right)\, d\mathbf{y} \ .$$

2.2. Strategy of Proof. The above calculations provide reasonable heuristics for the result contained in the main theorem of this paper. For a rigorous proof, however, decoupling the diffusion step from the homogenization step does not lead to satisfactory bounds, and we shall proceed in a different way. The heuristics for this different approach is based on the use of the parameter η: first one performs the limit as ϵ and α simultaneously tend to zero (for a fixed η) and then one takes the limit as η tends to zero. The first limit leads to a decomposition of the form

$$f^{\varepsilon,\eta} \sim n^\eta(t,x)F^\eta(\mathbf{y},\mathbf{v}) \ ,$$

where F^η is the solution to $\eta\mathbf{v}\cdot\nabla_{\mathbf{y}} F^\eta = Q(F^\eta)$, $\int_{Y\times V} F^\eta = 1$, and n^η satisfies

$$\partial_t n^\eta - \mathrm{div}_{\mathbf{x}} \mathbf{D}^\eta \nabla_{\mathbf{x}} n^\eta = 0 \quad \text{with} \quad \mathbf{D}^\eta = \int_V \int_Y \mathbf{X}^{\eta *}(\mathbf{y},\mathbf{v}) \otimes \mathbf{v} F^\eta(\mathbf{y},\mathbf{v}) d\mathbf{y} d\nu(\mathbf{v}) \ .$$

Here $\mathbf{X}^{\eta *}$ solves an equation similar to F^η, namely $-\eta\mathbf{v}\cdot\nabla_{\mathbf{y}} X^{\eta *} = Q^*(X^{\eta *}) + \mathbf{v}$. To arrive at the second limit one may use expansions of F^η and $X^{\eta *}$ in powers of η. A similar strategy was employed in [10] to analyze the diffusion limit in the presence of a very high magnetic field.

Our rigorous proof leading directly from the Boltzmann equation to the homogenized diffusion equation combines these two steps into one. First we perform a careful study of the cell problems for the equilibrium profile F^η and the auxiliary function $X^{\eta *}$, and derive their limiting expansions (in Section 3). These expansions are then used directly to find the two-scale limit (in Section 4).

3. Study of the cell equations

As in [6] and [21], two cell problems are involved in the proof. However, in our context these cell problems depend on the parameter $\eta = \frac{\varepsilon}{\alpha}$. In this section we give expansions of the corresponding solutions with respect to this parameter. More precisely, we define F^η and $\mathbf{X}^{\eta *}$ as the solutions to

$$T^\eta(F^\eta) = 0 \ , \qquad \int F^\eta \, d\mathbf{y}\, d\nu(\mathbf{v}) = 1 \ , \tag{3.1}$$

and

$$T^{\eta *}(\mathbf{X}^{\eta *}) = \mathbf{v} \ , \qquad \int \mathbf{X}^{\eta *} \, d\mathbf{y}\, d\nu(\mathbf{v}) = 0 \ , \tag{3.2}$$

where

$$T^\eta = \eta\mathbf{v}\cdot\nabla_{\mathbf{y}} - Q, \quad \text{on } \mathcal{D} \subset L^2_{per}(I\!R^d_{\mathbf{y}} \times V) \ , \tag{3.3}$$

and

$$T^{\eta *} = -\eta \mathbf{v} \cdot \nabla_{\mathbf{y}} - Q^* \tag{3.4}$$

is the formal adjoint with respect to the scalar product of $L^2_{per}(I\!R^d_{\mathbf{y}} \times V)$. For the definition of the domain of definition $\mathcal{D}$, see Proposition 3.4. Q^* is given by the formula

$$Q^*[\mathbf{y}](f)(\mathbf{v}) = \int_V \sigma(\mathbf{y}, \mathbf{v}', \mathbf{v})[f(\mathbf{v}') - f(\mathbf{v})]\, d\nu(\mathbf{v}') \; . \tag{3.5}$$

3.1. Preliminary results. We now list three propositions that will be useful for the proof of the main theorem.

PROPOSITION 3.1. *Let L be the unbounded operator on $L^2_{per}(I\!R^d_{\mathbf{y}})$ with domain $H^2_{per}(I\!R^d_{\mathbf{y}})$, defined by (2.6), and let L^* be its adjoint, defined by*

$$L^*(n) = -div_{\mathbf{y}}(D^\top(\mathbf{y})\nabla_{\mathbf{y}} n) - U(\mathbf{y}) \cdot \nabla_{\mathbf{y}} n \quad \textit{with periodic boundary conditions.} \tag{3.6}$$

The matrix-valued function D and the vector field U are given by (2.2). The following statements are true

1. $Im(L) = \{u \in L^2_{per}(I\!R^d_{\mathbf{y}}), \quad s.t. \quad \int_Y u\, d\mathbf{y} = 0\}$.
2. *There exists a unique, positive function $\rho(\mathbf{y}) \in H^2_{per}(I\!R^d_{\mathbf{y}})$ such that $L(\rho) = 0$ and $\int_Y \rho\, d\mathbf{y} = 1$.*
3. *For any function $v \in Im(L)$, there exists a unique solution to $L(u) = v$, $\int_Y u\, d\mathbf{y} = 0$ with $u \in H^2_{per}(I\!R^d_{\mathbf{y}})$. This solution will be denoted $u = L^{-1}(v)$, and L^{-1} will be referred to as the pseudo inverse of L.*
4. $Im(L^*) = \{u \in L^2_{per}(I\!R^d_{\mathbf{y}}), \quad s.t. \quad \int_Y u(\mathbf{y})\rho(\mathbf{y})\, d\mathbf{y} = 0\}$.
5. *The kernel of L^* is the set of constant functions (in the variable $\mathbf{y}$).*
6. *For any function $v^* \in Im(L^*)$, there exists a unique solution to $L^*(u^*) = v^*$, $\int_Y u^*(\mathbf{y})\, d\mathbf{y} = 0$ with $u^* \in H^2_{per}(I\!R^d_{\mathbf{y}})$. This solution will be denoted $u^* = L^{*-1}(v^*)$, and L^{*-1} will be referred to as the pseudo inverse of L^*.*
7. *$D(\mathbf{y})$, $U(\mathbf{y})$ and $\rho(\mathbf{y})$ have the following symmetry properties*
   - *The diffusion matrix $D(\mathbf{y})$ is even with respect to $\mathbf{y}$. The Flux $U(\mathbf{y})$ is odd with respect to $\mathbf{y}$. The equilibrium function $\rho(\mathbf{y})$ is even.*
   - *L^{-1} and L^{*-1} leave invariant the set of even as well as the set of odd functions in $\mathbf{y}$.*
8. *The diffusion matrix $D(\mathbf{y})$ is in $C^\infty_{per}(I\!R^d_{\mathbf{y}})$ and satisfies*

   $$D(y)\xi \cdot \xi = 0 \iff \xi = 0 \quad \textit{and furthermore} \quad D(y)\xi \cdot \xi \geq \beta |\xi|^2 \; , \tag{3.7}$$

   for some constant $\beta > 0$.
9. *There exists a constant C such that*

   $$\begin{cases} \|L^{*-1}v^*\|_{H^2_{per}} \leq C\|v^*\|_{L^2_{per}}, & \|L^{-1}v\|_{H^2_{per}} \leq C\|v\|_{L^2_{per}}, \\ \|L^{*-1}v^*\|_{L^2_{per}} \leq C\|v^*\|_{H^{-2}_{per}}, & \|L^{-1}v\|_{L^2_{per}} \leq C\|v\|_{H^{-2}_{per}} \; , \end{cases} \tag{3.8}$$

   for all $v \in Im(L)$ and $v^* \in Im(L^*)$.

Proof. We only give the proof of item 5 (thus also item 2) and item 8, leaving the others to the reader. Let c be positive and sufficiently large. The function 1 is a positive eigenvector for $(L^* + c)^{-1}$ (with corresponding eigenvalue c^{-1}). Furthermore, since $(L^*+c)^{-1}$ is a strongly positive operator (in the sense that it maps positive functions to positive functions) the Krein-Rutmann theorem [26] [16] asserts that a positive eigenvector can only be associated with the principal eigenvalue (the eigenvalue that equals the spectral radius). This theorem also asserts that the principal eigenvalue is simple, that it is at the same time the simple, principal eigenvalue for the adjoint ($[(L^*+c)^{-1}]^* = (L+c)^{-1}$), and that the corresponding eigenspace is generated by a positive eigenvector. It follows that c is the lowest eigenvalue for $L+c$ and L^*+c, and that the corresponding eigenspaces are simple and generated by positive eigenfunctions. Consequently 0 is the lowest eigenvalue for L and L^* and the eigenspaces are simple and generated by positive eigenfunctions.

The dissipation relation gives

$$\begin{aligned} D(\mathbf{y})\xi\cdot\xi &= \int_V \mathbf{X}^*(\mathbf{y},\mathbf{v})\cdot\xi\mathbf{v}\cdot\xi M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) \\ &= \frac{1}{2}\int_V\int_V \sigma(\mathbf{y},\mathbf{v},\mathbf{v}')M(\mathbf{y},\mathbf{v})|\mathbf{X}^*(\mathbf{y},\mathbf{v}')\cdot\xi-\mathbf{X}^*(\mathbf{y},\mathbf{v})\cdot\xi|^2 d\nu(\mathbf{v})d\nu(\mathbf{v}')\geq 0\ , \end{aligned}$$

see [21]. Equality implies that $\mathbf{X}^*(\mathbf{y},\cdot)\cdot\xi$ does not depend on $\mathbf{v}$, and therefore it belongs to the kernel of Q^*. Since $Q^*[\mathbf{y}](\mathbf{X}^*\cdot\xi)(\mathbf{v}) = -\mathbf{v}\cdot\xi$, we get $\mathbf{v}\cdot\xi=0$ for all $\mathbf{v}$, and so $\xi=0$. The coercivity estimate of (3.7) (with $\beta>0$, independent of $\mathbf{y}$) follows by continuity.

Proposition 3.2. *Assume hypotheses* **(H0)**, **(H1)** *and* **(H2)** *are satisfied, then*

1. *The operator $Q[\mathbf{y}]$ is a bounded operator on $L^2(V)$ with a uniformly bounded norm (when $\mathbf{y}$ varies in Y).*
2. *The kernel of $Q[\mathbf{y}]$ is a one dimensional space spanned by the positive function $M(\mathbf{y},\mathbf{v})$ (the local Maxwellian).*
3. *The range of $Q[\mathbf{y}]$ is the set of functions $f\in L^2(V)$ such that $\int f(\mathbf{v})\,d\nu(\mathbf{v})=0$.*
4. *Suppose $f=f(\mathbf{y},\mathbf{v})$ is a function in $L^p(V;H^k_{per}(I\!R^d_{\mathbf{y}}))$, with $p=2$ or $+\infty$ and k an integer, and suppose f has zero velocity average (pointwise in y). Then there exists a unique function u, denoted $Q[\mathbf{y}]^{-1}(f)$, satisfying*

$$Q[\mathbf{y}](u)=f,\qquad \int_V u(\mathbf{y},\mathbf{v})\,d\nu(\mathbf{v})=0\ ,\quad a.e.\ in\ Y\ .$$

Moreover u lies in $L^p(V;H^k_{per}(I\!R^d_{\mathbf{y}}))$, and we have the estimate

$$\|u\|_{L^p(V;H^k_{per})}\leq C_k\|f\|_{L^p(V;H^k_{per})}\ . \tag{3.9}$$

Analogously, the adjoint operator $Q^*[\mathbf{y}]$ (in $L^2(V)$) has the following properties

5. *The operator $Q^*[\mathbf{y}]$ is a bounded operator on $L^2(V)$ with a uniformly bounded norm (when $\mathbf{y}$ varies in Y).*
6. *The kernel of $Q^*[\mathbf{y}]$ is the set of constant functions (in $\mathbf{v}$).*
7. *The range of $Q^*[\mathbf{y}]$ is the set of functions $f\in L^2(V)$ such that $\int f(\mathbf{v})M(\mathbf{y},\mathbf{v})\,d\nu(\mathbf{v})=0$.*
8. *Suppose $f=f(\mathbf{y},\mathbf{v})$ is a function in $L^p(V;H^k_{per}(I\!R^d_{\mathbf{y}}))$, with $p=2$ or $+\infty$ and k an integer, and suppose $\int f(\mathbf{y},\mathbf{v})M(\mathbf{y},\mathbf{v})\,d\nu(\mathbf{v})=0$ a.e. in Y. Then there exists a unique function u^*, denoted $Q[\mathbf{y}]^{*-1}(f)$, satisfying*

$$Q^*[\mathbf{y}](u^*)=f\ ,\qquad \int_V u^*(\mathbf{y},\mathbf{v})\,d\nu(\mathbf{v})=0,\quad a.e.\ in\ Y\ .$$

Moreover u^* lies in $L^p(V;H^k_{per}(I\!R^d_{\mathbf{y}}))$, and we have the estimate

$$\|u^*\|_{L^p(V;H^k_{per})}\leq C_k\|f\|_{L^p(V;H^k_{per})}\ . \tag{3.10}$$

Finally, we have the following symmetry invariants

9. *The operators* $Q[\mathbf{y}]^{-1}$ *and* $Q[\mathbf{y}]^{*-1}$ *leave invariant the set of even functions, as well as the set of odd functions in* $\mathbf{v}$*.*

Proof. We start by proving items 2 and 6. The fact that M satisfies $Q[\mathbf{y}](M) = 0$ is a simple consequence of the assumption **(H2)**. The well known dissipation relation (with $Q[\mathbf{y}]$ abbreviated Q) states that

$$2Q(g)\frac{g}{f} = Q(\frac{g^2}{f}) + Q(f)(\frac{g}{f})^2 - \int_V \sigma(\mathbf{y},\mathbf{v},\mathbf{v}')f(\mathbf{v}')|\frac{g}{f}(\mathbf{v}') - \frac{g}{f}(\mathbf{v})|^2 d\nu(\mathbf{v}') \ ,$$

which, by insertion of $f = M$ and integration, leads to

$$\frac{1}{2}\int_{V\times V} \sigma(\mathbf{y},\mathbf{v},\mathbf{v}')M(\mathbf{y},\mathbf{v}')|\frac{g}{M}(\mathbf{y},\mathbf{v}') - \frac{g}{M}(\mathbf{y},\mathbf{v})|^2 d\nu(\mathbf{v})d\nu(\mathbf{v}') = -\int_V Q[\mathbf{y}](g)\frac{g}{M(\mathbf{y},\mathbf{v})}d\nu(\mathbf{v}) \ . \tag{3.11}$$

For any g with $Q[\mathbf{y}](g) = 0$ this implies that g/M does not depend on $\mathbf{v}$ and so $g = CM(\mathbf{y},\mathbf{v})$ a.e. in $\mathbf{v}$. This verifies 2. To prove 6, we already know that 1 belongs to the kernel of $Q^*[\mathbf{y}]$. If we insert $g = M\tilde{g}$ into (3.11) then we arrive at

$$\frac{1}{2}\int_{V\times V} \sigma(\mathbf{y},\mathbf{v},\mathbf{v}')M(\mathbf{y},\mathbf{v}')|\tilde{g}(\mathbf{v}') - \tilde{g}(\mathbf{v})|^2 d\nu(\mathbf{v})d\nu(\mathbf{v}') = -\int_V M(\mathbf{y},\mathbf{v})\tilde{g}Q^*[\mathbf{y}](\tilde{g})d\nu(\mathbf{v}) \ .$$

For any $\tilde{g}$ with $Q^*[\mathbf{y}](\tilde{g}) = 0$, this implies that $\tilde{g}$ does not depend on $\mathbf{v}$, and so item 6 is verified. The items 1,3, 5, 7 and 9 are quite elementary, and their proof is left to the reader. It thus remains to verify item 4 and 8. We shall only prove the estimate (3.9) and note that (3.10) follows by a similar argument. We first consider the case $p = 2$. The starting point is the bound $\|u(\mathbf{y},\cdot)\|_{L^2(V)} \le \|Q[\mathbf{y}]^{-1}\|\|f(\mathbf{y},\cdot)\|_{L^2(V)}$. Since $\|Q[\mathbf{y}]^{-1}\|$ is bounded uniformly in Y, we obtain (after taking the L^2 norm in $\mathbf{y}$) that

$$\|u\|_{L^2(V\times Y)} \le C\|f\|_{L^2(V\times Y)} \ .$$

To obtain estimates on the derivatives, we differentiate the equation with respect to an arbitrary coordinate y_i which yields $Q(\partial_{y_i}u) = -\partial_{y_i}Q(u) + \partial_{y_i}f$, where we denote by $\partial_{y_i}Q$ the operator in which σ is replaced by $\partial_{y_i}\sigma$. An application of the zeroth order inequality leads to

$$\begin{aligned}\|\partial_{y_i}u\|_{L^2(V\times Y)} \le C(\|\partial_{y_i}Q(u)\|_{L^2(V\times Y)} + \|\partial_{y_i}f\|_{L^2(V\times Y)}) &\le C'\|u\|_{L^2(V\times Y)} + C\|\partial_{y_i}f\|_{L^2(V\times Y)} \\ &\le C_1\|f\|_{L^2(V;H^1(Y))} \ .\end{aligned}$$

Differentiating successively with respect to $\mathbf{y}$ and proceeding by induction, we arrive at the inequality

$$\|u\|_{L^2(V;H^k(Y))} \le C_k\|f\|_{L^2(V;H^k(Y))} \ .$$

In order to prove the result in L^∞, we write

$$u(\mathbf{y},\mathbf{v}) = \frac{1}{\Sigma(\mathbf{y},\mathbf{v})}\left(-f(\mathbf{y},\mathbf{v}) + \int_V \sigma(\mathbf{y},\mathbf{v},\mathbf{v}')u(y,\mathbf{v}')d\nu(\mathbf{v}')\right) \ ,$$

where $\Sigma(\mathbf{y},\mathbf{v}) = \int \sigma(\mathbf{y},\mathbf{v}',\mathbf{v})\, d\nu(\mathbf{v}')$. As Σ is also bounded from below by a positive constant, we have the pointwise estimate

$$|u(\mathbf{y},\mathbf{v})| \le C|f(\mathbf{y},\mathbf{v})| + C\|u(\mathbf{y},\cdot)\|_{L^2(V)} \le C|f(\mathbf{y},\mathbf{v})| + C\|f(\mathbf{y},\cdot)\|_{L^2(V)} \ .$$

Hence

$$\|u(\cdot,\mathbf{v})\|_{L^2(Y)} \le C\|f(\cdot,\mathbf{v})\|_{L^2(Y)} + C\|f\|_{L^2(Y\times V)} \le C\|f(\cdot,\mathbf{v})\|_{L^2(Y)} + C\|f\|_{L^\infty(V;L^2(Y))} \ .$$

Taking supremum over all $\mathbf{v}$, we finally get

$$C\|u\|_{L^\infty(V;L^2(Y))} \le C_0\|f\|_{L^\infty(V;L^2(Y))} \ .$$

Applying this inequality to the equation $Q(\partial_{y_i} u) = -\partial_{y_i} Q(u) + \partial_{y_i} f$ we are led to the estimate

$$\|u\|_{L^\infty(V;H^1(Y))} \leq C_1 \|f\|_{L^\infty(V;H^1(Y))} \ .$$

As was the case in the L^2 case, the estimate for arbitrary Sobolev index k follows by induction. □

REMARK 3.3. *We note that M is bounded from below since*

$$M(\mathbf{y},\mathbf{v}) = \frac{1}{\Sigma(\mathbf{y},\mathbf{v})} \int \sigma(\mathbf{y},\mathbf{v},\mathbf{v}') M(\mathbf{y},\mathbf{v}') d\nu(\mathbf{v}') \geq c \int_V M(\mathbf{y},\mathbf{v}') d\nu(\mathbf{v}') = c \ .$$

Therefore $\frac{1}{M}$ belongs to $L^\infty(V; C^\infty_{per}(Y))$.

We now proceed to the analysis of the operators T^η and $T^{\eta *}$ defined by (3.3) and (3.4). Parts of the following proposition are taken from [21].

PROPOSITION 3.4. *The operator $T^\eta = \eta \mathbf{v} \cdot \nabla_{\mathbf{y}} - Q[\mathbf{y}](\cdot)$ is un unbounded operator on $L^2_{per}(I\!R^d_{\mathbf{y}} \times V)$ with domain*

$$\mathcal{D} = \{ u \in L^2_{per}(I\!R^d_{\mathbf{y}} \times V), \quad \textit{such that } \mathbf{v} \cdot \nabla_{\mathbf{y}} u \in L^2_{per}(I\!R^d_{\mathbf{y}} \times V) \ \} \ .$$

Let $T^{\eta^*} = -\eta \mathbf{v} \cdot \nabla_{\mathbf{y}} - Q^*[\mathbf{y}](\cdot)$ be the adjoint of T^η . Then

1. *The kernel of T^η is a one dimensional space spanned by a positive function $F^\eta(\mathbf{y},\mathbf{v})$, that we normalize by $\int_{V\times Y} F^\eta d\nu(\mathbf{v}) d\mathbf{y} = 1$. The function F^η satisfies*

   (3.12) $$F^\eta(-\mathbf{y},-\mathbf{v}) = F^\eta(\mathbf{y},\mathbf{v}), \quad \textit{a.e. in } \mathbf{y} \textit{ and } \mathbf{v} \ .$$

2. *The range of T^η is the set of functions $g \in L^2_{per}(I\!R^d_{\mathbf{y}} \times V)$ such that $\int_{V\times Y} g(\mathbf{y},\mathbf{v})\, d\nu(\mathbf{v}) d\mathbf{y} = 0$.*

3. *The adjoint $T^{\eta *}$ has the same domain $\mathcal{D}$. Its range is the set of functions g such that*

   $$\int_{V\times Y} F^\eta(\mathbf{y},\mathbf{v})\, g(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) d\mathbf{y} = 0 \ .$$

   Its kernel is the set of constant functions (in $\mathbf{y}$ and $\mathbf{v}$).

4. *For $g \in L^2_{per}(I\!R^d_{\mathbf{y}} \times V)$ satisfying $\int_{V\times Y} g(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) d\mathbf{y} = 0$ there exists a unique function $R^\eta \in \mathcal{D}$ such that*

   (3.13) $$T^\eta(R^\eta) = g \quad \textit{and} \quad \int_{V\times Y} R^\eta(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) d\mathbf{y} = 0 \ .$$

   We shall denote $R^\eta = (T^\eta)^{-1}(g)$. There exists $\eta_0 > 0$ such that R^η satisfies the following bound

   (3.14) $$||R^\eta||_{L^2(Y\times V)} \leq \frac{C}{\eta^2} ||g||_{L^2(Y\times V)} \ , \qquad 0 < \eta < \eta_0 \ .$$

   Moreover, the following symmetry implications hold true

   (3.15) $$\begin{cases} \textit{If } g(-\mathbf{y},-\mathbf{v}) = \ \ g(\mathbf{y},\mathbf{v}) & \textit{a.e.} \quad \text{then} \quad R^\eta(-\mathbf{y},-\mathbf{v}) = \ \ R^\eta(\mathbf{y},\mathbf{v}) \quad \textit{a.e.} \\ \textit{If } g(-\mathbf{y},-\mathbf{v}) = -g(\mathbf{y},\mathbf{v}) & \textit{a.e.} \quad \text{then} \quad R^\eta(-\mathbf{y},-\mathbf{v}) = -R^\eta(\mathbf{y},\mathbf{v}) \quad \textit{a.e.} \end{cases}$$

5. *For* $g^* \in L^2_{per}(I\!R^d_{\mathbf{y}} \times V)$ *satisfying* $\displaystyle\int_{V\times Y} F^\eta(\mathbf{y},\mathbf{v})g^*(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})d\mathbf{y} = 0$ *there exists a unique function* $R^{\eta *} \in \mathcal{D}$ *such that*

$$T^{\eta *}(R^{\eta *}) = g^* \quad \text{and} \quad \int_{V\times Y} R^{\eta *}(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})d\mathbf{y} = 0 \ . \tag{3.16}$$

We shall denote $R^{\eta *} = (T^{\eta *})^{-1}(g^*)$. *There exist* $\eta_0 > 0$ *such that* $R^{\eta *}$ *satisfies the following bound*

$$||R^{\eta *}||_{L^2(Y\times V)} \le \frac{C}{\eta^2}||g^*||_{L^2(Y\times V)} \ , \qquad 0 < \eta < \eta_0 \ . \tag{3.17}$$

Moreover, the following symmetry implications hold true

$$\begin{cases} \text{If } g^*(-\mathbf{y},-\mathbf{v}) = \ \ g^*(\mathbf{y},\mathbf{v}) & a.e. \quad \text{then} \quad R^{\eta *}(-\mathbf{y},-\mathbf{v}) = \ \ R^{\eta *}(\mathbf{y},\mathbf{v}) \quad a.e. \\ \text{If } g^*(-\mathbf{y},-\mathbf{v}) = -g^*(\mathbf{y},\mathbf{v}) & a.e. \quad \text{then} \quad R^{\eta *}(-\mathbf{y},-\mathbf{v}) = -R^{\eta *}(\mathbf{y},\mathbf{v}) \quad a.e. \end{cases} \tag{3.18}$$

Proof. Parts 1 thru 3 of this proposition are proven in [21] and require assumption **(A1)** on the probability measure $d\nu(v)$. Of the remaining two items we shall only prove 4. Suppose $\displaystyle\int_{V\times Y} g(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})d\mathbf{y} = 0$. Then (according to item 2) there exists a unique solution, R^η, to (3.13). We introduce $\tilde{R}^\eta = \eta^2 R^\eta$, $\gamma^\eta(\mathbf{y}) = \displaystyle\int_V \frac{\tilde{R}^\eta(\mathbf{y},\mathbf{v})}{M(\mathbf{y},\mathbf{v})}d\nu(\mathbf{v})$ and perform a first order Chapman-Enskog expansion

$$\tilde{R}^\eta(\mathbf{y},\mathbf{v}) = \gamma^\eta(\mathbf{y})M(\mathbf{y},\mathbf{v}) + r^\eta(\mathbf{y},\mathbf{v}) \quad \text{with} \quad \int_V \frac{r^\eta(\mathbf{y},\mathbf{v})}{M(\mathbf{y},\mathbf{v})}d\nu(\mathbf{v}) = 0 \ .$$

Following the same argument as in [23], we prove by contradiction that $||\tilde{R}^\eta||_{L^2(Y\times V)} \le C||g||_{L^2(Y\times V)}$ for η sufficiently close to zero. This verifies the bound 3.14 on $R^\eta = \eta^{-2}\tilde{R}^\eta$. To arrive at a contradiction suppose

$$||\tilde{R}^\eta||_{L^2(Y\times V)} = 1 \quad \text{and} \quad ||g^\eta||_{L^2(Y\times V)} \to 0 \ ,$$

for some sequence $\eta \to 0$. By equation (3.11), we obtain

$$||\frac{g}{M} - \int_V \frac{g}{M}d\nu(\mathbf{v})||^2_{L^2(V)} \le -\int_V Q(g)\frac{g}{M}d\nu(\mathbf{v}).$$

Therefore

$$\begin{aligned} ||\tilde{R}^\eta - \gamma^\eta M||^2_{L^2(Y\times V)} &\le ||\tfrac{\tilde{R}^\eta}{M} - \gamma^\eta||^2_{L^2(Y\times V)} \\ &\le -\int_Y\int_V Q(\tilde{R}^\eta)\frac{\tilde{R}^\eta}{M}dvdy \\ &= \int_Y\int_V (\eta^2 g^\eta - \eta v\cdot\nabla_y \tilde{R}^\eta)\frac{\tilde{R}^\eta}{M}dvdy \\ &\le C(\eta^2||\tilde{R}^\eta||_{L^2(Y\times V)}||g^\eta||_{L^2(Y\times V)} + \eta||\tilde{R}^\eta||^2_{L^2(Y\times V)}) \ . \end{aligned}$$

This shows that $||r^\eta||_{L^2(Y\times V)} \to 0$. Since $\tilde{R}^\eta = \gamma^\eta M(\mathbf{y},\mathbf{v}) + r^\eta$, we have

$$\eta\mathbf{v}\cdot\nabla_{\mathbf{y}}(\gamma^\eta M(\mathbf{y},\mathbf{v})) + \eta\mathbf{v}\cdot\nabla_{\mathbf{y}}(r^\eta) = Q(\gamma^\eta M(\mathbf{y},\mathbf{v})) + Q(r^\eta) + \eta^2 g^\eta \ ,$$

and since $Q(M) = 0$, the equation satisfied by the remainder is

$$\eta \mathbf{v} \cdot \nabla_{\mathbf{y}}(r^\eta) - Q(r^\eta) = \eta^2 g^\eta - \eta \mathbf{v} \cdot \nabla_{\mathbf{y}}(\gamma^\eta(\mathbf{y}) M(\mathbf{y}, \mathbf{v})) \ .$$

Integration of this equation with respect to $\mathbf{v}$ leads to

$$\mathrm{div}_{\mathbf{y}}\Big(\int_V (\mathbf{v} r^\eta) d\nu(\mathbf{v})\Big) = \eta \int_V g^\eta d\nu(\mathbf{v}) \ . \tag{3.19}$$

For this identity we use the facts that $\int_V Q(r^\eta)\, d\nu(\mathbf{v}) = 0$ and $\int_V \mathbf{v} M(\mathbf{y}, \mathbf{v})\, d\nu(\mathbf{v}) = 0$. Since

$$\begin{aligned} \mathrm{div}_{\mathbf{y}}\Big(\int_V (\mathbf{v} r^\eta) d\nu(\mathbf{v})\Big) &= -\mathrm{div}_{\mathbf{y}}\Big(\int_V \mathbf{X}^* Q(r^\eta) d\nu(\mathbf{v})\Big) \\ &= -\eta \mathrm{div}_{\mathbf{y}}\Big(\int_V \mathbf{X}^* \mathbf{v} \cdot \nabla_{\mathbf{y}}(\gamma^\eta M) d\nu(\mathbf{v})\Big) - \eta \mathrm{div}_{\mathbf{y}}\Big(\int_V \mathbf{X}^* \mathbf{v} \cdot \nabla_{\mathbf{y}}(r^\eta) d\nu(\mathbf{v})\Big) \\ &\quad + \eta^2 \mathrm{div}_{\mathbf{y}}\Big(\textstyle\int_V g^\eta \mathbf{X}^* d\nu(\mathbf{v})\Big) \ , \end{aligned}$$

we deduce from (3.19) that γ^η satisfies the elliptic equation

$$L(\gamma^\eta) = \int_V g^\eta d\nu(\mathbf{v}) + \mathrm{div}_{\mathbf{y}}\Big(\int_V \mathbf{X}^* \mathbf{v} \cdot \nabla_{\mathbf{y}}(r^\eta) d\nu(\mathbf{v})\Big) - \eta \mathrm{div}_{\mathbf{y}}\Big(\int_V g^\eta \mathbf{X}^* d\nu(\mathbf{v})\Big) \ , \tag{3.20}$$

where L is defined in (2.6). By elliptic regularity this yields the convergence of the sequence (γ^η) in $L^2(Y)$. Indeed, $\gamma^\eta = \gamma_1^\eta + \gamma_2^\eta + C^\eta \rho$ where ρ satisfies

$$L(\rho) = 0 \ , \qquad \int_Y \rho(\mathbf{y}) d\mathbf{y} = 1 \ ,$$

and γ_1^η and γ_2^η satisfy

$$L(\gamma_1^\eta) = \int_V g^\eta d\nu(\mathbf{v}), \qquad \gamma_1^\eta \in (span\{1\})^\perp \ ,$$

$$L(\gamma_2^\eta) = \mathrm{div}_{\mathbf{y}}\Big(\int_V \mathbf{X}^* \mathbf{v} \cdot \nabla_{\mathbf{y}}(r^\eta) d\nu(\mathbf{v})\Big) - \eta \mathrm{div}_{\mathbf{y}}\Big(\int_V g^\eta \mathbf{X}^* d\nu(\mathbf{v})\Big) \ , \qquad \gamma_2^\eta \in (span\{1\})^\perp \ .$$

Since $||g^\eta||_{L^2(Y\times V)} \to 0$, $||\nabla_{\mathbf{y}}(\int_V \mathbf{X}^* \mathbf{v} \cdot \nabla_{\mathbf{y}}(r^\eta) d\nu(\mathbf{v}))||_{H^{-2}_{per}} \to 0$ and $||\eta \mathrm{div}_{\mathbf{y}}(\int_V g^\eta \mathbf{X}^* d\nu(\mathbf{v}))||_{H^{-1}_{per}} \to 0$ we conclude that $\gamma_1^\eta + \gamma_2^\eta$ converges strongly to 0 in $L^2_{per}(I\!R^d_{\mathbf{y}})$, as $\eta \to 0$. Thus

$$C^\eta - \int_Y \gamma^\eta d\mathbf{y} = - \int_Y (\gamma_1^\eta + \gamma_2^\eta)\, d\mathbf{y} \to 0 \ .$$

At the same time

$$\int_Y \gamma^\eta d\mathbf{y} = \int_Y \int_V \gamma^\eta M(\mathbf{y}, \mathbf{v}) d\mathbf{y} d\nu(\mathbf{v}) = - \int_V \int_Y r^\eta(\mathbf{y}, \mathbf{v}) d\mathbf{y} d\nu(\mathbf{v}) \to 0 \ ,$$

and so $C^\eta \to 0$. In summary: (γ^η) converges strongly to zero in $L^2_{per}(I\!R^d_{\mathbf{y}})$. The formula $\tilde{R}^\eta = \gamma^\eta M(\mathbf{y}, \mathbf{v}) + r^\eta$ implies that $\tilde{R}^\eta$ converges strongly to zero in $L^2_{per}(I\!R^d_{\mathbf{y}} \times V)$. This is a direct contradiction to the fact that $||\tilde{R}^\eta||_{L^2(Y\times V)} = 1$. □

3.2. Expansion of the two-scale equilibrium. Note that the equation (3.1) is in fact a steady state Boltzmann equation on the unit periodic cell, with η representing a "cell mean free path". It is therefore not surprising that, in the limit $\eta \to 0$, we have a steady state diffusion approximation, as asserted by the following proposition.

PROPOSITION 3.5. *Assume Hypotheses* **(H0)**, **(H1)** *and* **(H2)** *hold true. Let M be the local Maxwellian, and let $\mathbf{X}$ and λ be uniquely defined by*

$$Q[\mathbf{y}](\mathbf{X}) = -\mathbf{v}M \quad \text{and} \quad Q[\mathbf{y}](\lambda) = \mathbf{v}\cdot\nabla_{\mathbf{y}} M \ , \qquad \int_V \mathbf{X}(\mathbf{y},\mathbf{v})\, d\nu(\mathbf{v}) = \int_V \lambda(\mathbf{y},\mathbf{v})\, d\nu(\mathbf{v}) = 0 \ . \tag{3.21}$$

The solution, F^η, to (3.1) can be writen

$$F^\eta(\mathbf{y},\mathbf{v}) = \rho(\mathbf{y})M(\mathbf{y},\mathbf{v}) + \eta\left(-\mathbf{X}(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{y}}\rho(\mathbf{y}) + \lambda(\mathbf{y},\mathbf{v})\rho(\mathbf{y})\right) + \eta^2 R^\eta(\mathbf{y},\mathbf{v}) \ ,$$

where the remainder term satisfies $||R^\eta||_{L^2(Y\times V)} \le C$, uniformly in η , and ρ is the unique 1-periodic solution to

$$-div_{\mathbf{y}}\left(D(\mathbf{y})\nabla_{\mathbf{y}}\rho\right) + div_{\mathbf{y}}\left(U(\mathbf{y})\rho\right) = 0 \quad \text{with} \quad \int_Y \rho(\mathbf{y})d\mathbf{y} = 1 \ .$$

Here

$$D(\mathbf{y}) = \int_V \mathbf{v}\otimes\mathbf{X}(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = \int_V \mathbf{X}^*(\mathbf{y},\mathbf{v})\otimes\mathbf{v}M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) \ ,$$
$$U(\mathbf{y}) = \int_V \mathbf{v}\lambda(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = -\int_V (\mathbf{v}\cdot\nabla_{\mathbf{y}}M)\mathbf{X}^*\, d\nu(\mathbf{v}) \ .$$

Proof. The first step of the proof consists in writing a finite expansion of F^η in powers of η. Namely, we write

$$\begin{aligned} F^\eta(\mathbf{y},\mathbf{v}) &= F_0(\mathbf{y},\mathbf{v}) + \eta F_1(\mathbf{y},\mathbf{v}) + \eta^2 F_2(\mathbf{y},\mathbf{v}) + \eta^3 F_3(\mathbf{y},\mathbf{v}) + \tilde{R}^\eta(\mathbf{y},\mathbf{v}) \\ &= F_0(\mathbf{y},\mathbf{v}) + \eta F_1(\mathbf{y},\mathbf{v}) + \eta^2 R^\eta \ , \end{aligned} \tag{3.22}$$

with $R^\eta = F_2(\mathbf{y},\mathbf{v}) + \eta F_3(\mathbf{y},\mathbf{v}) + \eta^{-2}\tilde{R}^\eta(\mathbf{y},\mathbf{v})$. Insertion of this expansion in (3.1), and a match of powers of η leads to the following equations

$$\begin{aligned} Q[\mathbf{y}](F_0) &= 0 \ , \quad \textstyle\int_{V\times Y} F_0 d\nu(\mathbf{v})d\mathbf{y} = 1 \ , \\ Q[\mathbf{y}](F_1) &= \mathbf{v}\cdot\nabla_{\mathbf{y}}F_0 \ , \quad \textstyle\int_{V\times Y} F_1 d\nu(\mathbf{v})d\mathbf{y} = 0 \ , \\ Q[\mathbf{y}](F_2) &= \mathbf{v}\cdot\nabla_{\mathbf{y}}F_1 \ , \quad \textstyle\int_{V\times Y} F_2 d\nu(\mathbf{v})d\mathbf{y} = 0 \ , \\ Q[\mathbf{y}](F_3) &= \mathbf{v}\cdot\nabla_{\mathbf{y}}F_2 \ , \quad \textstyle\int_{V\times Y} F_3 d\nu(\mathbf{v})d\mathbf{y} = 0 \ . \end{aligned}$$

We may satisfy the last three integral constraints by requiring that $\int_V F_i d\nu(\mathbf{v}) = 0$ a.e. in $\mathbf{y}$, for $1 \le i \le 3$. The first set of equations leads to $F_0 = \rho(\mathbf{y})M(\mathbf{y},\mathbf{v})$, where $M(\mathbf{y},\mathbf{v})$ is the Maxwellian. For the second set of equations to be solvable, the right hand side needs to be in the range of the operator Q, which means

$$\int_V \mathbf{v}\cdot\nabla_{\mathbf{y}}F_0 d\nu(\mathbf{v}) = 0 \ .$$

This is satisfied since M is even with repect to $\mathbf{v}$. For the third set of equations to have a solution, we need

$$\int_V \mathbf{v}\cdot\nabla_{\mathbf{y}}F_1 = 0 \ .$$

This will provide us (as is usual for diffusion limits) a diffusion equation for the function $\rho(\mathbf{y})$. Indeed, if $\mathbf{X}(\mathbf{y},\mathbf{v})$ and $\lambda(\mathbf{y},\mathbf{v})$ are as defined by (3.21), then we calculate $F_1(\mathbf{y},\mathbf{v}) = -\mathbf{X}(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{y}}\rho(\mathbf{y}) + \lambda(\mathbf{y},\mathbf{v})\rho(\mathbf{y})$. The equation $\int_V \mathbf{v}\cdot\nabla_{\mathbf{y}} F_1 = 0$ (together with $\int_{V\times Y} F_0 d\nu(\mathbf{v})d\mathbf{y} = 1$) now becomes

$$-\mathrm{div}_{\mathbf{y}}\,(D(\mathbf{y})\nabla_{\mathbf{y}}\rho) + \mathrm{div}_{\mathbf{y}}\,(U(\mathbf{y})\rho) = 0 \quad \text{with} \quad \int_Y \rho(\mathbf{y})d\mathbf{y} = 1\ , \tag{3.23}$$

a steady state diffusion equation for ρ. By Proposition 3.2, $Q[\mathbf{y}]^{-1}$ leaves invarient the set of even, as well as the set of odd functions in $\mathbf{v}$. Since we already know that F_1 is odd it follows that F_2 is even. Therefore the compatiblity condition $\int_v \mathbf{v}\cdot\nabla_{\mathbf{y}} F_2 d\nu(\mathbf{v}) = 0$ holds and this leads to the existence of F_3.

The term remainder term $\tilde{R}^\eta$ will now satisfy

$$\eta\mathbf{v}\cdot\nabla_{\mathbf{y}}\tilde{R}^\eta - Q[\mathbf{y}](\tilde{R}^\eta) = -\eta^4\mathbf{v}\cdot\nabla_{\mathbf{y}} F_3\ , \quad \text{with} \int_{V\times Y}\tilde{R}^\eta d\nu(\mathbf{v})d\mathbf{y} = 0\ . \tag{3.24}$$

Note that this equation can be solved since $\displaystyle\int_Y\int_V \mathbf{v}\cdot\nabla_{\mathbf{y}} F_3 d\mathbf{y}d\nu(\mathbf{v}) = 0$.

The final step of the proof consists in providing an estimate for $||R^\eta||_{L^2(Y\times V)}$ as η goes to zero. To this end, we observe that $||\mathbf{v}\cdot\nabla_{\mathbf{y}} F_3||_{L^2(Y\times V)} \le C$ due to the smoothness of the coefficients of $Q[\mathbf{y}]$ (see Proposition 3.2). By Proposition 3.4 $||\tilde{R}^\eta||_{L^2(Y\times V)} \le C\eta^2||\mathbf{v}\cdot\nabla_{\mathbf{y}} F_3|||_{L^2(Y\times V)}$, and so

$$||R^\eta||_{L^2(Y\times V)} \le C||F_2||_{L^2(Y\times V)} + \eta||F_3||_{L^2(Y\times V)} + \frac{1}{\eta^2}||\tilde{R}^\eta||_{L^2(Y\times V)} \le C\ .$$

This concludes the proof of Proposition 3.5. ☐

3.3. Expansion of the auxiliary function $\mathbf{X}^{\eta *}$. As with F^η, we need to expand $\mathbf{X}^{\eta *}$ (the solution to (3.2)) with respect to η. This will be substantially more technical and will involve more compatibility conditions. The largest term in the expansion turns out to be of the order $1/\eta$. Therefore, we write

$$\mathbf{X}^{\eta *} = \sum_{i=-1}^{+\infty}\eta^i\mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v})\ .$$

The above series does not converge, but truncating it at an arbitrary order (say I) we obtain a remainder of the subsequent order (η^{I+1}). The final choice of the index I will be guided by a Sobolev Imbedding Lemma, and chosen larger for larger dimensions. The equations that need to be satisfied by the successive terms are

$$\begin{cases} Q^*(\mathbf{X}^*_{[-1]}) &=& 0,\\ Q^*(\mathbf{X}^*_{[0]}) &=& -\mathbf{v} - (\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[-1]},\\ Q^*(\mathbf{X}^*_{[i]}) &=& -(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i-1]}, \quad i = 1,2,\cdots\ , \end{cases} \tag{3.25}$$

with Q^* being shorthand for $Q^*[\mathbf{y}]$. Furthermore $\int X^*_{[i]} d\mathbf{y}d\nu(\mathbf{v}) = 0\ ,\ \ i = -1, 0, \dots$.

1. Step $[-1]$ *Determination of* $X^*_{[-1]}$.
Since the kernel of Q^* is the set of constant functions in $\mathbf{v}$, the first equation of (3.25) implies that

$$\mathbf{X}^*_{[-1]} = \mathbf{\Theta}^{[-1]}(\mathbf{y})\ ,$$

where $\mathbf{\Theta}^{[-1]}$ is a vector valued function to be defined. The second equation defines $\mathbf{X}^*_{[0]}$ if the right hand side is in the range of Q^*, which means that it should be orthogonal to M in L^2_V. In other words

$$-\int_V (\mathbf{v} + (\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}))M(\mathbf{y},\mathbf{v})\,d\mathbf{v} = 0\ .$$

This identity is satisfied since M is even with respect to $\mathbf{v}$. We therefore have

$$\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v}) = Q^{*-1}(-\mathbf{v} - (\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y})) + \mathbf{\Theta}^{[0]}(\mathbf{y}) \ ,$$

where $\mathbf{\Theta}^{[0]}(\mathbf{y})$ belongs to the kernel of Q^*, and will be precisely defined later. We introduce $\mathbf{X}^* = -Q^{*-1}(\mathbf{v})$, in other words

$$Q^*(\mathbf{X}^*) = -\mathbf{v} \ , \quad \int_V \mathbf{X}^* d\nu(\mathbf{v}) = 0 \ .$$

With this definition

$$\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v}) = \mathbf{X}^* + (\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{\Theta}^{[0]}(\mathbf{y}) \ ,$$

where the functions $\mathbf{\Theta}^{[-1]}$ and $\mathbf{\Theta}^{[0]}$ still need to be determined. We also define

$$\overline{\mathbf{X}}^*_{[0]} = \mathbf{X}^* + (\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}) \ , \tag{3.26}$$

so that

$$\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v}) = \overline{\mathbf{X}}^*_{[0]}(\mathbf{y},\mathbf{v}) + \mathbf{\Theta}^{[0]}(\mathbf{y}) \ . \tag{3.27}$$

We proceed to consider $\mathbf{X}^*_{[1]}$. Since $Q^*(\mathbf{X}^*_{[1]}) = -(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[0]}$, a necessary and sufficient condition for the existence of $\mathbf{X}^*_{[1]}$ is that $\int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v})\,d\mathbf{v} = 0$, which is equivalent to $\int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\overline{\mathbf{X}}^*_{[0]}(\mathbf{y},\mathbf{v})\,d\mathbf{v} = 0$. Replacing $\overline{\mathbf{X}}^*_{[0]}$ by the right hand side of (3.26), we obtain the following elliptic equation for $\mathbf{\Theta}^{[-1]}(\mathbf{y}))$:

$$L^*(\mathbf{\Theta}^{[-1]}) = S^*_{[-1]} \ , \quad \int \mathbf{\Theta}^{[-1]}(\mathbf{y}) \ d\mathbf{y} = 0 \ . \tag{3.28}$$

Here L^* is the operator $-\mathrm{d}iv_{\mathbf{y}}(D^\top(y)\nabla_{\mathbf{y}}) - (U(y)\cdot\nabla_{\mathbf{y}})$, defined in (3.6) (with periodic boundary conditions), and

$$S^*_{[-1]}(\mathbf{y}) = \int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\,\mathbf{X}^*(\mathbf{y},\mathbf{v})d\mathbf{v} \ . \tag{3.29}$$

Proposition 3.1 item 4 implies the existence and uniqueness of $\mathbf{\Theta}^{[-1]} = L^{*-1}(S^*_{[-1]})$. Indeed, $\mathbf{X}^*$ is even with respect to $\mathbf{y}$ and odd with respect to $\mathbf{v}$. Therefore $S^*_{[-1]}(\mathbf{y})$ is odd, which implies that $\int_Y S^*_{[-1]}(\mathbf{y})\rho(\mathbf{y})\,d\mathbf{y} = 0$, the function ρ being even. As a matter of fact, the function $\mathbf{\Theta}^{[-1]}$ is odd with respect to $\mathbf{y}$. To summarize the information collected at this stage

$$\left\{\begin{array}{lcll}
\mathbf{\Theta}^{[-1]}(\mathbf{y}) & = & L^{*-1}(S^*_{[-1]}) \ , & S^*_{[-1]}(\mathbf{y}) = \int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\,\mathbf{X}^*(\mathbf{y},\mathbf{v})d\mathbf{v} \ , \\
\mathbf{X}^*_{[0]} & = & \overline{\mathbf{X}}^*_{[0]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[0]}(\mathbf{y}) \ , & \overline{\mathbf{X}}^*_{[0]} = \mathbf{X}^* + (\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}) \ , \\
\mathbf{X}^*_{[1]} & = & \overline{\mathbf{X}}^*_{[1]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[1]}(\mathbf{y}) \ , & \overline{\mathbf{X}}^*_{[1]}(\mathbf{y},\mathbf{v}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[0]}) \ , \\
S^*_{[-1]}(-\mathbf{y}) & = & -S^*_{[-1]}(\mathbf{y}) \ , & \overline{\mathbf{X}}^*_{[0]}(-\mathbf{y},\mathbf{v}) = \overline{\mathbf{X}}^*_{[0]}(\mathbf{y},\mathbf{v}), \quad \overline{\mathbf{X}}^*_{[0]}(\mathbf{y},-\mathbf{v}) = -\overline{\mathbf{X}}^*_{[0]}(\mathbf{y},\mathbf{v}) \ .
\end{array}\right.$$

The functions $\mathbf{\Theta}^{[0]}$ and $\mathbf{\Theta}^{[1]}$ are not determined yet. The equation that determines $\mathbf{\Theta}^{[-1]}(\mathbf{y})$ is the compatibity condition that ensures the existence of $\overline{\mathbf{X}}^*_{[1]}(\mathbf{y},\mathbf{v})$.

2. Step [i] : Determination of $\mathbf{X}^*_{[i]}$, $i \geq 0$.
We now describe how to iteratively determine the functions $\mathbf{X}^*_{[i]}$. Suppose at the end of step $[i-1]$ we already

know that

$$
\begin{cases}
\mathbf{X}^*_{[i-1]}(\mathbf{y},\mathbf{v}) = \overline{\mathbf{X}}^*_{[i-1]}(\mathbf{y},\mathbf{v}) + \mathbf{\Theta}^{[i-1]}(\mathbf{y}) , & \mathbf{\Theta}^{[i-1]}(\mathbf{y}) = L^{*-1}(S^*_{[i-1]}) \text{ is odd w.r.t. } \mathbf{y} , \\
\mathbf{X}^*_{[i]} = \overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[i]}(\mathbf{y}) , & \overline{\mathbf{X}}^*_{[i]} \text{ is odd w.r.t. } (\mathbf{y},\mathbf{v}) \text{ , i.e. } , \mathbf{X}^*_{[i]}(-\mathbf{y},-\mathbf{v}) = -\mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) , \\
\mathbf{X}^*_{[i+1]} = \overline{\mathbf{X}}^*_{[i+1]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[i+1]}(\mathbf{y}) , & \overline{\mathbf{X}}^*_{[i+1]}(\mathbf{y},\mathbf{v}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i]}) .
\end{cases}
\tag{3.30}
$$

For $i=0$ this is satisfied with $\overline{\mathbf{X}}^*_{[-1]} = 0$, $S^*_{[-1]}(\mathbf{y}) = \int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\,\mathbf{X}^*(\mathbf{y},\mathbf{v})d\mathbf{v}$ and $\overline{\mathbf{X}}^*_{[0]} = X^* + (X^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y})$ according to step [-1]. At this stage $\mathbf{\Theta}^{[i-1]}$, $S^*_{[i-1]}$, $\overline{\mathbf{X}}^*_{[i-1]}$ and $\overline{\mathbf{X}}^*_{[i]}$ are known, but $\mathbf{\Theta}^{[i]}$ and $\mathbf{\Theta}^{[i+1]}$ are not yet determined. We advance one step and look for $\mathbf{X}^*_{[i+2]}$, a solution to

$$Q^*(\mathbf{X}^*_{[i+2]}) = -(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i+1]} .$$

This equation has a solution if and only if $\int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i+1]}\,d\nu(\mathbf{v})$ vanishes, which is equivalent to

$$\int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\overline{\mathbf{X}}^*_{[i+1]}\,d\nu(\mathbf{v}) = 0 . \tag{3.31}$$

As a consequence

$$\mathbf{X}^*_{[i+2]} = \overline{\mathbf{X}}^*_{[i+2]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[i+2]}(\mathbf{y}) , \qquad \overline{\mathbf{X}}^*_{[i+2]}(\mathbf{y},\mathbf{v}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i+1]}) ,$$

where $\mathbf{\Theta}^{[i+2]}(\mathbf{y})$ is to be determined. Using the formula

$$\overline{\mathbf{X}}^*_{[i+1]}(\mathbf{y},\mathbf{v}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i]}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})(\overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v}) + \mathbf{\Theta}^{[i]}(\mathbf{y}))) ,$$

the constraint (3.31) becomes

$$L^*(\mathbf{\Theta}^{[i]}) = S^*_{[i]} , \quad S^*_{[i]}(\mathbf{y}) = -\int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})(\overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v})))\,d\nu(\mathbf{v}) .$$

For $\mathbf{\Theta}^{[i]}$ to exist, one needs that $\int S^*_{[i]}(\mathbf{y})\rho(\mathbf{y})\,d\mathbf{y} = 0$. Since $\overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v})$ is odd with respect to $(\mathbf{y},\mathbf{v})$, and since this property is preserved under the action of the operators $\mathbf{v}\cdot\nabla_{\mathbf{y}}$ and Q^{*-1}, the integrated function $S^*_{[i]}(\mathbf{y})$ is odd with respect to $\mathbf{y}$. As $\rho(\mathbf{y})$ is even, we deduce that $\int S^*_{[i]}(\mathbf{y})\rho(\mathbf{y})\,d\mathbf{y} = 0$. In summary

$$
\begin{cases}
S^*_{[i]}(\mathbf{y}) = -\int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})(\overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v}))\,d\nu(\mathbf{v}) , & \\
\mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) = \overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v}) + \mathbf{\Theta}^{[i]}(\mathbf{y}) , & \mathbf{\Theta}^{[i]}(\mathbf{y}) = L^{*-1}(S^*_{[i]}) \text{ is odd w.r.t. } \mathbf{y} , \\
\mathbf{X}^*_{[i+1]} = \overline{\mathbf{X}}^*_{[i+1]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[i+1]}(\mathbf{y}) , & \overline{\mathbf{X}}^*_{[i+1]} = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i]}) , \\
\mathbf{X}^*_{[i+2]} = \overline{\mathbf{X}}^*_{[i+2]}(\mathbf{y},\mathbf{v})) + \mathbf{\Theta}^{[i+2]}(\mathbf{y}) , & \overline{\mathbf{X}}^*_{[i+2]}(\mathbf{y},\mathbf{v}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i+1]}) .
\end{cases}
\tag{3.32}
$$

The function $\overline{\mathbf{X}}^*_{[i+1]}$ is an odd function with respect to $(\mathbf{y},\mathbf{v})$. To see this, we note that

$$\overline{\mathbf{X}}^*_{[i+1]} = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i]}) = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\overline{\mathbf{X}}^*_{[i]}) - Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[i]}(\mathbf{y})).$$

We also note that the arguments of Q^{*-1} in the very right hand side of the above formula are odd functions in the variable $(\mathbf{y},\mathbf{v})$, and that this property is preserved by the operator Q^{*-1}. At this point we have derived the equivalent of (3.30) with $i-1$ replaced by i, and so the induction step is complete. The formulas (3.32) therefore determine $\mathbf{X}^*_{[i]}$ iteratively for all $i \geq 0$.

We are now in a position to prove the following proposition

PROPOSITION 3.6. *Under the Hypotheses* **(H0)**, **(H1)**, *and* **(H2)** *the auxiliary function* $\mathbf{X}^{\eta *}$, *the solution of (3.2), has the asymptotic expansion*

$$\mathbf{X}^{\eta *}(\mathbf{y},\mathbf{v}) = \sum_{i=-1}^{I-1} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) \ + R_I(\mathbf{y},\mathbf{v}) \ ,$$

for any $I \geq 0$. *The remainder* R_I *satisfies*

$$\|R_I\|_{L^2(Y\times V)} \leq C\eta^I \quad .$$

The terms of the expansion are uniquely defined by

$$\mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) = \overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v}) + \mathbf{\Theta}^{[i]}(\mathbf{y}) \ , \qquad \mathbf{\Theta}^{[i]}(\mathbf{y}) = L^{*-1}(S^*_{[i]}) \ ,$$

with

$$\overline{\mathbf{X}}^*_{[-1]}(\mathbf{y},\mathbf{v}) = 0 \ , \qquad S^*_{[-1]}(\mathbf{y}) = \int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\,\mathbf{X}^*(\mathbf{y},\mathbf{v})d\mathbf{v} \ ,$$

$$\overline{\mathbf{X}}^*_{[0]} = \mathbf{X}^* + (\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}) \ , \qquad S^*_{[0]}(\mathbf{y}) = -\int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})(\overline{\mathbf{X}}^*_{[0]}(\mathbf{y},\mathbf{v})))\,d\nu(\mathbf{v}) \ ,$$

and

$$\overline{\mathbf{X}}^*_{[i]} = -Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[i-1]}) \ , \qquad S^*_{[i]}(\mathbf{y}) = -\int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})Q^{*-1}((\mathbf{v}\cdot\nabla_{\mathbf{y}})(\overline{\mathbf{X}}^*_{[i]}(\mathbf{y},\mathbf{v})))\,d\nu(\mathbf{v}) \ ,$$

for $i = 1, 2, \cdots$. *Here* $\mathbf{X}^*$ *denotes the function* $\mathbf{X}^* = -Q^{*-1}(\mathbf{v})$. *All the terms* $\mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v})$ *of the expansion belong to* $L^\infty(V; C^\infty_{per}(I\!R^d_{\mathbf{y}}))$.

Proof. First, the regularity of the terms in the expansion is a direct consequence of the cross section regularity (see Hypothesis **(H0)**). Let now I be an arbitrary integer, and write

$$\mathbf{X}^{\eta *} = \sum_{i=-1}^{I+1} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) + \tilde{R}_I.$$

If we prove that $\|\tilde{R}_I\|_{L^2} = O(\eta^I)$ this would lead us to the estimate

$$\mathbf{X}^{\eta *} = \sum_{i=-1}^{I-1} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) + (\eta^I \mathbf{X}^*_{[I]} + \eta^{I+1}\mathbf{X}^*_{[I+1]} + \tilde{R}_I) = \sum_{i=-1}^{I-1} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v}) + O(\eta^I) \ ,$$

as desired. To this end, the equation satisfied by $\tilde{R}^I$ reads

$$T^{\eta *}(\tilde{R}_I) = \eta^{I+2}\mathbf{v}\cdot\nabla_{\mathbf{y}}\mathbf{X}^*_{[I+1]} \ .$$

By applying Proposition 3.4 Item 6. (see (3.17)), we obtain the inequality $\|\tilde{R}_I\|_{L^2} \leq C\eta^I \|(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[I+1]}\|_{L^2}$, and the proof is complete. □

COROLLARY 3.7. *The asymptotic expansion* $\mathbf{X}^{\eta *}(\mathbf{y},\mathbf{v}) = \sum_{i=-1}^{+\infty} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y},\mathbf{v})$ *is valid in* $L^\infty_{per}(I\!R^d_{\mathbf{y}} \times V)$, *i.e.,*

$$\|\mathbf{X}^{\eta *} - \sum_{i=-1}^{I-1} \eta^i \mathbf{X}^*_{[i]}\|_{L^\infty(Y\times V)} = \|R_I\|_{L^\infty(Y\times V)} \leq C\eta^I \quad .$$

In particular

$$\mathbf{X}^{\eta *} = \frac{1}{\eta}\mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{X}^*_{[0]} + O(\eta) = \frac{1}{\eta}\mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{X}^* + (\mathbf{X}^* \cdot \nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{\Theta}^{[0]}(\mathbf{y}) + O(\eta) \ ,$$

with $O(\eta)$ bounded by $C\eta$ uniformly in $Y \times V$.

Proof. As in the proof of Proposition 3.6 we consider an arbirary integer I, and write

$$\mathbf{X}^{\eta *} = \sum_{i=-1}^{I+1} \eta^i \mathbf{X}^*_{[i]}(\mathbf{y}, \mathbf{v}) + \tilde{R}_I \ , \quad \text{with} \quad T^{\eta *}(\tilde{R}_I) = \eta^{I+2}\mathbf{v}\cdot\nabla_{\mathbf{y}}\mathbf{X}^*_{[I+1]} \ .$$

We already know that $\|\tilde{R}^I\|_{L^2} = O(\eta^I)$, and that $\mathbf{X}^*_{[I+1]}$ is in $L^\infty(V, C^\infty_{per}(Y))$. Differentiating the equation for $\tilde{R}_I$ with respect to y_i, we obtain

$$T^{\eta *}(\partial_{y_i}\tilde{R}_I) = \partial_{y_i}Q^*(\tilde{R}_I) + \eta^{I+2}\mathbf{v}\cdot\nabla_{\mathbf{y}}\partial_{y_i}\mathbf{X}^*_{[I+1]},$$

where $\partial_{y_i}Q^*$ has the same expression as Q^*, with σ replaced by $\partial_{y_i}\sigma$. Differentiating once more, we obtain

$$T^{\eta *}(\partial^2_{y_iy_j}\tilde{R}_I) = \partial^2_{y_iy_j}Q^*(\tilde{R}_I) + \partial_{y_j}Q^*(\partial_{y_i}\tilde{R}_I) + \partial_{y_i}Q^*(\partial_{y_j}\tilde{R}_I) + \eta^{I+2}(\mathbf{v}\cdot\nabla_{\mathbf{y}})\partial_{y_iy_j}\mathbf{X}^*_{[I+1]} \ ,$$

where $\partial^2_{y_iy_j}Q^*$ are defined analogously to $\partial_{y_i}Q^*$. Proposition 3.4 leads to the following estimates for the remainder and its derivatives up to second order

(i) $||\tilde{R}_I||_{L^2} \leq C\left(\eta^I||(\mathbf{v}\cdot\nabla_{\mathbf{y}})\mathbf{X}^*_{[I+1]}||_{L^2}\right)$,

(ii) $||\partial_{y_i}\tilde{R}_I||_{L^2} \leq C\left(\frac{1}{\eta^2}||\tilde{R}_I||_{L^2(Y\times V)} + \eta^I||(\mathbf{v}\cdot\nabla_{\mathbf{y}})\partial_{y_i}\mathbf{X}^*_{[I+1]}||_{L^2}\right) \leq C_1\eta^{I-2}\|\mathbf{X}^*_{[I+1]}||_{L^2(V,H^2(Y))}$,

(iii) $||\partial_{y_i,y_j}\tilde{R}_I||_{L^2} \leq C\left(\frac{1}{\eta^2}(||\tilde{R}_I||_{L^2} + ||\nabla_{\mathbf{y}}\tilde{R}_I||_{L^2}) + \eta^I||(\mathbf{v}\cdot\nabla_{\mathbf{y}})\partial_{y_iy_i}\mathbf{X}^*_{[I+1]}||_{L^2}\right) \leq C_2\eta^{I-4}\|\mathbf{X}^*_{[I+1]}||_{L^2(V,H^3(Y))}$.

Proceeding by induction, we obtain for the $\mathbf{k}^{th}$ derivative (with $\mathbf{k} = (k_1, k_2, \cdots, k_d)$)

$$||\partial^{\mathbf{k}}_{\mathbf{y}}\tilde{R}_I||_{L^2} \leq C_{\mathbf{k}}\eta^{I-2|k|}\|\mathbf{X}^*_{[I+1]}||_{L^2(V,H^{|k|+1}(Y))} \ .$$

Let $m > \frac{d}{2}$, so that $H^m_{per}(I\!R^d_{\mathbf{y}}) \subset L^\infty_{per}(I\!R^d_{\mathbf{y}})$, and let $I = 3m$ and $|k| = 0, 1, 2, \cdots, m$. The above estimate gives that

$$\|\tilde{R}_{3m}\|_{L^2(V,L^\infty(Y))} \leq C\|\tilde{R}_{3m}\|_{L^2(V,H^m(Y))} \leq C\eta^m\|\mathbf{X}^*_{[3m+1]}||_{L^2(V,H^{m+1}(Y))} \leq C_m\eta^m \ .$$

Since m is arbitrary, this implies that the asymptotic expansion holds true in $L^2(V; L^\infty_{per}(I\!R^d_{\mathbf{y}}))$. In order to see that this expansion is also asymptotic in $L^\infty_{per}(I\!R^d_{\mathbf{y}} \times V)$, we write

$$-\eta\mathbf{v}\cdot\nabla_{\mathbf{y}}\tilde{R}_{3m} + \Sigma\tilde{R}_{3m} = \int_V \sigma(\mathbf{y}, \mathbf{v}, \mathbf{v}')\tilde{R}_{3m}(\mathbf{y}, \mathbf{v}')d\nu(\mathbf{v}') + \eta^{3m+2}\mathbf{v}\cdot\nabla_{\mathbf{y}}\mathbf{X}^*_{[3m+1]} \ ,$$

where $\Sigma(\mathbf{y}, \mathbf{v}) = \int \sigma(\mathbf{y}, \mathbf{v}, \mathbf{v}')\, d\nu(\mathbf{v}')$. The right hand side is an $O(\eta^I)$ in $L^\infty_{per}(I\!R^d_{\mathbf{y}}, V)$ (since R_{3m} is integrated in velocity). As Σ is bounded from below by a positive constant, this implies by inversion of the operator $-\eta\mathbf{v}\cdot\nabla_{\mathbf{y}} + \Sigma$ that $\tilde{R}_{3m}$ is bounded by $O(\eta^m)$ in $L^\infty_{per}(I\!R^d_{\mathbf{y}}, V)$. [2] □

[2] The L^∞ boundedness of the operator $(-\eta\mathbf{v}\cdot\nabla_{\mathbf{y}} + \Sigma)^{-1}$ follows immediately from the formula $(-\eta\mathbf{v}\cdot\nabla_{\mathbf{y}} + \Sigma)^{-1}h = \eta^{-1}\int_0^\infty \exp\left(-\frac{1}{\eta}\int_0^\sigma \Sigma(\mathbf{y} + \tau\mathbf{v}, \mathbf{v})\, d\tau\right) h(\mathbf{y} + \sigma\mathbf{v}, \mathbf{v})\, d\sigma$.

4. Study of the limiting process

We consider the full equation (1.3) with all three parameters ε, η and $\alpha = \varepsilon/\eta$ going to zero. The main result is a convergence result for $f^{\varepsilon,\eta}$ in a sense of two-scale convergence [1],[30], the two scales being 1 and $\alpha = \frac{\varepsilon}{\eta}$.

4.1. Existence of a two-scale limit. Our notion of two-scale convergence for a sequence of positive measures is as follows.

Definition 4.1. **Two-scale convergence.** *Let α_k be a sequence with the property that $\alpha_k \to 0$ as $k \to \infty$. A sequence of positive, regular, Borel measures μ^k is said to two-scale converge (at the scales 1 and α_k) towards μ if and only if: for any test function $\phi \in \mathcal{C}^0_{c,per}([0,T], I\!R^d_{\mathbf{x}}, I\!R^d_{\mathbf{y}}, V)$ we have*

$$\lim_{k\to\infty} \int_{I\!R^+} \int_{I\!R^d} \int_V \phi(t, \mathbf{x}, \frac{\mathbf{x}}{\alpha_k}, \mathbf{v}) \mu^k(dt, d\mathbf{x}, d\nu(\mathbf{v})) = \int_{I\!R^+} \int_{I\!R^d} \int_Y \int_V \phi(t, \mathbf{x}, y, \mathbf{v}) \mu(dt, dx, d\mathbf{y}, d\nu(\mathbf{v})) \ .$$

$\mathcal{C}^0_{c,per}(I\!R^+, I\!R^d_{\mathbf{x}}, I\!R^d_{\mathbf{y}}, V)$ denotes the set of continuous functions on $I\!R^+ \times I\!R^d \times I\!R^d \times V$ that are compactly supported in t and $\mathbf{x}$, and periodic in $\mathbf{y}$.

Remark 4.2. *By inserting test functions of the form $\phi(t, \mathbf{x}, \frac{\eta\mathbf{x}}{\varepsilon}, \mathbf{v})$ in (1.5), we obtain*

$$\int_{I\!R^d \times V} f_{ini}(\mathbf{x}, \mathbf{v}) \phi(0, \mathbf{x}, \frac{\eta \mathbf{x}}{\varepsilon}, \mathbf{v})\, d\mathbf{x}\, d\nu(\mathbf{v})$$
$$+ \int_{I\!R^+} \int_{I\!R^d \times V} f^{\varepsilon,\eta} (\partial_t \phi + \frac{\mathbf{v} \cdot \nabla_{\mathbf{x}} \phi}{\varepsilon} + \frac{\eta \mathbf{v} \cdot \nabla_{\mathbf{y}} \phi}{\varepsilon^2} + \frac{Q^*[\mathbf{y}](\phi)}{\varepsilon^2})|_{\mathbf{y} = \frac{\eta\mathbf{x}}{\varepsilon}}\, d\mathbf{x} d\nu(\mathbf{v}) dt = 0 \ . \tag{4.1}$$

To pass to the limit, we shall apply the energy method introduced by Tartar (see [34] and the other references about homogenization quoted in the introduction). This method here consists in testing the equation against suitably chosen functions, depending on the time and space variables, and also on the fast variable $\frac{\eta\mathbf{x}}{\varepsilon}$. The intermediate results obtained by applying these test functions are summarized in the following proposition.

Proposition 4.3.

1. *For any test function $\phi \in \mathcal{C}^1_{c,per}([0,T], I\!R^d_{\mathbf{x}}, I\!R^d_{\mathbf{y}}, V)$, i.e., for any function that is periodic in $\mathbf{y}$, C^1 in the variables $(t, \mathbf{x}, \mathbf{y})$, continuous in $\mathbf{v}$, and has compact support in t and $\mathbf{x}$*

$$\lim_{\varepsilon \ll \eta \to 0} \int_{I\!R^+} \int_{I\!R^d} \int_V f^{\varepsilon,\eta}(t, \mathbf{x}, \mathbf{v}) Q^*[\mathbf{y}](\phi(t, \mathbf{x}, \mathbf{y}, \cdot))|_{\mathbf{y} = \frac{\eta\mathbf{x}}{\varepsilon}}(\mathbf{v}) dx d\nu(\mathbf{v}) dt = 0 \ , \tag{4.2}$$

$$\lim_{\varepsilon \ll \eta \to 0} \int_{I\!R^+} \int_{I\!R^d} \int_V f^{\varepsilon,\eta}(t, \mathbf{x}, \mathbf{v}) T^{*\eta}(\phi(t, \mathbf{x}, \cdot, \cdot))(\mathbf{y}, \mathbf{v})|_{\mathbf{y} = \frac{\eta\mathbf{x}}{\varepsilon}} dx d\nu(\mathbf{v}) dt = 0 \ . \tag{4.3}$$

2. *If $J^{\varepsilon,\eta} = \int_V \mathbf{v} f^{\varepsilon,\eta} d\nu(\mathbf{v})$ then*

$$\lim_{\varepsilon \ll \eta \to 0} \int_{I\!R^+} \int_{I\!R^d} J^{\varepsilon,\eta}(t, \mathbf{x}) \cdot \nabla_{\mathbf{x}} \varphi(t, \mathbf{x}) dt dx = 0 \ , \tag{4.4}$$

for any test function $\varphi \in C^0_c([0,T], I\!R^d)$.

The notation $\varepsilon << \eta \to 0$ signifies that the limit is obtained along any sequence ε, η, with $\varepsilon \to 0$, $\eta \to 0$, and $\varepsilon/\eta \to 0$.

Proof. The first identity of Item 1 is obtained by multiplication of (4.1) by ε^2, and passage to the limit, using the $L^1(I\!R \times I\!R^d \times V)$ bound on $f^{\varepsilon,\eta}$. The second identity of Item 1 is a simple consequence of the first. In order to verify Item 2 we consider (4.1) with $\phi(t,\mathbf{x},\mathbf{y},\mathbf{v}) = \varphi(t,\mathbf{x}) + \varepsilon\psi(t,\mathbf{x},\mathbf{y},\mathbf{v})$. After multiplication by ε, this leads to the limit

$$\lim_{\varepsilon \ll \eta \to 0} \int_{I\!R^+}\int_{I\!R^d}\int_V f^{\varepsilon,\eta}\left(\mathbf{v}\cdot\nabla\varphi(t,\mathbf{x}) + Q^*[\mathbf{y}](\psi(t,\mathbf{x},\mathbf{y},\cdot))\right)|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}(\mathbf{v})dtdxd\nu(\mathbf{v}) = 0 \ .$$

As a consequence of Item 1, we get

$$\lim_{\varepsilon \ll \eta \to 0} \int_{I\!R^+}\int_{I\!R^d}\int_V f^{\varepsilon,\eta}\left(\mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi\right)dtd\mathbf{x}d\nu(\mathbf{v}) = 0 \ ,$$

which is exactly (4.4). ☐

Proposition 4.4. *Given a sequence of parameters ε and η with $\varepsilon \to 0$, $\eta \to 0$ and $\frac{\varepsilon}{\eta} \to 0$, let $f^{\varepsilon,\eta}$ be the weak solution to (1.3) and $n^{\varepsilon,\eta} = \int_V f^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{v})d\nu(\mathbf{v})$. There exists a subsequence (ε_k,η_k), a positive, regular Borel measure $\mathcal{F}(dt,dx,d\mathbf{y},d\nu(\mathbf{v})) \in \mathcal{M}([0,\infty)\times I\!R^d \times Y \times V)$ and a positive, regular Borel measure $N(dt,dx,d\mathbf{y}) \in \mathcal{M}([0,\infty)\times I\!R^d\times Y)$ such that, for any test function $\phi \in \mathcal{C}^0_{c,per}([0,T], I\!R^d_{\mathbf{x}}, I\!R^d_{\mathbf{y}}, V)$ we have*

$$\lim_{k\to\infty}\int_{I\!R^+}\int_{I\!R^d}\int_V \phi(t,\mathbf{x},\frac{\eta_k\mathbf{x}}{\varepsilon_k},\mathbf{v})f^{\varepsilon_k,\eta_k}(t,\mathbf{x},\mathbf{v})dt,d\mathbf{x},d\nu(\mathbf{v}) = \int_{I\!R^+}\int_{I\!R^d}\int_Y\int_V \phi(t,\mathbf{x},y,\mathbf{v})\mathcal{F}(dt,dx,d\mathbf{y},d\nu(\mathbf{v})) \ ,$$

and for any test function $\varphi \in \mathcal{C}^0_{c,per}([0,T], I\!R^d_{\mathbf{x}}, I\!R^d_{\mathbf{y}})$ we have

$$\lim_{k\to\infty}\int_{I\!R^+}\int_{I\!R^d}\varphi(t,\mathbf{x},\frac{\eta_k\mathbf{x}}{\varepsilon_k})n^{\varepsilon_k,\eta_k}(t,\mathbf{x})dt,d\mathbf{x} = \int_{I\!R^+}\int_{I\!R^d}\int_Y \varphi(t,\mathbf{x},y)N(dt,dx,d\mathbf{y}) \ .$$

Proof. As the measure $\mu^{\varepsilon,\eta}(dt,dx,d\mathbf{y},d\nu(\mathbf{v})) = f^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{v})\delta(\mathbf{y}-\frac{\eta\mathbf{x}}{\varepsilon})dtdxd\nu(\mathbf{v})$ is bounded in the dual norm on $[C^0([0,T]\times I\!R^d\times Y\times V)]^*$, it follows from the weak* compactness of the unit ball that there exists a subsequence $\mu_k = \mu^{\varepsilon_k,\eta_k}$ and a regular Borel measure $\mathcal{F}$ so that $\mu_k \to \mathcal{F}$ weak* in $[C^0([0,T]\times I\!R^d\times Y\times V)]^*$. Thus

$$\begin{aligned}\int_{I\!R^+}\int_{I\!R^d}\int_V \phi(t,\mathbf{x},\frac{\eta_k\mathbf{x}}{\varepsilon_k},\mathbf{v})f^{\varepsilon_k,\eta_k}(t,\mathbf{x},\mathbf{v})dtd\mathbf{x}d\nu(\mathbf{v}) &= \langle\mu_k,\phi(t,\mathbf{x},\mathbf{y},\mathbf{v})\rangle \\ &\to \int_{I\!R^+}\int_{I\!R^d}\int_Y\int_V \phi(t,\mathbf{x},\mathbf{y},\mathbf{v})\mathcal{F}(t,\mathbf{x},\mathbf{y},\mathbf{v})dtd\mathbf{x}d\mathbf{y}d\nu(\mathbf{v}).\end{aligned}$$

The case of $n^{\varepsilon,\eta}$ is handled by similar arguments. ☐

Remark 4.5. *Proposition 4.4 is a compactness result. It automatically translates into a convergence result once we demonstrate that the limits $\mathcal{F}$ and N are independent of the subsequence. This independence follows from the structure result we prove in the following section.*

4.2. Identification of the two-scale limits. Proof of Theorem 1.2. We now provided a structural characterization of the limits $\mathcal{F}$ and N.

Proposition 4.6.

1. *Given a sequence of parameters ε, η, with $\varepsilon \to 0$, $\eta \to$, and $\varepsilon/\eta \to 0$, the limiting measures $\mathcal{F}(dt,d\mathbf{x},d\mathbf{y},d\nu(\mathbf{v}))$ and $N(dt,d\mathbf{x},d\mathbf{y})$ from Proposition 4.4 satisfy $\mathcal{F}(dt,d\mathbf{x},d\mathbf{y},d\nu(\mathbf{v})) = M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})N(dt,d\mathbf{x},d\mathbf{y})$, where $M(\mathbf{y},\mathbf{v})$ is the local Maxwellian. M is the unique solution to the cell problem*

$$Q[\mathbf{y}](M) = 0 \ , \quad \textit{with} \quad \int_V M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v}) = 1 \ . \tag{4.5}$$

2. *The particle density $N(dt, d\mathbf{x}, d\mathbf{y})$ can be written $N(dt, d\mathbf{x}, d\mathbf{y}) = \rho(\mathbf{y})d\mathbf{y}n(dt, d\mathbf{x})$, where $\rho(\mathbf{y})$ is the unique periodic solution to the elliptic problem*

$$
\begin{aligned}
&-div_{\mathbf{y}}(D(\mathbf{y})\nabla_{\mathbf{y}}\rho(\mathbf{y})) + div_{\mathbf{y}}(U(\mathbf{y})\rho(\mathbf{y})) = 0 \ , \quad \int_Y \rho(\mathbf{y}) = 1 \ , \\
&\text{with} \quad D(\mathbf{y}) = \int_V \mathbf{X}^*(\mathbf{y},\mathbf{v}) \otimes \mathbf{v} M(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) = \int_V \mathbf{v} \otimes \mathbf{X}(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) \ , \\
&\text{and} \quad U(\mathbf{y}) = \int_V \mathbf{v}\lambda(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}) \ .
\end{aligned}
\tag{4.6}
$$

Here X^*, X and λ are the solutions to

$$Q^*[\mathbf{y}](X^*) = -\mathbf{v} \ , \quad Q[\mathbf{y}](X) = -\mathbf{v}M \ , \quad \text{and} \quad Q[\mathbf{y}](\lambda) = \mathbf{v}\cdot\nabla_{\mathbf{y}} M \ ,$$

respectively.

3. *The macroscopic density $n(dt, d\mathbf{x})$ has the form $n(dt, d\mathbf{x}) = n(t, x)dt\, d\mathbf{x}$ where $n(t, x)$ is the unique solution to the diffusion equation*

$$\partial_t n - div_{\mathbf{x}}(\mathbf{D}\nabla_{\mathbf{x}} n) = 0 \quad , \quad n(t=0,\mathbf{x}) = n_{ini}(\mathbf{x}) = \int_V f_{ini}(\mathbf{x},\mathbf{v})\, d\nu(\mathbf{v}) \ , \tag{4.7}$$

$$\text{with} \qquad \mathbf{D} = \int_Y \Big(\rho(\mathbf{y})D(\mathbf{y}) - \mathbf{\Theta}^{[-1]}(\mathbf{y}) \otimes H(\mathbf{y})\Big)\, d\mathbf{y} \ . \tag{4.8}$$

H and $\mathbf{\Theta}^{[-1]}$ are defined by

$$H(\mathbf{y}) = div_{\mathbf{y}}(\rho(\mathbf{y})D^\top(\mathbf{y})) + D(\mathbf{y})\nabla_{\mathbf{y}}\rho - \rho(\mathbf{y})U(\mathbf{y}) \ , \quad \mathbf{\Theta}^{[-1]}(\mathbf{y}) = L^{*-1}(\int M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})\, \mathbf{X}^*(\mathbf{y},\mathbf{v}) d\nu(\mathbf{v}))$$

Proof. The three items of this Proposition will be proven by testing the Boltzmann equation against three types of test functions.

Step 1. $\mathcal{F}(dt, d\mathbf{x}, d\mathbf{y}, d\mathbf{v}) = N(dt, d\mathbf{x}, d\mathbf{y})M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})$.

First, we recall that for any test function $\phi(t, \mathbf{x}, \mathbf{y}, \mathbf{v})$,

$$\lim_{\varepsilon \ll \eta \to 0} \int_{\mathbb{R}^+} \int_{\mathbb{R}^d} \int_V f^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{v}) Q^*[\mathbf{y}](\phi(t,\mathbf{x},\mathbf{y},\cdot))|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}(\mathbf{v}) d\nu(\mathbf{v}) d\mathbf{x} dt = 0 \ .$$

Let now $\psi(t, \mathbf{x}, \mathbf{y}, \mathbf{v})$ be an arbitrary test function, and write

$$\psi(t,\mathbf{x}\,\mathbf{y},\mathbf{v}) = \overline{\psi}(t,\mathbf{x},\mathbf{y}) + [\psi(t,\mathbf{x},\mathbf{y},\mathbf{v}) - \overline{\psi}(t,\mathbf{x},\mathbf{y})] \ , \quad \text{with} \quad \overline{\psi} = \int \psi(t,\mathbf{x},\mathbf{y},\mathbf{v}) M(\mathbf{y},\mathbf{v})\, d\nu(\mathbf{v}) \ .$$

We note that the second term $\psi(t, \mathbf{x}, \mathbf{y}, \mathbf{v}) - \overline{\psi}(t, \mathbf{x}, \mathbf{y})$ lies in $Im(Q^*)$ since

$$\int M(\mathbf{y},\mathbf{v})[\psi(t,\mathbf{x}\,\mathbf{y},\mathbf{v}) - \overline{\psi}(t,\mathbf{x},\mathbf{y})]\, d\nu(\mathbf{v}) = 0 \ .$$

Therefore, along the subsequence ε_k, η_k from Proposition 4.4

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^d}\int_Y\int_V \psi\mathcal{F}(dt,d\mathbf{x},d\mathbf{y},d\nu(\mathbf{v})) \quad = \quad \lim_{k\to\infty} \int_{\mathbb{R}^+}\int_{\mathbb{R}^d}\int_V f^{\varepsilon_k,\eta_k}(t,\mathbf{x},\mathbf{v})\psi(t,\mathbf{x},\frac{\eta_k\mathbf{x}}{\varepsilon_k},\mathbf{v}) d\nu(\mathbf{v}) d\mathbf{x} dt \ .$$

By 4.2, and the fact that $\psi - \overline{\psi} = Q^*[\mathbf{y}](\phi)$ we get

$$\begin{aligned}\int_{I\!R^+}\int_{I\!R^d}\int_Y\int_V \psi \mathcal{F} d\nu(\mathbf{v})d\mathbf{y}dxdt &= \lim_{k\to\infty}\int_{I\!R^+}\int_{I\!R^d}\int_V f^{\varepsilon_k,\eta_k}(t,\mathbf{x},\mathbf{v})\overline{\psi}(t,\mathbf{x},\frac{\eta_k\mathbf{x}}{\varepsilon_k})d\nu(\mathbf{v})d\mathbf{x}dt\\ &= \lim_{k\to\infty}\int_{I\!R^+}\int_{I\!R^d} n^{\varepsilon_k,\eta_k}(t,\mathbf{x})\overline{\psi}(t,\mathbf{x},\frac{\eta_k\mathbf{x}}{\varepsilon_k})d\mathbf{x}dt\\ &= \int_{I\!R^+}\int_{I\!R^d}\int_Y \overline{\psi}(t,\mathbf{x},\mathbf{y})N(dt,d\mathbf{x},d\mathbf{y})\\ &= \int_{I\!R^+}\int_{I\!R^d}\int_Y\int_V \psi(t,\mathbf{x},\mathbf{y},\mathbf{v})M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})N(dt,d\mathbf{x},d\mathbf{y})\ .\end{aligned}$$

This shows that

$$\mathcal{F}(dt,d\mathbf{x},d\mathbf{y},d\nu(\mathbf{v})) = N(dt,d\mathbf{x},d\mathbf{y})M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})\ ,$$

where $N(dt,d\mathbf{x},d\mathbf{y})$ is the two-scale limit of $n^{\varepsilon_k,\eta_k}(t,\mathbf{x})dtd\mathbf{x}$.

Step 2. $N(dt,d\mathbf{x},d\mathbf{y}) = n(dt,d\mathbf{x})\rho(\mathbf{y})d\mathbf{y}$.

Let us consider the weak formulation (4.1) with the following family of test functions. For any $\psi(t,\mathbf{x},\mathbf{y})$, define

$$z = \mathbf{X}^*\cdot\nabla_{\mathbf{y}}\psi\ ,\quad w = \mathbf{X}^*\cdot\nabla_{\mathbf{x}}\psi\ ,$$

so that

$$Q^*[\mathbf{y}](\psi) = 0\ ,\quad Q^*[\mathbf{y}](z) = -\mathbf{v}\cdot\nabla_{\mathbf{y}}\psi\ ,\quad Q^*[\mathbf{y}](w) = -\mathbf{v}\cdot\nabla_{\mathbf{x}}\psi\ .$$

Then introduce

$$\phi^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{y},\mathbf{v}) = \psi(t,\mathbf{x},\mathbf{y}) + \eta z(t,\mathbf{x},\mathbf{y},\mathbf{v}) + \varepsilon w(t,\mathbf{x},\mathbf{y},\mathbf{v})\ .$$

We have

$$\int_V\int_{I\!R^d} f_{ini}\phi^{\varepsilon,\eta}(0,\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})d\mathbf{x}d\nu(\mathbf{v}) + \int_{I\!R^+}\int_V\int_{I\!R^d} f^{\varepsilon,\eta}S^{\varepsilon,\eta}(t,\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})\, d\mathbf{x}d\nu(\mathbf{v})dt = 0\ ,$$

where

$$\begin{aligned}S^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{y},\mathbf{v}) &= \partial_t\psi + \eta\partial_t z + \varepsilon\partial_t w + \frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}}\psi + \frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}}z + \mathbf{v}\cdot\nabla_{\mathbf{x}}w\\ &\quad + \frac{\eta}{\varepsilon^2}\mathbf{v}\cdot\nabla_{\mathbf{y}}\psi + \frac{\eta^2}{\varepsilon^2}\mathbf{v}\cdot\nabla_{\mathbf{y}}z + \frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{y}}w + \frac{1}{\varepsilon^2}Q^*(\psi) + \frac{\eta}{\varepsilon^2}Q^*(z) + \frac{1}{\varepsilon}Q^*(w)\\ &= \partial_t\psi + \eta\partial_t z + \varepsilon\partial_t w + \frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}}z + \mathbf{v}\cdot\nabla_{\mathbf{x}}w + \frac{\eta^2}{\varepsilon^2}\mathbf{v}\cdot\nabla_{\mathbf{y}}z + \frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{y}}w\ .\end{aligned}$$

This leads to

$$\begin{aligned}\int_V\int_{I\!R^d} f_{ini}\phi^{\varepsilon,\eta}(0,\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})d\mathbf{x}d\nu(\mathbf{v}) + \int_{I\!R^+}\int_V\int_{I\!R^d} f^{\varepsilon,\eta}\Bigg(\partial_t\psi + \eta\partial_t z + \varepsilon\partial_t w + \frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}}z\\ + \mathbf{v}\cdot\nabla_{\mathbf{x}}w + \frac{\eta^2}{\varepsilon^2}\mathbf{v}\cdot\nabla_{\mathbf{y}}z + \frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{y}}w)\Bigg)d\mathbf{x}d\nu(\mathbf{v})dt = 0\ .\end{aligned}$$

Multiplication of this equation by ε^2/η^2, and passage to the limit along the sequence ε_k , η_k now yields

$$\int_{I\!R^+}\int_{I\!R^d}\int_Y\int_V \mathbf{v}\cdot\nabla_{\mathbf{y}} z\,\mathcal{F}(dt,d\mathbf{x},d\mathbf{y},d\nu(\mathbf{v}))=0\ .$$

Replacing $\mathcal{F}(dt,d\mathbf{x},d\mathbf{y},d\nu(\mathbf{v}))$ by $N(dt,d\mathbf{x},d\mathbf{y})M(\mathbf{y},\mathbf{v})d\nu(\mathbf{v})$, and z by $\mathbf{X}^*\cdot\nabla_{\mathbf{y}}\psi$, we obtain

$$\int_{I\!R^+}\int_{I\!R^d}\int_Y\int_V M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})(\mathbf{X}^*(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{y}}\psi(t,\mathbf{x},\mathbf{y}))d\nu(\mathbf{v})N(dt,dx,d\mathbf{y})=0\ , \tag{4.9}$$

which is equivalent to

$$\int_{I\!R^+}\int_{I\!R^d}\int_Y L^*(\psi(t,\mathbf{x},\mathbf{y}))\,N(dt,dx,d\mathbf{y})=0\ . \tag{4.10}$$

For any test function $\phi(t,\mathbf{x},\mathbf{y})$, the function

$$\chi(t,\mathbf{x},\mathbf{y})=\phi(t,\mathbf{x},\mathbf{y})-\int_Y\rho(\mathbf{y})\phi(t,\mathbf{x},\mathbf{y})d(\mathbf{y})\quad \text{lies in } \mathrm{Im}L^*\ ,$$

since $\int_Y\rho(\mathbf{y})\chi(t,\mathbf{x},\mathbf{y})dy=0$. As a consequence of this and (4.10)

$$\int_{I\!R^+}\int_{I\!R^d}\int_Y\phi(t,\mathbf{x},\mathbf{y})\,N(dt,dx,d\mathbf{y})=\int_{I\!R^+}\int_{I\!R^d}\int_Y\rho(\mathbf{y})\phi(t,\mathbf{x},\mathbf{y})\,d\mathbf{y}\int_Y N(dt,dx,d\mathbf{y}')\ .$$

Therefore $N(dt,d\mathbf{x},d\mathbf{y})=n(dt,d\mathbf{x})\rho(\mathbf{y})d\mathbf{y}$, where $\rho(\mathbf{y})$ is the generator of the kernel of L and $n(dt,dx)=\int_Y N(dt,d\mathbf{x},d\mathbf{y}')$.

Step 3. $n(dt,d\mathbf{x})$ solves a diffusion equation.

Up to now, we did not use the expansion of F^η and $\mathbf{X}^{\eta *}$ in the proof. We shall use it in this step in order to exhibit the equation satisfied by $n(dt,d\mathbf{x})$. Given a smooth function $\varphi(t,\mathbf{x})$, let $\psi^\eta(t,\mathbf{x},\mathbf{y},\mathbf{v})$ be defined by

$$\psi^\eta=T^{\eta*-1}(\mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi)=\mathbf{X}^{\eta*}\cdot\nabla_{\mathbf{x}}\varphi\ ,$$

so that

$$-\eta\mathbf{v}\cdot\nabla_{\mathbf{y}}\psi^\eta-Q^*(\psi^\eta)=\mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi\ .$$

The two-scale weak formulation (4.1) with

$$\phi^{\varepsilon,\eta}(t,\mathbf{x},\mathbf{y},\mathbf{v})=\varphi(t,\mathbf{x})+\varepsilon\psi^\eta(t,\mathbf{x},\mathbf{y},\mathbf{v})$$

now takes the form

$$\begin{aligned}
&-\int_V\int_{I\!R^d} f_{ini}\phi^{\varepsilon,\eta}(0,\mathbf{x},\frac{\eta\mathbf{x}}{\varepsilon},\mathbf{v})d\mathbf{x}d\nu(\mathbf{v})\\
&=\int_{I\!R^+}\int_V\int_{I\!R^d} f^{\varepsilon,\eta}[\partial_t\varphi+\varepsilon\partial_t\psi^\eta+\frac{1}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{x}}\varphi+\\
&\qquad+\mathbf{v}\cdot\nabla_{\mathbf{x}}\psi^\eta+\frac{\eta}{\varepsilon}\mathbf{v}\cdot\nabla_{\mathbf{y}}\psi^\eta+\frac{1}{\varepsilon}Q^*(\psi^\eta)]|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt\\
&=\int_{I\!R^+}\int_V\int_{I\!R^d} f^{\varepsilon,\eta}[\partial_t\varphi+\varepsilon\partial_t\psi^\eta+\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{X}^{\eta*}(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{x}}\varphi)]|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt\\
&=\int_{I\!R^+}\int_V\int_{I\!R^d} f^{\varepsilon,\eta}[\partial_t\varphi+\varepsilon\partial_t\psi^\eta+\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{x}}\varphi)]|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt\\
&\quad+\frac{1}{\eta}\int_{I\!R^+}\int_V\int_{I\!R^d} f^{\varepsilon,\eta}\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt+O(\eta)\ ,
\end{aligned}$$

since $||\mathbf{X}^{\eta *} - \frac{1}{\eta}\mathbf{\Theta}^{[-1]}(\mathbf{y}) - \mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v})||_{L^\infty(Y\times V)} = 0(\eta)$. Passing to the limit in this identity along the sequence ε_k, η_k, we obtain

$$
\begin{aligned}
(4.11)\qquad & -\int_{I\!R^d} n_{ini}(\mathbf{x})\varphi(0,\mathbf{x})d\mathbf{x} \\
&= \int_{I\!R^+}\int_{I\!R^d}\int_Y\int_V \rho(\mathbf{y})M(\mathbf{y},\mathbf{v})[\partial_t\varphi(t,\mathbf{x}) + \mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{X}^*_{[0]}(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{x}}\varphi(t,\mathbf{x}))]d\mathbf{y}d\nu(\mathbf{v})\,n(dt,d\mathbf{x}) \\
&\quad + \lim_{k\to\infty}\frac{1}{\eta_k}\int_{I\!R^+}\int_V\int_{I\!R^d} f^{\varepsilon_k,\eta_k}\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)|_{\mathbf{y}=\frac{\eta_k\mathbf{x}}{\varepsilon_k}}d\mathbf{x}d\nu(\mathbf{v})dt\ ,
\end{aligned}
$$

with $n_{ini}(\mathbf{x}) = \int_V f_{ini}(\mathbf{x},\mathbf{v})d\nu(\mathbf{v})$. Let us now evaluate the last term of the the above identity. For this purpose we introduce $\Psi(x,\mathbf{y},\mathbf{v}) = \mathbf{X}^*\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)$, which satisfies

$$Q^*[\mathbf{y}](\Psi) = -\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)\ .$$

Using Ψ as a test function in (4.1), we immediately obtain the identity

$$
\begin{aligned}
&\frac{1}{\eta}\int_{I\!R^+}\int_{V\times I\!R^d} f^{\varepsilon,\eta}\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt \\
&\quad= -\frac{1}{\eta}\int_{I\!R^+}\int_{V\times I\!R^d} f^{\varepsilon,\eta}Q^*[\mathbf{y}](\Psi)d\mathbf{x}d\nu(\mathbf{v})dt \\
&\quad= \frac{\varepsilon^2}{\eta}\int_{V\times I\!R^d} f_{ini}(\mathbf{x},\mathbf{v})\Psi(0,\mathbf{x},\mathbf{y},\mathbf{v})|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v}) \\
&\qquad + \int_{I\!R^+}\int_{V\times I\!R^d} f^{\varepsilon,\eta}(\tfrac{\varepsilon^2}{\eta}\partial_t\Psi + \tfrac{\varepsilon}{\eta}\mathbf{v}\cdot\nabla_{\mathbf{x}}\Psi + \mathbf{v}\cdot\nabla_{\mathbf{y}}\Psi)|_{\mathbf{y}=\frac{\eta\mathbf{x}}{\varepsilon}}d\mathbf{x}d\nu(\mathbf{v})dt\ .
\end{aligned}
$$

Therefore, along the sequence ε_k, η_k (with $\varepsilon_k\to 0$, $\eta_k\to 0$ and $\varepsilon_k/\eta_k\to 0$)

$$
\begin{aligned}
&\lim_{k\to\infty}\frac{1}{\eta_k}\int_{I\!R^+}\int_V\int_{I\!R^d} f^{\varepsilon_k,\eta_k}\mathbf{v}\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi)|_{\mathbf{y}=\frac{\eta_k\mathbf{x}}{\varepsilon_k}}d\mathbf{x}d\nu(\mathbf{v})dt \\
&\quad= \int_{I\!R^+}\int_{I\!R^d}\int_V\int_Y M(\mathbf{y},\mathbf{v})\rho(\mathbf{y})(\mathbf{v}\cdot\nabla_{\mathbf{y}})(\mathbf{X}^*\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi(t,\mathbf{x})))d\mathbf{y}d\nu(\mathbf{v})\,n(dt,d\mathbf{x})\ .
\end{aligned}
$$

Inserting this identity into (4.11), and using the formula $\mathbf{X}^*_{[0]} = \mathbf{X}^* + (\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{\Theta}^{[0]}(\mathbf{y})$, we find

$$
\begin{aligned}
&\int_{I\!R^d} n_{ini}(\mathbf{x})\varphi(0,\mathbf{x})d\mathbf{x} + \int_{I\!R^+}\int_{I\!R^d}\partial_t\varphi(t,\mathbf{x})n(dt,d\mathbf{x}) \\
&\quad + \int_{I\!R^+}\int_{I\!R^d}\int_Y\int_V \rho(\mathbf{y})M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{x}})[(\mathbf{X}^* + (\mathbf{X}^*\cdot\nabla_{\mathbf{y}})\mathbf{\Theta}^{[-1]}(\mathbf{y}))\cdot\nabla_{\mathbf{x}}\varphi(t,\mathbf{x})]d\mathbf{y}d\nu(\mathbf{v})\,n(dt,d\mathbf{x}) \\
&\quad + \int_{I\!R^+}\int_{I\!R^d}\int_Y\int_V \rho(\mathbf{y})M(\mathbf{y},\mathbf{v})(\mathbf{v}\cdot\nabla_{\mathbf{y}})(\mathbf{X}^*\cdot\nabla_{\mathbf{x}}(\mathbf{\Theta}^{[-1]}(\mathbf{y})\cdot\nabla_{\mathbf{x}}\varphi(t,\mathbf{x})))d\mathbf{y}d\nu(\mathbf{v})\,n(dt,d\mathbf{x}) = 0\ .
\end{aligned}
$$

This is exactly a weak formulation of the diffusion problem

$$\partial_t n - \mathrm{div}_{\mathbf{x}}(\mathbf{D}\nabla_{\mathbf{x}})n = 0, \quad n(0,\mathbf{x}) = n_{ini}(\mathbf{x}) = \int f_{ini}(\mathbf{x},\mathbf{v})\,d\nu(\mathbf{v})\ ,$$

with

$$\begin{aligned}\mathbf{D} &= \int_Y \int_V (\mathbf{X}^*(\mathbf{y},\mathbf{v}) + \mathbf{X}^*(\mathbf{y},\mathbf{v})\cdot\nabla_{\mathbf{y}}\mathbf{\Theta}^{[-1]}(\mathbf{y}))\otimes\mathbf{v}\rho(\mathbf{y})M(\mathbf{y},\mathbf{v}) \\ &\qquad -\mathbf{\Theta}^{[-1]}(\mathbf{y})\otimes\mathbf{X}^*(\mathbf{y},\mathbf{v})\mathbf{v}\cdot\nabla_{\mathbf{y}}(\rho(\mathbf{y})M(\mathbf{y},\mathbf{v}))\,d\nu(\mathbf{v})d\mathbf{y} \\ &= \int_Y\int_V \mathbf{X}^*(\mathbf{y},\mathbf{v})\otimes\mathbf{v}\rho(\mathbf{y})M(\mathbf{y},\mathbf{v}) - \mathbf{\Theta}^{[-1]}(\mathbf{y})\otimes\Big[\mathrm{div}_{\mathbf{y}}(\mathbf{v}\otimes\mathbf{X}^*(\mathbf{y},\mathbf{v})M(\mathbf{y},\mathbf{v})\rho(\mathbf{y})) \\ &\qquad +\mathbf{X}^*(\mathbf{y},\mathbf{v})\otimes\mathbf{v}M(\mathbf{y},\mathbf{v})\nabla_{\mathbf{y}}\rho(\mathbf{y}) + \mathbf{X}^*(\mathbf{y},\mathbf{v})\mathbf{v}\cdot\nabla_{\mathbf{y}}M(\mathbf{y},\mathbf{v})\rho(\mathbf{y})\Big]\,d\nu(\mathbf{v})d\mathbf{y} \\ &= \int_Y D(\mathbf{y})\rho(\mathbf{y}) - \mathbf{\Theta}^{[-1]}(\mathbf{y})\otimes[\mathrm{div}_{\mathbf{y}}(D^\top(\mathbf{y})\rho(\mathbf{y})) + D(\mathbf{y})\nabla_{\mathbf{y}}\rho(\mathbf{y}) - U(\mathbf{y})\rho(\mathbf{y})]d\mathbf{y}\ .\end{aligned}$$

We note that $\mathbf{D}$ is as introduced in (1.7). We also note that, due to the positivity of the constant matrix $\mathbf{D}$ (see Remark 4.9), $n(dt, d\mathbf{x})$ is unique, and it has the form $n(t,\mathbf{x})dt\,d\mathbf{x}$, where the function $n(t,\mathbf{x})$ is a classical solution to the diffusion problem.
□

In view of the Definition 4.1, *The proof of Theorem 1.2* now follows by a simple combination of the results in Proposition 4.4 and Proposition 4.6. The uniquenss of the limit (Proposition 4.6) implies convergence along any sequence with $\varepsilon \to 0$, $\eta \to 0$ and $\varepsilon/\eta \to 0$.

REMARK 4.7. *We need a uniformly convergent asymptotic formula for $\mathbf{X}^{\eta*}$, because we do not work in the L^2 framework but in the L^1 framework. Indeed, when the Maxwellian profile depends on y, the usual dissipation relation does not give a L^2 bound for $f^{\varepsilon,\eta}$. We could obtain a weighted L^2 norm $||\frac{f^{\varepsilon,\eta}}{F^\eta}||_{L^2(\mathbb{R}\times\mathbb{R}^d\times V)}$ and prove that F^η is bounded from below independently of η. It would certainly be possible to show this for sufficiently small η, by demonstrating C^0 convergence of F^η towards ρM, but it would imply the same technical difficulties (and the same kind of assumptions) as required for the expansion of $\mathbf{X}^{\eta*}$.*

REMARK 4.8. *Note that, as we suggested in the section about the strategy proof, one could formally obtain the same limiting result, starting from the work of [3] or [21], and introducing the parameter η in the effective diffusion equation. In that case one arrives at an η dependent density $n^\eta(t,\mathbf{x})$ satisfying*

$$\partial_t n^\eta - \mathrm{div}_{\mathbf{x}}\mathbf{D}^\eta\nabla_{\mathbf{x}}n^\eta = 0 \quad \textit{with} \quad \mathbf{D}^\eta = \int_V\int_Y \mathbf{X}^{\eta*}(\mathbf{y},\mathbf{v})\otimes\mathbf{v}F^\eta(\mathbf{y},\mathbf{v})d\mathbf{y}d\nu(\mathbf{v})\ .$$

By expanding F^η and $\mathbf{X}^{\eta*}$ as above, and passing to the limit $\eta\to 0$,

$$\begin{aligned}\mathbf{D}^\eta &= \int_V\int_Y \left[\frac{1}{\eta}\mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{X}^* + \mathbf{X}^*\cdot\nabla_{\mathbf{y}}\mathbf{\Theta}^{[-1]}(\mathbf{y}) + \mathbf{\Theta}^{[0]}(\mathbf{y}) + O(\eta)\right] \\ &\qquad \otimes\mathbf{v}\left[\rho(\mathbf{y})M(\mathbf{y},\mathbf{v}) + \eta Q^{-1}(\mathbf{v}\cdot\nabla_{\mathbf{y}}(\rho(\mathbf{y})M(\mathbf{y},\mathbf{v}))) + O(\eta^2)\right]d\mathbf{y}d\nu(\mathbf{v}) \\ &= \int_V\int_Y \Big[(\mathbf{X}^* + \mathbf{X}^*\cdot\nabla_{\mathbf{y}}\mathbf{\Theta}^{[-1]}(\mathbf{y}))\otimes\mathbf{v}\rho(\mathbf{y})M(\mathbf{y},\mathbf{v}) \\ &\qquad +\mathbf{\Theta}^{[-1]}(\mathbf{y})\otimes\mathbf{v}Q^{-1}(\mathbf{v}\cdot\nabla_{\mathbf{y}}(\rho(\mathbf{y})M(\mathbf{y},\mathbf{v})))\Big]d\mathbf{y}d\nu(\mathbf{v}) + O(\eta) \\ &= \int_V\int_Y \Big[(\mathbf{X}^* + \mathbf{X}^*\cdot\nabla_{\mathbf{y}}\mathbf{\Theta}^{[-1]}(\mathbf{y}))\otimes\mathbf{v}\rho(\mathbf{y})M(\mathbf{y},\mathbf{v}) \\ &\qquad -\mathbf{\Theta}^{[-1]}(\mathbf{y})\otimes\mathbf{X}^*\mathbf{v}\cdot\nabla_{\mathbf{y}}(\rho(\mathbf{y})M(\mathbf{y},\mathbf{v}))\Big]d\mathbf{y}d\nu(\mathbf{v}) + O(\eta)\ ,\end{aligned}$$

in other words

$$\lim_{\eta\to 0} \mathbf{D}^\eta = \mathbf{D} \ ,$$

and one recovers, formally, the result of our main theorem.

REMARK 4.9. *The fact that*

$$\mathbf{D}\xi\cdot\xi = \lim_{\eta\to 0} \mathbf{D}^\eta\xi\cdot\xi$$

(see the previous Remark) in combination with the dissipation relation

$$\mathbf{D}^\eta\xi\cdot\xi = \frac{1}{2}\int_V\int_V\int_Y \sigma(\mathbf{y},\mathbf{v},\mathbf{v}')F^\eta(\mathbf{y},\mathbf{v})|\mathbf{X}^{\eta *}(\mathbf{y},\mathbf{v}')\cdot\xi - \mathbf{X}^{\eta *}(\mathbf{y},\mathbf{v})\cdot\xi|^2 d\nu(\mathbf{v}')d\nu(\mathbf{v})d\mathbf{y}$$

(see [21]) immediately yields

$$\mathbf{D}\xi\cdot\xi = \frac{1}{2}\lim_{\eta\to 0}\int_V\int_V\int_Y \sigma(\mathbf{y},\mathbf{v},\mathbf{v}')F^\eta(\mathbf{y},\mathbf{v})|\mathbf{X}^{\eta *}(\mathbf{y},\mathbf{v}')\cdot\xi - \mathbf{X}^{\eta *}(\mathbf{y},\mathbf{v})\cdot\xi|^2 d\nu(\mathbf{v}')d\nu(\mathbf{v})d\mathbf{y} \ . \tag{4.12}$$

Thanks to Corollary 3.7 we have

$$\begin{aligned}
&|\mathbf{X}^{\eta *}(\mathbf{y},\mathbf{v}')\cdot\xi - \mathbf{X}^{\eta *}(\mathbf{y},\mathbf{v})\cdot\xi|^2 \\
&\quad = |((\mathbf{X}^{\eta *}(\mathbf{y},\mathbf{v}') - \frac{1}{\eta}\Theta^{[-1]}(y) - \Theta^{[0]}(y))\cdot\xi - (\mathbf{X}^{\eta *}(\mathbf{y},\mathbf{v}) - \frac{1}{\eta}\Theta^{[-1]}(y) - \Theta^{[0]}(y))\cdot\xi|^2 \\
&\quad \to |(\mathbf{X}^*(\mathbf{y},\mathbf{v}') + \mathbf{X}^*(\mathbf{y},\mathbf{v}')\cdot\nabla_\mathbf{y}\Theta^{[-1]}(\mathbf{y}))\cdot\xi - (\mathbf{X}^*(\mathbf{y},\mathbf{v}) + \mathbf{X}^*(\mathbf{y},\mathbf{v})\cdot\nabla_\mathbf{y}\Theta^{[-1]}(\mathbf{y}))\cdot\xi|^2 \ ,
\end{aligned}$$

in $L^\infty_{per}(I\!R^d_\mathbf{y}\times V)$ and, thanks to Proposition 3.5, we have

$$F^\eta(\mathbf{y},\mathbf{v}) \to \rho(\mathbf{y})M(\mathbf{y},\mathbf{v}) \quad \text{in } L^2(Y\times V) \ , \quad \text{as } \eta\to 0 \ .$$

In view of these two convergence statements, and the fact that $\sigma(\mathbf{y},\mathbf{v},\mathbf{v}')\rho(\mathbf{y})M(\mathbf{y},\mathbf{v})$ is positive, the identity (4.12) implies

$$\begin{aligned}
\mathbf{D}\xi\cdot\xi = \frac{1}{2}\int_V\int_V\int_Y \sigma(\mathbf{y},\mathbf{v},\mathbf{v}')\rho(\mathbf{y})M(\mathbf{y},\mathbf{v})|(\mathbf{X}^*(\mathbf{y},\mathbf{v}') + \mathbf{X}^*(\mathbf{y},\mathbf{v}')\cdot\nabla_\mathbf{y}\Theta^{[-1]}(\mathbf{y}))\cdot\xi \\
-(\mathbf{X}^*(\mathbf{y},\mathbf{v}) + \mathbf{X}^*(\mathbf{y},\mathbf{v})\cdot\nabla_\mathbf{y}\Theta^{[-1]}(\mathbf{y}))\cdot\xi|^2 d\nu(\mathbf{v}')d\nu(\mathbf{v})d\mathbf{y} \geq 0 \ .
\end{aligned} \tag{4.13}$$

Furthermore, it follows that if $\mathbf{D}\xi\cdot\xi$ vanishes, then the expression

$$(\mathbf{X}^*(\mathbf{y},\mathbf{v}) + \mathbf{X}^*(\mathbf{y},\mathbf{v})\cdot\nabla_\mathbf{y}\Theta^{[-1]}(\mathbf{y}))\cdot\xi$$

must be independent of $\mathbf{v}$, and so it must belong to the kernel of $Q^*[\mathbf{y}]$. An application of $Q^*[\mathbf{y}]$ now yields

$$(\mathbf{v} + \mathbf{v}\cdot\nabla_\mathbf{y}\Theta^{[-1]}(\mathbf{y}))\cdot\xi = 0 \ ,$$

which, after integration in $\mathbf{y}$, gives

$$\mathbf{v}\cdot\xi = 0 \quad , \text{ for a.e. } \mathbf{v}\in V \ .$$

We conclude that $\xi = 0$ and, as a consequence of (4.13), that

$$\mathbf{D}\xi\cdot\xi \geq \beta|\xi|^2 \quad \text{ for some } \beta > 0 \ .$$

This positivity property of the (constant) matrix $\mathbf{D}$ was used to ensure the uniquenss of the density $n(dt, d\mathbf{x})$.

5. Acknowledgements

N. Ben Abdallah and M. Puel were partially supported by ANR project BLAN07-2 212988. M.S. Vogelius was partially supported by NSF grant DMS-0604999. The authors would also like to thank Guillaume Bal and Yves Capdeboscq for many helpful comments on this work.

6. Postscript

Throughout much of the work on this project Naoufel Ben Abdallah was the moving force. However, the manuscript was completed after he passed away on July the fifth, 2010. We (Marjolaine Puel and Michael Vogelius) very much hope he would have approved of this final version.

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