Math 250–Section #5   Hourly #1 Review

Name: ______________________

Note: The total points in this review add to more than 100 [in the test it will total 100]. You SHOULD also review all the material in the quizzes, including Quiz # 3, whose answers will be posted by Thursday [if not before]. Some of the problems below have written answers, the others will be discussed in class on Tuesday, before the hourly, or at office hours.

Answer:
1. [12 pts] Given the matrix

\[ A = \begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & 1 \\ 1 & -1 & -2 & 2 \\ -4 & 2 & 3 & 1 \\ 1 & -1 & -1 & 3 \end{bmatrix} \]

(a) Find its reduced echelon form \( R \) of \( A \).
(b) What are the rank and the nullity of \( A \).
(c) Argue that the rows of \( R \) with pivots are linearly independent.
(d) Argue that the columns of \( A \) with pivots are linearly independent.

Answer:
2. [10 pts] Determine a value of $r$ for which the set of vectors
\[
\begin{bmatrix}
-2 \\
0 \\
1
\end{bmatrix},
\begin{bmatrix}
1 \\
1 \\
-3
\end{bmatrix},
\begin{bmatrix}
-1 \\
1 \\
r
\end{bmatrix}
\]
is linearly dependent.
3. [12 pts] Find all the values for $t$ for which the resulting system of equations (a) has no solution, (b) a unique solution, and (c) infinitely many solutions.

\[
\begin{align*}
  x + y - z &= 2 \\
  x + 2y + z &= 3 \\
  x + y + (t^2 - 5)z &= t
\end{align*}
\]

Answer: Let us Gauss-Jordan’s on this system

\[
\begin{array}{ccc|c}
  1 & 1 & -1 & 2 \\
  1 & 2 & 1 & 3 \\
  1 & 1 & t^2 - 5 & t \\
\end{array}
\]

Using the 1 in the position (1, 1) as a pivot, we get

\[
\begin{array}{ccc|c}
  1 & 1 & -1 & 2 \\
  0 & 1 & 2 & 1 \\
  0 & 0 & t^2 - 4 & t - 2 \\
\end{array}
\]

We are now ready: If $t \neq \pm 2$, we get a unique solution since we have 3 pivots. If $t = 2$, we have an infinite number of solutions. If $t = -2$, it is impossible.
4. [10 pts] Find the inverse of the matrix

\[ A = \begin{bmatrix} 
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 
\end{bmatrix} \]

Answer: Using either of the methods we know [Gauss-Jordan algorithm or the formula involving the adjoint matrix] you should get:

\[ A = \begin{bmatrix} 
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 
\end{bmatrix} \]
5. [12 pts] (a) Explain what is a *linear combination* of vectors.
(b) Find out whether the vector (1, 0, 1, 2) is a linear combination of 
(2, 1, 3, 0), (7, 3, 1, −1) and (0, 1, 4, 3).

Answer: (a) A linear combination of the *vectors* \( v_1, \ldots, v_n \) is a vector

\[ v = c_1 v_1 + \cdots + c_n v_n, \]

where \( c_1, \ldots, c_n \) are scalars.

(b): To answer this item, you must check whether it is possible to solve for \( c_1, c_2 \) and \( c_3 \) the equation

\[ (1, 0, 1, 2) = c_1(2, 1, 3, 0) + c_2(7, 3, 1, −1) + c_3(0, 1, 4, 3). \]

Using Gaussian elimination, it turns out to be impossible.
6. [10 pts] Answer (a) or (b):
(a) Let $A$ and $C$ be square matrices of the same sizes.
(i) Find examples such that $AC = 0$ but $CA \neq 0$.
(ii) Argue that this cannot occur if one of the matrices (either one) is invertible.

(b) If a square matrix $A$ satisfies the relation
\[ A^3 - A + I = 0 \]
show that it is invertible.

Answer:

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ \quad $C = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$.

Then:

$AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ \quad $CA = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$

If $A$ is invertible, $A^{-1}AC = IC = 0$ implies $C = 0$ implies $CA = 0$.

(b) $A(A^2 - I) = -I$. Thus $A^{-1} = I - A^2$. 

7. [10 pts] Given 5 countries, assume each maintains diplomatic relations with SOME of the others. To organize these relationships we use a $5 \times 5$ matrix $A$, here we set $a_{ii} = 0$ and $a_{ij} = 1$ if the countries $i$ and $j$ have diplomatic relation; if not we set $a_{ij} = 0$:

$$A = \begin{bmatrix}
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0
\end{bmatrix}$$

(a) Which pairs of countries maintain diplomatic relations with one another.
(b) How many countries link country 1 with country 3?
(c) Give an interpretation of the (1, 4)-entry of $A^3$. 
8. [10 pts] Let $B$ and $C$ be two $3 \times 3$ matrices. Argue that the columns of $BC$ are linear combinations of the columns of $B$. Can you say the same with regard to the rows of $C$ vis-à-vis the rows of $BC$? Can we argue from this that $\text{rank } B \geq \text{rank } BC$ (and similarly for $C$)?
9. [10 pts] Prove that the inverse of a lower triangular matrix $A$ (if it exists) is lower.
10. [10 pts] If $A$ and $B$ are $2 \times 2$ symmetric matrices [what are these anyway?] show that $AB$ may not be symmetric.
11. [$\infty$ pts] What is your favorite matrix? (Warning: this is a deep psychological inquiry...)
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