Multiplicities and the Number of Generators of Cohen–Macaulay Ideals

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Abstract. Let \((R, m)\) be a Cohen–Macaulay local ring of dimension \(d\) and let \(I\) be a Cohen–Macaulay ideal of codimension \(g\). There are several invariants of \(R\) and of \(I\) (multiplicities, embedding dimension, index of nilpotency, etc.) that may affect the number of generators \(\nu(I)\) of \(I\). After reviewing results of Sally, Valla and others, we derive estimates for \(\nu(I)\) depending on a new mix of some of these invariants.

1. Introduction

Let \((R, m)\) be a Cohen–Macaulay local ring of dimension \(d\) and let \(I\) be an ideal of height \(g > 0\). If \(I\) is a Cohen–Macaulay ideal, there are several approaches that can be used to bound the minimal number of generators of \(I\) in terms of the multiplicity data, or some other degree,

\[ \nu(I) \leq f(\deg(R), \deg(R/I), e(I), g, d, \text{ embed}(R), \ldots), \]

where \(f\) is a linear or quadratic polynomial.

For our purpose here, we set ourselves in the framework of three previous discussions of this issue: \([2]\), \([8]\), and \([10]\). A different approach to link \(\nu(I)\) to mixed multiplicities is undertaken in \([5]\). After reviewing quickly a few of these results, we first introduce some variations on them, and embark on an approach that emphasizes the computation of the lengths of chains

\[ I \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = m \]

of irreducible ideals containing \(I\), together with an analysis of the number of generators of the subideals in a composition series of \(R/I\) containing subchains of irreducible ideals. A more intrusive use of the integral closure \(\overline{I}\) of \(I\) was sought but it has yet to be realized.

Our two most contrasting results are the following. We state the results for \(m\)-primary ideals, since in the Cohen–Macaulay case the reduction to primary ideals is always possible (unexplained notation defined in the body of the paper).

- Proposition 3.1(b): If \(I\) is \(m\)-primary of multiplicity \(e(I)\), and \(I \subseteq m^2\), then

\[ \nu(I) \leq e(I) \leq d! \cdot \ell(R/\overline{I}) \deg(R). \]

- Theorem 3.7(b): If \(e = \text{ embed}(R) \geq 2\), \(I \subseteq m^2\), \(\ell(R/I) > 4\) and \(t\) is the index of nilpotency of \(R/I\), then

\[ \nu(I) \leq (\ell(R/I) - t - \frac{e}{2} + 2)(e - 1). \]

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The first of these is an assemblage of known parts. The other bound is novel. Together with already established results they suggest that a more comprehensive understanding could be at play. In a short final section we make some observations on the non-primary ideal case.

2. Estimating with multiplicities

Let us begin by setting the stage. Throughout $R$ is a local ring of dimension $d$, with infinite residue field and multiplicity $\deg(R)$. $I$ is a Cohen–Macaulay ideal of height $g$ and multiplicity $\deg(R/I)$. Other elements to appear are the embedding dimension $\text{embdim}(R)$ of $R$ and indices of nilpotency.

A very general approach to estimate number of generators with multiplicities uses the method of extended multiplicities (see [6]). These are numerical functions (denoted by $\text{Deg}(\cdot)$) on modules that coincide with the classical multiplicity of local rings or graded modules (denoted by $\deg(\cdot)$). A typical expression from [6] is:

$$\nu(I) \leq \deg(R) + (g - 1) \deg(R/I) + (d - r - 1)(\text{Deg}(R/I) - \deg(R/I)),$$

where $\text{Deg}(\cdot)$ is any extended degree function and $r = \text{depth } R/I$.

If $I$ is a Cohen–Macaulay ideal, $\text{Deg}(R/I) = \deg(R/I)$, we get a benchmark for comparison with all other approaches (see also [10]):

**Proposition 2.1.** If $I$ is a Cohen–Macaulay ideal of codimension $g$, then

$$\nu(I) \leq \deg(R) + (g - 1)\deg(R/I).$$

The case $g \leq 1$ is part of a very general result valid for all Cohen–Macaulay modules ([3, Corollary 4.7.11]).

The following two results map different aspects of the problem. The first result comes from [8]:

**Theorem 2.2.** Let $(R, m)$ be a Cohen–Macaulay local ring of dimension $d > 0$.

(a) Let $I$ be an $m$–primary ideal of nilpotency index $t$. Then $\nu(I) \leq t^{d-1} \deg(R) + d - 1$.

(b) Let $I$ be a Cohen–Macaulay ideal of height $g > 0$. Then $\nu(I) \leq \deg(R/I)^{g-1} \deg(R) + g - 1$.

Exploring the Koszul complex, Valla ([10]) derived the following different bounds:

**Theorem 2.3.** Let $(R, m)$ be a Cohen–Macaulay local ring of multiplicity $e$. Let $I$ be a Cohen–Macaulay ideal of codimension $g$ such that the multiplicity of $R/I$ is $\delta$. Setting $m = \min\{e, \delta\}$, then

(a) If $g > 0$, then

$$\nu(I) \leq e + \frac{\delta(g - 1)^2}{g} + \frac{m(g - 1)}{g}.$$

(b) If $g \geq 2$ and $I \subset m^2$, then

$$\nu(I) \leq e + \frac{\delta(g - 1)^2}{g} + \frac{\min\{m + g, \delta\}}{g} - \binom{g}{2}.$$

In some cases, instead of $\deg(R/I)$ one has other invariants of the ideal appearing in these estimates. Here is one from [2], whose technique will be used again:

**Theorem 2.4.** Let $R$ be a Cohen–Macaulay local ring and let $I$ be a Cohen–Macaulay ideal of codimension 2. If $R/I$ has Cohen–Macaulay type $r$, then

$$\nu(I) \leq (r + 1)\deg(R).$$
Proof. The proof of [2] places no restriction on $R$, but here we assume that $R$ contains an appropriate field so that we may complete $R$ and assume the existence of a regular local subring $S \subseteq R$ over which $R$ is finite. Since $R$ is Cohen–Macaulay, this means that $R$ is a free $S$–module of rank $\deg(R)$. $R/I$ has a minimal free resolution over $S$,

$$0 \to S^n \to S^m \to S^n \to R/I \to 0,$$

in which $n \leq \deg(R)$. The number $m$ is the minimal number of generators of $I$ as an $S$–module. Since $m = n + p$, we need to bound $p$. Applying the functor $\text{Hom}_S(\cdot, S)$, we get that the canonical module $\omega_{R/I} = \text{Ext}^2_S(R/I, S)$ of $R/I$ can be generated minimally by $p$ elements. But $\omega_{R/I}$ is generated by $r$ elements as an $R/I$–module and therefore can be generated by $r \cdot \deg(R)$ over $S$, and we get our estimate. \qed

The next method has a different character: Let $J_0$ be an ideal generated by a regular sequence of $g - 1$ elements, $J_0 \subseteq I$. Preferably $J_0$ should be part of a minimal reduction of $I$. We have

$$(3) \quad \nu(I) \leq \nu(I/J_0) + \nu(J_0) \leq \deg(R/J_0) + g - 1,$$

since $I/J_0$ is a maximal Cohen–Macaulay module of rank 1 over the Cohen–Macaulay ring $R/J_0$. The issue is to relate $\deg(R/J_0)$ to $\deg(R/I)$.

Let us consider two applications of the method to the case of an ideal $I$ that is equimultiple. The other approach is somewhat not sensitive to this additional information. First, let $R$ be a Gorenstein ring and suppose $J = (J_0, x)$ is a minimal reduction of $I$. Since $x$ is regular modulo $J_0$, we have $\deg(R/J_0) \leq \deg(R/J)$,

$$\nu(I) \leq \deg(R/J) + g - 1.$$

Consider the exact sequence

$$0 \to (J : I)/J \to R/J \to R/(J : I) \to 0,$$

where $(J : I)/J$ is the canonical module of $R/I$ and therefore the equality $\deg((J : I)/J) = \deg(R/I)$ yields

$$\deg(R/J) = \deg(R/I) + \deg((J : I)/J).$$

If, for instance, $I$ is equimultiple of reduction number 1, that is $I^2 = JI$, $I \subset J : I$ and therefore $\deg(R/I) \geq \deg((J : I)/J)$, since they are both modules of dimension $d - g$. We thus obtain the estimate

$$\nu(I) \leq 2\deg(R/I) + g - 1$$

which is, usually, for this limited class of ideals, considerably better than the bound (2). We extend this further to arbitrary Cohen–Macaulay rings.

Proposition 2.5. Let $R$ be a Cohen–Macaulay local ring of type $r$, and let $I$ be a Cohen–Macaulay ideal of height $g \geq 1$. If $I$ is equimultiple and has reduction number $\leq 1$, then

$$(4) \quad \nu(I) \leq (r + 1)\deg(R/I) + g - 1.$$

Proof. Taking into account the considerations above, it will suffice to show that $\deg(J : I/J) \leq r \deg(R/I)$. We may assume (passing to the completion if required) that $R$ has a canonical module, say $\omega$. The ring $R/J$ has $\overline{\omega} = \omega / J\omega$ for its canonical module. $\overline{\omega}$ is a faithful $R/J$, generated by $r$ elements. In the usual manner, we can build an embedding

$$R/J \hookrightarrow \overline{\omega}^\oplus r.$$ 

Applying the functor $\text{Hom}(R/J, \cdot)$, we obtain an embedding of $J : I/J$ into the direct sum of $r$ copies of $\text{Hom}(R/J, \overline{\omega})$. Since this is the canonical module of $R/I$, it has the same multiplicity as $R/I$. \qed
3. Number of generators of primary ideals

Let \((R, m)\) be a Cohen–Macaulay local ring and let \(I\) be an \(m\)-primary ideal. We seek bounds between the number of generators of \(I\) and other quantities attached to the ideal:

\[
\begin{align*}
\nu(I) & \leftrightarrow \\
\quad & d = \dim R \\
\quad & e = \nu(m) \\
\quad & s = \ell(R/I), \text{ the colength of } I \\
\quad & e(I) = \text{multiplicity of } I \\
\quad & t = \text{nil}(R/I), \text{ smallest } n \text{ such that } m^n \subset I
\end{align*}
\]

One relation between some of these quantities is put together as follows (see also [1]):

**Proposition 3.1.** Let \((R, m)\) be a Cohen–Macaulay local ring of dimension \(d\) and let \(I\) be an \(m\)-primary ideal of multiplicity \(e(I)\).

(a) Then \(\nu(I) \leq e(I) + d - 1\).

(b) If \(I \subset m^2\), then \(\nu(I) < e(I) \leq d! \cdot \ell(R/I) \cdot \deg(R)\).

**Proof.** We may assume that \(R\) has an infinite residue field. Let \(J\) be a minimal reduction of \(I\). From the inequalities exhibited by the diagram

\[
\begin{array}{c}
\begin{array}{ccc}
\nu(I) & \rightarrow & \ell(I/J) \\
\downarrow & & \downarrow \\
mI & \cap & mJ \\
\downarrow & & \downarrow \\
J & \cap & mJ \\
\end{array}
\end{array}
\]

\[
\begin{align*}
\nu(I) &= \ell(I/mI) \leq \ell(I/mJ) \\
&= \ell(I/J) + \ell(J/mJ) \\
&= \ell(R/J) - \ell(R/I) + d,
\end{align*}
\]

the first assertion follows, since \(e(I) = \ell(R/J)\) as \(R\) is Cohen–Macaulay. For the other bound, just observe that with \(I \subset m^2\), \(\ell(R/I) \geq \ell(R/m^2) \geq e + 1 \geq d + 1\).

The final inequality arises from a well-known theorem of Lech ([7]):

**Theorem 3.2.** Let \((R, m)\) be a Noetherian local ring of dimension \(d\) and let \(I\) be an \(m\)-primary ideal. Then

\[
e(I) \leq d! \cdot \ell(R/I) \cdot \deg(R).
\]

It gets slightly improved when we replace \(I\) by its integral closure \(\overline{I}\), since \(e(I) = e(\overline{I})\), which we then use in the estimation. \(\Box\)

**Number of generators and the socle.** We will now develop new expressions for bounds on the number of generators of ideals that integrate efficiencies usually associated with Gorenstein ideals.

**Proposition 3.3.** Let \((R, m)\) be a Noetherian local ring and let \(I\) be an \(m\)-primary ideal.

(a) If \(I \subset m^2\), \(z \in (I : m) \setminus I\), \(I\) is irreducible and \(e \geq 2\), then \(\nu(I) = \nu(I, z) - 1\).

(b) [H. Schoutens] In general, \(\nu(I) \leq \nu(I, z) + e - 1\).
A related class of ideals of reduction number one is taken from [4]; it will not require that $R$ be Cohen-Macaulay.

We start by quoting [4, Lemma 3.5] about the existence of a dual basis.

**Lemma 3.4.** Let $(R, m)$ be a Noetherian local ring with embedding dimension $n$ at least two. Let $I$ be an $m$-primary irreducible ideal contained in $m^2$. If $m \subseteq (x_1, \ldots, x_n)$ then there exist $y_1, \ldots, y_n$ such that for all $1 \leq i, j \leq n$

(6) \[ x_i y_j \equiv \delta_{ij} \Delta \mod I, \]

where $\Delta$ is the lift in $R$ of the socle generator of $R/I$ and $\delta_{ij}$ denotes Kronecker’s delta.

**Proof.** The first assertion is part of Lemma 3.4: It follows from the equality $m(I, z) = mI$.

For the other assertion, suppose $I$ is minimally generated by $a_1, \ldots, a_r$; adding $z$ may result in a set (after notation change of the $a_i$) of generators for $(I, z)$ of the form $a_1, \ldots, a_r, z$. This means (after another rewrite) that $a_i = b_iz$, for $i \geq r + 1$. As $z$ lies in the socle of $I$, we have $n - r \leq \epsilon$, thus proving the assertion. \( \square \)

**Remark 3.5.** We are going to use slightly modified versions to take into account the possibility that $I$ is not contained in $m^2$. Denote by $\epsilon_0$ the dimension of the vector space $(I + m^2)/m^2$. (For later usage, we denote it $\epsilon_0(I, z)$.) If $\epsilon_0 > 0$, say $\{x_1, \ldots, x_n\} \subseteq I$ is part of a minimal set of generators of $I$, we can mod out these elements, we may still apply Proposition 3.3.a, provided $\epsilon - \epsilon_0 \geq 2$ and still get the equality $\nu(I) = \nu(L) - 1$. As for Proposition 3.3.b, we get the improved bound $\nu(L) \leq \nu(L) + \epsilon - \epsilon_0 - 1$, since $\epsilon - \epsilon_0$ is the embedding dimension of the actual ring.

The second assertion gives a mechanism to bound $\nu(I)$ in terms of $s$ and of $\epsilon$. On the other hand, the first assertion allows for a possible decreased count for the number of generators. This suggests that one should go from $I$ to $m$ by steps that meet the largest possible number of irreducible ideals. We look at this issue now.

The question reduces to the Artinian case. Let $I$ be an ideal and consider sequences

\[ I \subseteq I_1 \subseteq I_2 \subseteq \cdots \subseteq I_n = m, \]

where each $I_i$ is an irreducible ideal.

**Proposition 3.6.** The supremum of the length of these sequences is at most $t$, the nilpotency index of $R/I$. Furthermore, if $A = R[x]_{(m, x)}$ then for the ideal $IA$ the supremum is exactly $\text{nil}(R/I) = \text{nil}(A/IA)$.

**Proof.** We first assume that $R$ is a Gorenstein ring and let $L = \text{ann}(I)$. By duality, we get a chain of ideals

\[ \text{ann}(I) \supset \text{ann}(I_1) \supset \text{ann}(I_2) \supset \cdots \supset \text{ann}(I_n) = \text{ann}(m), \]

where $\text{ann}(I_i) = Rx_i$ is principal for $i = 1, \ldots, n$. In particular we obtain that $x_n = rx_1x_2 \cdots x_{n-1}$, which shows that $n - 1 < t$.

Let us show that the lengths of these chains, in the ring $A$, attain the bound $t$. From $m^t \subseteq I$, we have that $Lm^t = 0$. Suppose we had $Lm^p = 0$ for $p < t$. This would imply $m^p \subseteq 0 : L = I$, by duality, contrary to the definition of $t$.

If $L$ is principal, $L = (e)$, from $em^{t-1} \neq 0$ we would build a sequence $x_1, \ldots, x_t$ of elements of $m$ with $ex_1 \cdots x_t \neq 0$, which would give rise to a corresponding sequence of irreducible ideals $I_t = \text{ann}(ex_1 \cdots x_t)$, of length $t$, as desired.
If \( I \) is not irreducible, \( L \) is not principal, \( L = (c_1, \ldots, c_m) \), pass to the ring \( A \) and consider the element
\[
e = \sum_{j=1}^{m} c_j x^j.
\]
It clear that \( m^e \neq 0 \). We can now build a sequence of irreducible ideals containing \( I \), of length \( t \).

If \( R \) is not Gorenstein, let \( E \) be the injective envelope of the residue field of \( R \) and set \( S \) to be the trivial extension of \( R \) by \( E \).

\[
S = R \times E, \quad \text{multiplication defined by } (a, b)(c, d) = (ac, ad + cb).
\]

\( S \) is a Gorenstein local ring of maximal ideal \( M = m \times E \). If we put \( P' = I \times E \), \( \text{nil}(S/P') = \text{nil}(R/I) \) and we can obviously apply the argument above to the pair \((S, P')\) to derive the same assertion for the pair \((R, I)\).

Let us now derive our first bound for \( \nu(I) \). Since none of the measures introduced above are affected by the passage to \( A \), we can assume the existence of chains of irreducible ideals containing \( I \) with \( t \) elements (actually \( t + 1 \) if \( t > 1 \)). We make a few additional observations. The approach we take is to build \( m \) starting from \( I \) by adding successively socle elements along convenient chains–here we will take the chains of irreducible ideals. If anywhere in the process we get an ideal not contained in \( m^2 \), we can reduce the dimension of \( R \) and the embedding dimension by 1, at least for as long as \( d > 0 \). Finally, we remark that if \( \ell(R/I) \) is very low, then the number of generators can be easily estimated. For example, if \( \ell(R/I) = 3 \), then \( \ell(m/I) = 2 \), from which we get \( \nu(I) \leq e + 1 \). The next case is not much more complicated, and one gets that if \( \ell(R/I) = 4 \) then \( \nu(I) \leq e + 3 \).

A version of Part (a) below is due to H. Schoutens ([9]).

**Theorem 3.7.** Let \((R, m)\) be a Noetherian local ring of embedding dimension \( e \geq 2 \), and let \( I \) be an \( m \)-primary ideal contained in \( m^2 \) of colength \( s \).

(a) If \( \ell(R/I) > 4 \) then
\[
\nu(I) \leq (\ell(R/I) - 4)(e - 1) - \frac{e(e - 7)}{2}.
\]

(b) If \( t \) is the nilpotency index of \( R/I \) then
\[
\nu(I) \leq (\ell(R/I) - t)(e - 1) - \frac{e^2 - 5e + 2}{2} - 1 \leq (s - t - \frac{e}{2} + 2)(e - 1).
\]

**Proof.** (a) We are going to, starting from \( I \), successively add socle elements along the chain of irreducible ideals. If at some point, we add to the ideal \( J \) a socle element \( z \in J : \) \( m \) so that \( L = (J, z) \) is also contained in the chain, we want to use Proposition 3.3 to bound \( \nu(J) \) in terms of \( \nu(L) \).

The argument here is a simplified version of what is needed in Part (b). We pick a chain of ideals
\[
I = L_0 \subset L_1 \subset \cdots \subset L_r \subset L_{r+1} \subset m,
\]
so that \( L_{i+1} = (L_i, z_i) \), where \( z_i \) is a nonzero socle element for \( R/L_i \); furthermore we assume that \( \ell(R/L_{r+1}) = 4 \).

Using Proposition 3.3.a and the remark that follows it, we have the following relationship between \( \nu(L_i) \) and \( \nu(L_{i+1}) \): If \( e_0 = e_0(L_i) > 0 \), then
\[
\nu(L_i) \leq \nu(L_{i+1}) + e - e_0 - 1.
\]
This means that when we reach \( L_\tau \), that satisfies \( e_0(L_\tau) \leq 4 \) since \( \ell(m/L_\tau) = 4 \), from the combined count

\[
\nu(I) \leq \nu(L_{\tau+1}) + (\ell(R/4) - 4)(\epsilon - 1),
\]

we can further take away

\[
1 + 2 + \cdots + (\epsilon - 4) = \frac{(\epsilon - 3)}{2},
\]

without breaking the inequality. Since \( \nu(L_{\tau+1}) \leq \epsilon + 3 \), collecting we get the desired estimate.

(b) The assertions are not changed if we reduce \( R \) mod \( mJ \), so that we may assume that \( R \) is an Artinian local ring. After passage to the extension \( A \), we may assume that we have a chain of irreducible ideals

\[
I \subset I_1 \subset I_2 \subset \cdots \subset I_t = m.
\]

To apply the proposition properly, we want to distinguish the cases when \( J \) is irreducible into two subcases. An irreducible ideal \( J \neq m \) is said to be of type 1 if

\[
\dim m/(J + m^2) = 1;
\]

otherwise \( J \) is said to be of type 2.

We now proceed with the proof. If \( J \) is not irreducible, then we have

\[
\nu(J) \leq \nu(L) + \epsilon - 1,
\]

which is taken into the ledger. If \( J \) is irreducible, say \( J = I_{\alpha_0} \), and of type 2, then \( J \) has a set of minimal generators of the form

\[
J = (x_1, \ldots, x_m, y_1, \ldots, y_n),
\]

where \( x_i \in m \setminus m^2 \), and \( y_j \in m^2 \). Furthermore, because \( J \) is of type 2, \( R' = R/(x_1, \ldots, x_m) \) has embedding dimension at least two and \( JR' \) is an irreducible ideal contained in the square of its maximal ideal. We can again apply Proposition 3.3.b to obtain that \( \nu(J) = \nu(L) - 1 \).

In case \( J \) is of type 1, \( R/J \) has embedding dimension 1, and in particular this is a principal ideal ring. Thus \( J \) and all ideals containing it are generated by \( \epsilon \) elements. Suppose this first occur at the irreducible ideal \( I_{\alpha_0}, t_0 \leq t \). This means that \( I_{\alpha_0} \) and all ideals containing it are generated by \( \epsilon \) elements. The number of “socle” extensions from \( I \) until \( I_{\alpha_0} \) is given by \( \ell(R/I) - \ell(R/I_{\alpha_0}) \). This means that we have

\[
\nu(I) \leq (\ell(R/I) - \ell(R/I_{\alpha_0}) - t_0 + 1)(\epsilon - 1) + \nu(I_{\alpha_0}) - t_0 - \delta,
\]

where \( \delta \) represents the discrepancy in using \( \epsilon - 1 \) instead of \( \epsilon - \epsilon_0 - 1 \), as done in Part (a). We claim that this contributes at least

\[
\delta \geq 1 + 2 + \cdots + (\epsilon - 2) = \frac{(\epsilon - 1)(\epsilon - 2)}{2}.
\]

This is the case because whenever we reach an irreducible ideal \( J \), the socle element \( z \) does not affect \( \epsilon_0 \), that is \( \epsilon_0(J) = \epsilon_0(J, z) \), unless \( \epsilon_0(J) = \epsilon - 1 \), since socle elements of irreducible ideals in rings of embedding dimension at least two are contained in the square of the maximal ideal.

Finally, considering that \( \ell(R/I_{\alpha_0}) = t - t_0 + 1 \), we obtain

\[
\nu(I) \leq (\ell(R/I) - t)(\epsilon - 1) + \epsilon - \frac{(\epsilon - 1)(\epsilon - 2)}{2} - t_0
\]

\[
\leq (\ell(R/I) - t)(\epsilon - 1) - \frac{\epsilon^2 - 5\epsilon + 2}{2} - t_0.
\]

Since \( t_0 \geq 1 \), the estimate is established after a straightforward rewrite. \( \square \)
Table 1. Bounds for the number of generators of \( m \)-primary ideals \( (s = \ell(R/I), \\ t = \mathrm{nil}(R/I), \epsilon = \mathrm{embdim}(R) \geq 2) \).

<table>
<thead>
<tr>
<th>expression</th>
<th>lower bound</th>
<th>upper bound</th>
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</table>
| \( \deg(R) + (d - 1)\ell(R/I) \) | 2.1 | (s - \( \frac{d}{2} \) + 2)(\epsilon - 1) | (3.7(b))
| \( e(I) + d - 1 \) | 3.1(a) | \( d! \cdot \ell(R/I) \deg(R) \) if \( I \subseteq m^2 \) | (3.1(b))
| \( e(I) \leq d! \cdot \ell(R/I) \deg(R) \) if \( I \subseteq m^2 \) | \( d! \cdot \ell(R/I) \deg(R) \) if \( I \subseteq m^2 \) | \( d! \cdot \ell(R/I) \deg(R) \) if \( I \subseteq m^2 \) |

Remark 3.8. This kind of estimation is inherently uneven: If we apply it to the powers \( I^q \) of the ideal \( I \), for \( q \gg 0 \) the value \( \nu(I^q) \) is given by a polynomial in \( q \) of degree \( d - 1 \), while \( \ell(R/I^q) \) grows as a polynomial in \( q \) of degree \( d \). Given that \( \ell(I^q) \) is bounded by a linear polynomial, the expression is asymptotically quite imbalanced.

4. Non-primary Cohen–Macaulay ideals

Let \((R, m)\) be a Cohen–Macaulay local ring of dim \( d \geq 1 \), and let \( I \) be a Cohen–Macaulay ideal of codimension \( g < d \). We seek to extend to this class of ideals some of the improvements on the number of generators for \( I \) as measured against benchmarks bounds such as (2).

A first step consists in examining the class of equimultiple ideals (we assume that \( R \) has infinite residue throughout).

Proposition 4.1. Let \( I \) be an equimultiple Cohen–Macaulay ideal of codimension \( g \geq 1 \), \( J \) one of its minimal reductions.

(a) If \( \sqrt{I} \) is a Cohen–Macaulay ideal

\[ \nu(I) \leq \deg(R/J) + (g - 1)\deg(R/\sqrt{I}). \]

(b) Moreover, if \( I \subseteq (\sqrt{I})^2 \)

\[ \nu(I) \leq \deg(R/J) - \deg(R/\sqrt{I}). \]

Proof. The proof is similar to that of Proposition 3.1. Consider the exact sequences

\[ 0 \rightarrow J/J\sqrt{I} \cong (R/\sqrt{I})^g \rightarrow R/J\sqrt{I} \rightarrow R/J \rightarrow 0, \]

\[ 0 \rightarrow I/J\sqrt{I} \rightarrow R/J\sqrt{I} \rightarrow R/I \rightarrow 0. \]

From the first sequence we get that \( R/J\sqrt{I} \) is a Cohen–Macaulay module, which taken into the other sequence shows that \( I/J\sqrt{I} \) is also a Cohen–Macaulay module. Thus all the modules in these sequence are Cohen–Macaulay of dimension \( d - g \), so that we can compute multiplicities using the rules to compute lengths. From the second sequence we have

\[ \deg(I/J\sqrt{I}) = \deg(R/J\sqrt{I}) - \deg(R/I), \]

while from the other sequence,

\[ \deg(R/J\sqrt{I}) = g\deg(R/\sqrt{I}) + \deg(R/J). \]

It follows that

\[ \deg(I/J\sqrt{I}) = \deg(R/J) + (g - 1)\deg(R/\sqrt{I}) - \deg(\sqrt{I}/I). \]

If we drop the last summand and take into account that \( \deg(I/J\sqrt{I}) \geq \nu(I/J\sqrt{I}) \) (this is a Cohen–Macaulay module), we prove Part (a) since

\[ \nu(I/J\sqrt{I}) \geq \nu(I/I\sqrt{I}) \equiv \nu(I). \]
On the other hand, if $I \subset (\sqrt{I})^2$, 
\[ \deg(\sqrt{I}/I) \geq \deg(\sqrt{I}/(\sqrt{I})^2) \geq g\deg(R/\sqrt{I}), \]
and the estimation above shows that 
\[ \deg(I/J\sqrt{I}) \leq \deg(R/J) - \deg(R/\sqrt{I}), \]
which proves Part (b). \qed

 Remark 4.2. The requirement on $\sqrt{I}$ is strict. For example, if $I = (x^2y^2 + z^3w, x^3z + y^2z^2 + xyzw) \subset k[x, y, z, w]$, a computation with Macaulay will show that $\sqrt{I}$ is not Cohen–Macaulay.

The other approach to the estimation of $\nu(I)$, in the case $g < d$, is that of hyperplane sections: we can pick a regular sequence $z = z_1, \ldots, z_{d-g}$ that is superficial for both rings, $R/I$ and $R$, and if $J$ is a regular sequence, for $R/J$ as well. This has for effect, $\deg(R/I) = \ell(R/(I, z))$, and similar formulas for $\deg(R)$ and $\deg(R/J)$ (in case $J$ itself is Cohen–Macaulay, as when $I$ is equimultiple). At the same time, $\nu(I) = \nu(I, z) = (d - g)$. Note that if $I$ is equimultiple, 
\[ e(I, z) = e(J, z) = \deg(R/J). \]
Let us integrate these observations into Propositions 3.1(a):

**Proposition 4.3.** For these ideals, 
\[ \nu(I) \leq e(I, z) + g - 1. \]
	Moreover, if $I$ is equimultiple with a minimal reduction $J$ 
\[ \nu(I) \leq \deg(R/J) + g - 1. \]

**References**


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