Math 311: Advanced Calculus

Wolmer V. Vasconcelos

Set 6

Spring 2010
Main Goal

Understand

Study of Sequences and Series of Functions
Consider the function of last hourly

\[ G(x) = \int_0^x e^{t^2} \, dt. \]

**Question:** How to evaluate \( G(1) \)?

We are going to make use of something we know already

\[ e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \]

and do lots of reckless arithmetic:
\[ G(1) = \int_0^1 \left( \sum_{n=0}^{\infty} \frac{(t^2)^n}{n!} \right) dt \]

\[ = \sum_{n=0}^{\infty} \int_0^1 \frac{t^{2n}}{n!} dt \]

\[ = \sum_{n=0}^{\infty} \frac{1}{n!(2n+1)} \]

\[ = \]
## Outline

1. Main Goal
2. **Properties of Infinite Series**
3. Workshop #10
4. Uniform Convergence and Differentiability
5. Series of Functions
6. Power Series
7. Taylor Series
8. Workshop #11
9. Old Finals

---

Wolmer Vasconcelos
Advanced Calculus
Convergence of Series

Given the series

\[ \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots \]

there are two sequences associated to it

- The sequence of terms, \((a_n)\) and
- The sequence of partial sums, \((s_n)\),

\[ s_n = a_0 + a_1 + \cdots + a_n \]

We say the series converges to \(A \in \mathbb{R}\) if \(\lim s_n = A\). We write this as

\[ \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots = A \]
A cautionary tale

We pick the alternating harmonic series—which we know to be convergent—and carry out arithmetic operations: See what happens

\[
S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots
\]

\[
\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots
\]

\[
S + \frac{1}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \cdots
\]

Thus \(S + \frac{1}{2}S = \frac{3}{2}S\) is just a rearrangement of \(S\)! The arithmetic is saying instead that

\[
\frac{3}{2}S = S!
\]
Algebraic Limit Theorem for Series

**Theorem**

If \( \sum_{k=1}^{\infty} a_k = A \) and \( \sum_{k=1}^{\infty} b_k = B \), then:

1. \( \sum_{k=1}^{\infty} ca_k = cA \) for all \( c \in \mathbb{R} \) and
2. \( \sum_{k=1}^{\infty} (a_k + b_k) = A + B \).

**Proof.** (i) To show \( \sum_{k=1}^{\infty} ca_k = cA \), we consider the sequence of partial sums

\[
t_n = ca_1 + ca_2 + \cdots + ca_n.
\]

Since \( \sum_{k=1}^{\infty} a_k = A \), its sequence of partial sums

\[
s_n = a_1 + a_2 + \cdots + a_n
\]

converges to \( A \). By the Algebraic Limit Theorem for Sequences, \( \lim t_n = c \lim s_n = cA \).
(ii) To show that \( \sum_{k=1}^{\infty} (a_k + b_k) = A + B \), let \( r_n = a_1 + \cdots + a_n \), \( s_n = b_1 + \cdots + b_n \) be the partial sum terms of the series. The partial sum term of the addition of the two series is

\[
t_n = (a_1 + b_1) + \cdots + (a_n + b_n) = (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) = r_n + s_n.
\]

By the Algebraic Limit Theorem for Sequences,

\[
\lim t_n = \lim r_n + \lim s_n = A + B.
\]
Other operations are harder:

**Question:** Given two series, \( a_0 + a_1 + a_2 + \cdots + a_n + \cdots \) and \( b_0 + b_1 + b_2 + \cdots + b_n + \cdots \), what is

\[
(a_0 + a_1 + a_2 + \cdots + a_n + \cdots)(b_0 + b_1 + b_2 + \cdots + b_n + \cdots) = ?
\]

Part of the issue arises from the **distributive rule**. We will offer a partial fix later.
Cauchy Criterion for Series

**Definition**

A sequence \((a_n)\) is called a **Cauchy sequence** if, for every \(\epsilon > 0\), there is an \(N \in \mathbb{N}\) such that whenever \(m, n \geq N\) it follows that \(|a_n - a_m| < \epsilon\).

Recall:

**Theorem**

A sequence converges if and only if it is a Cauchy sequence.

We apply this criterion to the sequence \((s_n)\) of partial sums of a series \(\sum_{k=1}^{\infty} a_k\). Note that

\[
|s_m - s_n| = |a_{m+1} + \cdots + a_n|
\]
Cauchy Test for Series

**Theorem**

The series \( \sum_{k=1}^{\infty} a_k \) converges if and only if given \( \epsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that whenever \( n > m \geq N \) it follows that

\[
|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.
\]

**Proof.** Just observe

\[
|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon,
\]

and apply the Cauchy’s Criterion for sequences.

**Corollary**

If the series \( \sum_{k=1}^{\infty} a_k \) converges, then \( (a_k) \to 0 \).
Example

Consider the geometric series \((1 > q \geq 0)\)

\[1 + q + q^2 + \cdots + q^n + \cdots\]

The difference of partial sums \(s_n - s_m\) is

\[s_n - s_m = q^{m+1} + \cdots + q^n\]

\[= q^{m+1}(1 + q + \cdots + q^{n-m})\]

\[= q^{m+1} \frac{1 - q^{n-m+1}}{1 - q}\]

\[\leq q^{m+1} \frac{1}{1 - q} \leq q^N \frac{1}{1 - q}, \quad n, m \geq N\]
Converse?

**Question:** Is a series whose sequence of terms $a_n$ converges to 0 convergent? This one is easy:

**Answer:** No. The (harmonic) series

$$1 + 1/2 + 1/3 + \cdots + 1/n + \cdots$$

has $1/n \to 0$ but it is divergent.
Comparisons

Given two series $\sum_{k \geq 1} a_k$ and $\sum_{k \geq 1} b_k$ that loosely connected we seek to link their convergence/divergence:

**Theorem (Comparison Test)**

Assume $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.

1. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.
Proof. Both follow from Cauchy’s Criterion applied to the partial sums

\[ |a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + a_{m+2} + \cdots + b_n| \]

If, for instance, given \( \epsilon > 0 \) we can find \( N \) so that for \( n, m > N \)
\[ |b_{m+1} + a_{m+2} + \cdots + b_n| < \epsilon, \] then the same condition will apply to the \( a_n \).
We know that the **harmonic series**, \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges. It is clear that the same happens if we form the series \( \sum_{n=N}^{\infty} \frac{1}{n} \) where \( N \) is some fixed number \( N \geq 1 \).

If \( a \) and \( b \) are positive numbers, consider the series [called generalized harmonic series] whose terms are given by the rule:

\[
\frac{1}{a'}, \frac{1}{a+b'}, \frac{1}{a+2b'}, \ldots, \frac{1}{a+nb'}, \ldots
\]

We claim that this series is also divergent: We compare the terms to a multiple of the harmonic series

\[
\frac{1}{a+bn} \geq \frac{1}{n+bn} = \frac{1}{b+1} \cdot \frac{1}{n}, \quad n \geq a
\]
Absolute Convergence Test

If $\sum_{n=1}^{\infty} a_n$ is a series of non-negative terms, its partial sums

$$s_n = a_1 + a_2 + \cdots + a_n, \quad s_{n+1} = s_n + a_n$$

is a monotone sequence. Therefore, by the criterion, the series converges exactly when the sequence $(s_n)$ is bounded.

We make use of this:

**Theorem (Absolute Convergence Test)**

*If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} a_k$ converges as well.*
Proof of the Absolute Convergence Test

1. We make use of Cauchy criterion for series: Let $\epsilon > 0$. Since the series $\sum_{k=1}^{\infty} |a_k|$ converges, there exists $N$ so that

$$|a_{n+1}| + |a_{n+1}| + \cdots + |a_m| < \epsilon \quad m \geq n > N$$

2. By the triangle inequality (one that say $|a + b| \leq |a| + |b|$), we get

$$|a_{n+1} + a_{n+1} + \cdots + a_m| < \epsilon \quad m \geq n > N$$

3. Therefore the series $\sum_{k=1}^{\infty} a_k$ satisfies the Cauchy condition and therefore converges.
Converse?

The series

\[1 - \frac{1}{2} + \frac{1}{3} - \cdots (-1)^{n-1} \frac{1}{n} + \cdots\]

is convergent (alternating harmonic series) (the one that won a Grammy’s Award), but the series of the absolute values is

\[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots,\]

is divergent.
Alternating Series

An alternating series is one with consecutive terms have opposite signs. One group of them is easy to study:

**Theorem (Alternating Series Test)**

Let \((a_n)\) be a sequence satisfying

1. \(a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots\), and
2. \((a_n) \to 0\).

Then the alternating series \(\sum_{n=1}^{\infty} (-1)^{n+1} a_n\) converges.

In other words: If \((a_n)\) is a decreasing sequence of positive terms then

\[
\sum_{n=1}^{\infty} (-1)^{n+1} a_n \text{ converges if and only if } \lim_{n \to \infty} a_n = 0
\]
Proof. Observe the odd and even sequences of partial sums

\[ s_1 = a_1 \geq s_3 = a_1 - (a_2 - a_3) \geq s_5 = s_3 - (a_4 - a_5), \ldots \]

\[ s_2 = a_1 - a_2 \leq s_4 = s_2 + (a_3 - a_4) \leq s_5 = s_3 + (a_5 - a_6), \ldots \]

They are monotone and bounded: Since \((a_n) \to 0\), there exists \(a_n \leq K\), \(s_{2n} = s_{2n-1} + a_{2n} \leq s_{2n-1} + K \leq a_1 + K\), therefore the even sequence is increasing and bounded. Thus it has a limit \(\ell_1\). Similarly, the other sequence is decreasing and with a lower bound, so it has a limit \(\ell_2\). Since \(\pm a_n = s_n - s_{n-1}\) converges to 0, \(\ell_1 = \ell_2\).
Rearrangements

Definition

Let $\sum_{k \geq 1} a_k$ be a series. A series $\sum_{k \geq 1} b_k$ is said to be a rearrangement of $\sum_{k \geq 1} a_k$ if there exists a 1–1, onto function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $b_{f(k)} = a_k$ for all $k \in \mathbb{N}$.

Consider the geometric series of ratio $q$

$$1 + q + q^2 + q^3 + \cdots + q^n + \cdots$$

Now we shuffle the terms

$$q + 1 + q^3 + q^2 + q^5 + q^4 + \cdots$$

This is not a geometric series, but we should expect its fate linked to the first series. The next result says this.
Series of Positive Terms

Theorem (Dirichlet)

The sum of a series of positive terms [convergence/divergence] is the same in whatever order [rearrangement] the terms are taken.

Proof. Let \( a_0 + a_1 + a_2 + \cdots + a_n + \cdots \) be a series of positive terms of sum \( s \). Then any partial sum of rearrangement \( b_0 + b_1 + b_2 + \cdots + b_n + \cdots \) is bounded by \( s \). Thus the second is convergent and its sum \( t \) is bound by \( s \). We reverse the roles to obtain \( s \leq t \).
Product of Series

Question: Given two series, \(a_0 + a_1 + a_2 + \cdots + a_n + \cdots\) and \(b_0 + b_1 + b_2 + \cdots + b_n + \cdots\), what is

\[(a_0 + a_1 + a_2 + \cdots + a_n + \cdots)(b_0 + b_1 + b_2 + \cdots + b_n + \cdots) = ?\]
The issue is: we have all the products $a_m b_n$ that can be organized into many different series, and then grouped. For instance, if we list the $a_m b_n$ as the double array, we
We could try the following: **Define** the product as the series

\[ a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots \]

Makes sense? [Discuss] Will see another rearrangement soon.

\[
\begin{array}{cccccc}
  a_0 b_0 & a_1 b_0 & a_2 b_0 & a_3 b_0 & \ldots \\
  a_0 b_1 & a_1 b_1 & a_2 b_1 & a_3 b_1 & \ldots \\
  a_0 b_2 & a_1 b_2 & a_2 b_2 & a_3 b_2 & \ldots \\
  a_0 b_3 & a_1 b_3 & a_2 b_3 & a_3 b_3 & \ldots \\
  \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]
The partial sums remind us how polynomials are multiplied

\[(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)(b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m)\]

\[= \sum_{k=0}^{m+n} \left( \sum_{0 \leq i \leq k} a_i b_{k-i} \right) x^k\]

\[a_0 b_0, \ a_0 b_1 + a_1 b_0, \ a_0 b_2 + a_1 b_1 + a_2 b_2, \ldots\]

Another aspect of this definition is:

**Theorem**

If \(\sum_{n \geq 0} a_n\) and \(\sum_{n \geq 0} b_n\) are two convergent series of positive terms, and \(s\) and \(t\) are their respective sums, then the third series is convergent and has the sum \(st\).
Out of all products $a_m b_n$, the ‘product’ above is given in terms of the diagonals

\[
\begin{align*}
a_0 b_0 & \quad a_1 b_0 & \quad a_2 b_0 & \quad a_3 b_0 & \quad \cdots \\
a_0 b_1 & \quad a_1 b_1 & \quad a_2 b_1 & \quad a_3 b_1 & \quad \cdots \\
a_0 b_2 & \quad a_1 b_2 & \quad a_2 b_2 & \quad a_3 b_2 & \quad \cdots \\
a_0 b_3 & \quad a_1 b_3 & \quad a_2 b_3 & \quad a_3 b_3 & \quad \cdots \\
\cdots & \quad \cdots & \quad \cdots & \quad \cdots & \quad \cdots
\end{align*}
\]

$a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_2, \ldots$ whose partial sums don’t write conveniently:

\[
p_n = (a_0 b_0) + (a_1 b_0 + a_1 b_0) + (a_2 b_0 + a_1 b_1 + a_0 b_2) + \cdots
\]
We want to re-write the terms of the product series differently:

\[
\begin{align*}
   &a_0b_0 \quad a_1b_0 \quad a_2b_0 \quad a_3b_0 \quad \ldots \\
   &a_0b_1 \quad a_1b_1 \quad a_2b_1 \quad a_3b_1 \quad \ldots \\
   &a_0b_2 \quad a_1b_2 \quad a_2b_2 \quad a_3b_2 \quad \ldots \\
   &a_0b_3 \quad a_1b_3 \quad a_2b_3 \quad a_3b_3 \quad \ldots \\
   &\ldots \quad \ldots \quad \ldots \quad \ldots
\end{align*}
\]

\[a_0b_0, (a_0 + a_1)(a_0 + a_1) - a_0b_0,\]
\[(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) - (a_0 + a_1)(b_0 + b_1), \ldots\]
whose \(n\)th partial sum is

\[(a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n),\]

a sequence that converges to \(st\) by the Algebraic Limit Theorem.
Observe that

\[ p_n = (a_0 b_0) + (a_1 b_0 + a_0 b_1) + \cdots + (a_0 b_n + \cdots + a_n b_0) \leq \]

\[(a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n)\]

on one hand and

\[ p_n \geq (a_0 + a_1 + \cdots + a_{n/2})(b_0 + b_1 + \cdots + b_{n/2}) \]

Since the terms at the ends converge to \( st \), \((p_n) \rightarrow st\) as well.
Theorem

If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then any rearrangement of this series converges to the same limit.

Proof. Assume $\sum_{k\geq 1} a_k$ converges absolutely to $A$, and let $\sum_{k\geq 1} b_k$ be an rearrangement of $\sum_{k\geq 1} a_k$. Let

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

and

$$t_n = \sum_{k=1}^{n} b_k = b_1 + b_2 + \cdots + b_n$$

be the corresponding partial sums.

Let $\epsilon > 0$. Since $(s_n) \to A$, choose $N_1$ such that

$$|s_n - A| < \epsilon/2$$
Because the convergence is absolute, we can choose $N_2$ so that

$$\sum_{k=m+1}^{n} |b_k| < \frac{\epsilon}{2}$$

for all $n > m \geq N_2$. Take $N = \max\{N_1, N_2\}$. We know that the terms $\{a_1, a_2, \ldots, a_N\}$ must all appear in the rearranged series, and we move far out enough in the series $\sum_{k \geq 1} b_k$ that these terms are all included. Thus, choose $M = \max\{f(k) | 1 \leq k \leq N\}$.

It is clear that if $m \geq M$, then $(t_m - s_N)$ consists of a finite number of terms, the absolute values of which appear in the tail of $\sum_{k=N+1}^{\infty} |a_k|$. The earlier choice of $N_2$ guarantees $|t_m - s_N| < \frac{\epsilon}{2}$, and so

$$|t_m - A| = |t_m - s_N + s_N - A|$$

$$\leq |t_m - s_N| + |s_N - A| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
Theorem (Integral Test)

Let \( \sum_{n \geq 0} a_n \) be a series of positive terms. If there is a decreasing function \( f(x) \) such that \( a_n \leq f(n) \) for large \( n \) and

\[
\int_{x=1}^{\infty} f(x) \, dx < \infty,
\]

then \( \sum_{n \geq 0} a_n \) converges.

Proof. If \( a_n \leq f(n) \) for \( n \geq n_0 \), since \( f(x) \) is decreasing, \( a_n \leq \int_{n-1}^{n} f(x) \, dx, \quad n > n_0 \). From this, and the assumption that \( \int_{1}^{\infty} f(x) \, dx < \infty \), we get that the partial sums of the series \( \sum_{n \geq 0} a_n \) are bounded, and therefore converge by the theorem on bounded monotone sequences. \( \square \)
The series
\[ 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots, \]
for \( p > 1 \) will always converge. Its sum is denoted by \( \zeta(p) \).

For example, \( \zeta(2) = \frac{\pi^2}{6} \).
This function is actually defined for all complex numbers \( p \)
whose real part is \( > 1 \). It is known as Riemann zeta function. It
is probably the most famous function of Mathematics.
Let us show that

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots,$$

for $p > 1$ will always converge.

We are going to bound each term $1/n^p$ by the terms of another series, and then argue the new series converges.
Consider the function \( f(x) = \frac{1}{x^p}, \ x \geq 2 \). This is a decreasing function (draw the graph).

Observe

\[
\frac{1}{n^p} \leq \int_{x=n-1}^{n} \frac{1}{x^p} \, dx
\]

Therefore its partial sums are bounded by

\[
s_n \leq 1 + \int_{x=1}^{n} \frac{dx}{x^p} = 1 + \frac{1}{p-1} \left[ 1 - \frac{1}{n^{p-1}} \right] < 1 + \frac{1}{p-1}
\]
Examples

The series in earlier Workshop satisfies

\[ \sum_{n \geq 1} \frac{1}{n(n + 1)} \leq \sum_{n \geq 1} \frac{1}{n^2}, \]

which is convergent.

In the same manner, if

\[ \sum_{n \geq 1} \frac{p(n)}{q(n)}, \]

where \( p(n) \) and \( q(n) \) are positive polynomial expressions with \( \deg q \geq 2 + \deg p \), then the series converges by the same reason. Do it!
Exam Type Exercises

1. Show that
\[ \sum_{n \geq 0} (-1)^n \frac{2n + 3}{(n + 1)(n + 2)} = 1. \]

2. Determine the values of \( q \) for which the series
\[ q + 2q^2 + 3q^3 + \cdots + nq^n + \cdots \]
is convergent.

3. Show that \( \sum_{n \geq 2} \frac{1}{n(\ln n)^p} \) converges if \( p > 1 \), and diverges if \( p \leq 1 \).
Ratio Tests

There are very useful tests involving the ratio $a_{n+1}/a_n$ of two successive terms of a series. Sometimes we compare the ratio $a_{n+1}/a_n$ to another $b_{n+1}/b_n$. In these we suppose that $a_n$ and $b_n$ are strictly positive.

Suppose $a_n, b_n > 0$ and that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for sufficiently large $n$, that is for $n \geq n_0$.

Then

$$a_n = \frac{a_{n_0+1}}{a_{n_0}} \cdot \frac{a_{n_0+2}}{a_{n_0+1}} \cdot \ldots \cdot \frac{a_n}{a_{n-1}} a_{n_0} \leq \frac{b_{n_0+1}}{b_{n_0}} \cdot \frac{b_{n_0+2}}{b_{n_0+1}} \cdot \ldots \cdot \frac{b_n}{b_{n-1}} a_{n_0} = \frac{a_{n_0}}{b_{n_0}} b_n = Cb_n, \quad C = \frac{a_{n_0}}{b_{n_0}}.$$
Here are some applications:

Theorem

Let \( \sum a_n \) and \( \sum b_n \) be series of positive terms.

1. If for \( n \geq n_0 \)
   \[
   \frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n},
   \]
   and the series \( \sum b_n \) converges, then \( \sum a_n \) converges also.

2. If for \( n \geq n_0 \)
   \[
   \frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n},
   \]
   and the series \( \sum a_n \) diverges, then \( \sum b_n \) diverges also.
Theorem (d’Alambert Test)

The series $\sum a_n$ is convergent if $a_{n+1}/a_n \leq r$, where $r < 1$, for all sufficiently large $n$. 
Theorem

Given a series \( \sum_{n \geq 1} a_n \) with \( a_n \neq 0 \), if \( (a_n) \) satisfies

\[
\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,
\]

then the series converges absolutely.

Proof.

1. Let \( r' \) satisfy \( r < r' < 1 \). For \( \epsilon = r' - r \), there is \( N \) such that for \( n \geq N \) \( \left| \frac{a_{n+1}}{a_n} - r \right| < \epsilon \), and therefore

\[
\left| \frac{a_{n+1}}{a_n} \right| - r \leq \left| \frac{a_{n+1}}{a_n} - r \right| < \epsilon = r' - r,
\]

giving \( \left| a_{n+1} \right| \leq r' \left| a_n \right| \) for \( n \geq N \).

2. The above shows that the series \( \sum_{n=N}^{\infty} \left| a_n \right| \) satisfies

\[
\left| a_n \right| \leq \left| a_N \right| (r')^{n-N},
\]

a geometric series of ratio \( r' < 1 \), which converges.
A quick application of the ratio test:
We claim that the series

\[ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]

converges for all values of \( x \).

For the ratio of consecutive terms

\[ \frac{a_{n+1}}{a_n} = \frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x}{n+1} \]

so that for any \( x \), \( \lim a_{n+1}/a_n = 0 \).

This is a well used technique for power series.
Examples

1. For the series $\sum_{n \geq 1} \frac{n}{2^n}$ we invoke the ratio test:

$$\frac{a_{n+1}}{a_n} = \frac{n + 1}{2^{n+1}} \cdot \frac{\frac{n}{2^n}}{1} = \frac{n + 1}{n} \cdot \frac{1}{2}$$

which has limit $1/2 < 1$. So the series converges.

2. Decide [with justification] whether the series

$$\sum_{n \geq 1} \frac{n!}{n^n},$$

is convergent or divergent?
Exercises

1. Show that if \( a_n > 0 \) and \( \lim n a_n = L \), with \( L \neq 0 \), then the series \( \sum a_n \) diverges.

2. Show that if \( a_n > 0 \) and \( \lim n^2 a_n = L \), with \( L \neq 0 \), then the series \( \sum a_n \) converges.

3. Find examples of two series \( \sum a_n \) and \( \sum b_n \) both of which diverge but for which \( \sum \min\{a_n, b_n\} \) converges. To make it more difficult, choose examples where \( (a_n) \) and \( (b_n) \) are positive and decreasing.
Let $\sum_{n \geq 1} a_n$ be a series of positive terms. We are going to examine how the limit
\[ \lim_{n \to \infty} \sqrt[n]{a_n} \]
is used to decide convergence. We recall one special calculation of these limits: If $x > 0$
\[ \lim_{n \to \infty} \sqrt[n]{x} = 1 \]
Recall another limit: $\lim_{n \to \infty} \sqrt[n]{n} = 1$. 
Root Test

**Theorem**

If $\sum_{n \geq 1} a_n$ is a series of positive terms and $\lim_{n \to \infty} \sqrt[n]{a_n} = r < 1$, then the series converges.

**Proof.** Let $r < r' < 1$ and pick $\epsilon = r' - r$. This is the same subtle point we used above.

1. There is $N$ so that for $n > N$

   $$\left| \sqrt[n]{a_n} - r \right| < \epsilon$$

2. This implies that $\sqrt[n]{a_n} < r + \epsilon = r' < 1$ for $n > N$. As a consequence

   $$a_n < (r')^n$$
Example

Consider the series (for \( q > 0 \))

\[
1 + q + 2q^2 + \cdots + nq^n + \cdots
\]

We invoke the root test

\[
\lim_{n \to \infty} \sqrt[n]{nq^n} = q \quad \lim_{n \to \infty} \sqrt[n]{n} = q
\]

Therefore it converges if \( q < 1 \)
Let us calculate the sum of the series. For that we must have an inkling on how the series arose from the geometric series. At these times we replace \( q \) by \( x \) and recall:
Nice calculation

1. Differentiate the ‘equality’
\[
\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots
\]

2. To get almost our series
\[
\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots
\]

3. Now multiply by \(x\) and add 1
\[
1 + \frac{x}{(1-x)^2} = 1 + x + 2x^2 + \cdots + nx^n + \cdots
\]

4. Thus for \(0 < q < 1\) the series sums to \(1 + \frac{q}{(1-q)^2}\).
Outline

1. Main Goal
2. Properties of Infinite Series
3. Workshop #10
4. Uniform Convergence and Differentiability
5. Series of Functions
6. Power Series
7. Taylor Series
8. Workshop #11
9. Old Finals
1. If $a$ is a positive integer, prove that the series

$$\sum_{n \geq 1} \frac{1}{n(a + n)}$$

converges. Find its sum.

2. If $b > a > 0$, do the same for the series

$$\sum_{n \geq 1} \frac{1}{n(a + n)(b + n)}.$$
3: Argue by induction that for any sequence of integers
0 < a_1 < a_2 < \ldots < a_r, the series
\[
\sum_{n \geq 1} \frac{1}{n(a_1 + n)(a_2 + n) \cdots (a_r + n)}
\]
converges and its sum can be effectively computed.

4: Given the series
\[
\sum_{n \geq 0} \frac{1}{n^2 + 1}
\]
- Prove by comparison and by a direct application of the integral test that it converges.
- Try to find its sum somehow/somewhere.
- Google it.
Outline

1. Main Goal
2. Properties of Infinite Series
3. Workshop #10
4. **Uniform Convergence and Differentiability**
5. Series of Functions
6. Power Series
7. Taylor Series
8. Workshop #11
9. Old Finals
Sequences of Functions

Let $f_n : A \to \mathbb{R}$, $n \in \mathbb{N}$, be a set of functions. For each $x \in A$ they define a numerical sequence $(f_n(x))$. If $f_n(x) \to L$, we say that $(f_n)$ converges at $x$. We are greatly interested in case it converges to all $x \in A$, as the limit

$$f_n(x) \to f(x)$$

will define a function $f : A \to \mathbb{R}$.

1. If the $f_n$ are continuous, when is $f$ continuous?
2. If the $f_n$ are differentiable, when is $f$ differentiable?
Example

Let $f_n(x) = x^n$, $n \in \mathbb{N}$, be the sequence of powers of $x$ as functions on $[0, 1]$. For any $x$ in this interval, we have

$$
\lim_{n \to \infty} f_n(x) = 0, \quad 0 \leq x < 1
$$

$$
\lim_{n \to \infty} f_n(x) = 1, \quad x = 1
$$

Thus $\lim_{n \to \infty} f_n$ exists for all $x \in [0, 1]$, but it is not a continuous function on the interval.

We need a rule that guarantees that $\lim_{n \to \infty} f_n$ is continuous.
Pointwise and Uniform Convergence

**Definition**

The sequence of functions \((f_n(x))\) converges **pointwise** to \(f(x)\) if for every \(x\) \(f_n(x)\) converges to \(f(x)\). For a given \(x\), this means that given \(\epsilon > 0\) there is \(N = N(x) \in \mathbb{N}\) such that for \(n \geq N\),

\[|f_n(x) - f(x)| < \epsilon.\]

Another definition of convergence is much more restrictive:

**Definition**

The sequence of functions \((f_n(x))\) converges uniformly to \(f(x)\) if for every \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that for \(n \geq N\),

\[|f_n(x) - f(x)| < \epsilon.\]
Example: Let $f_n(x) = \frac{1}{n(1 + x^2)}$. Then $f(x) = \lim_{n \to \infty} f_n(x) = 0$. Given $\epsilon > 0$

$$|f_n(x) - f(x)| < \frac{1}{n}$$

Thus if $N \geq \frac{1}{\epsilon}$,

$$|f_n(x) - f(x)| < \epsilon$$

for $n \geq N$. 
Cauchy Criterion for Uniform Convergence

**Theorem**

A sequence of functions \((f_n(x))\) defined on a set \(A \subset \mathbb{R}\) converges uniformly on \(A\) if and only if for every \(\epsilon > 0\) there exists \(N \in \mathbb{N}\) such that \(|f_n(x) - f_m(x)| < \epsilon\) for all \(n, m \geq N\) and all \(x \in A\).

**Proof.** \(\Rightarrow:\) For each \(x \in A\), the numerical Cauchy sequence \((f_n(x))\) converges: Call the limit \(f(x)\). Now we argue that \(f_n\) converges to \(f\) uniformly. Let \(\epsilon > 0\) and let \(N\) be such that \(|f_n(x) - f_m(x)| < \epsilon\) for \(n, m \geq N\). Now we use the argument used in the numerical case.
Let $\epsilon > 0$. Because the sequence $f_n$ is Cauchy, there exists $N$ such that for all $n, m \geq N$ and all $x \in A$,

$$|f_n(x) - f_m(x)| < \epsilon/2.$$ 

On the other hand, for each $x \in A$ the sequence $f_n(x) \to f(x)$, so there is $N_K$

$$|f_{N_K}(x) - f(x)| < \epsilon/2.$$ 

Thus for all $x \in A$ and all $n \geq N_K$

$$|f_n(x) - f(x)| = |f_n(x) - f_{N_K}(x) + f_{N_K}(x) - f(x)|$$

$$\leq |f_n(x) - f_{N_K}(x)| + |f_{N_K}(x) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon$$
Theorem

If the sequence of continuous functions \((f_n(x))\) converges uniformly to \(f(x)\), then \(f(x)\) is continuous (on the same domain).

Proof. Let \(x = c\) be a point in the domain. Given \(\varepsilon > 0\), we must show that there exists \(\delta > 0\) such that if \(0 < |x - c| < \delta\), then \(|f(x) - f(c)| < \varepsilon\). The idea is to write

\[
    f(x) - f(c) = (f(x) - f_n(x)) + (f_n(x) - f_n(c)) + (f_n(c) - f(c))
\]

and use uniform convergence on the first and third terms and continuity on the second.
\[ |f(x) - f(c)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \]

\[ |f(x) - f_n(x)| < \frac{\epsilon}{3}, \quad n \geq N \]
\[ |f_n(x) - f_n(c)| < \frac{\epsilon}{3}, \quad 0 < |x - c| < \delta \]
\[ |f(c) - f_n(c)| < \frac{\epsilon}{3}, \quad n \geq N \]

Thus, for \( 0 < |x - c| < \delta \),

\[ |f(x) - f(c)| < \epsilon. \]
Theorem

Let $f_n \to f$ pointwise on interval $[a, b]$ and assume each $f_n$ is differentiable. If $(f'_n)$ converges uniformly on $[a, b]$ to a function $g$, then $f$ is differentiable and $f' = g$.

Proof. Let $\epsilon > 0$ and fix $c \in [a, b]$. We will argue that $f'(c)$ exists and it is equal to $g(c)$. We begin with

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

and claim we can find $\delta > 0$ so that for $0 < |x - c| < \delta$

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon.$$
\[
\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| \leq \left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \\
+ \left| \frac{f_n(x) - f_n(c)}{x - c} - f'(c) \right| + \left| f'(c) - g(c) \right|
\]

We will argue that we can find \( \delta \) so that each of the three terms \(< \epsilon/3. \)
Apply the MVT to $f_n - f_m$ on $[c, x]$: there exists $\alpha \in (c, x)$ such that

$$f'_n(\alpha) - f'_m(\alpha) = \frac{(f_n(x) - f_m(x)) - (f_n(c) - f_m(c))}{x - c}. $$

By Cauchy Criterion for Uniform Convergence, there exists $N \in \mathbb{N}$ such that for $n, m \geq N_1$,

$$|f'_n(\alpha) - f'_m(\alpha)| < \frac{\epsilon}{3}$$

Together we have

$$\left| \frac{f_n(x) - f_m(x)}{x - c} - \frac{f_n(c) - f_m(c)}{x - c} \right| < \frac{\epsilon}{3}$$

for all $m, n \geq N_1$, and all $x \in [a, b]$. If we take the limit $f_m \to f$ (making use of the Order Limit Theorem)
\[
\left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| \leq \frac{\epsilon}{3}
\]

Finally, choose \( N_2 \) large enough so that

\[
|f'_m(c) - g(c)| < \frac{\epsilon}{3}
\]

for all \( m \geq N_2 \), and let \( N = \max\{N_1, N_2\} \) Use that \( f_N \) is differentiable to produce \( \delta > 0 \) for which

\[
\left| \frac{f_N(x) - f_N(c)}{x - c} - f'_N(c) \right| < \frac{\epsilon}{3}
\]

whenever \( 0 < |x - c| < \delta \). Substituting in the original expression,

\[
\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon
\]
Theorem

Let \((f_n)\) be a sequence of differentiable functions defined on the interval \([a, b]\) and assume that \((f'_n)\) converges uniformly on \([a, b]\) to a function \(g\). If there exists a point \(x_0 \in [a, b]\) where \((f_n(x_0))\) is convergent, then \((f_n)\) converges uniformly on \([a, b]\).

Proof. For any \(x \in [a, b]\), we have

\[
|f_n(x) - f_m(x)| \leq |(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))| + |f_n(x_0) - f_m(x_0)|
\]

One reduces to the previous proof by applying the MVT to \(f_n - f_m\) on \([x_0, x]\): there exists \(\alpha \in (x_0, x)\) such that

\[
f'_n(\alpha) - f'_m(\alpha) = \frac{(f_n(x) - f_m(x)) - (f_n(x_0) - f_m(x_0))}{x - x_0}.
\]
Combining the two theorems we get

**Theorem**

Let \((f_n)\) be a sequence of differentiable functions defined on the interval \([a, b]\) and assume that \((f'_n)\) converges uniformly on \([a, b]\) to a function \(g\). If there exists a point \(x_0 \in [a, b]\) where \((f_n(x_0))\) is convergent, then \((f_n)\) converges uniformly on \([a, b]\). Moreover, the limit function \(f = \lim f_n\) is differentiable and satisfies \(f' = g\).
Let $f_n \to f$ pointwise on interval $[a, b]$ and assume each $f_n$ is differentiable. If $(f'_n)$ converges uniformly on $[a, b]$ to a function $g$, then $f$ is differentiable and $f' = g$. 

Let $(f_n)$ be a sequence of differentiable functions defined on the interval $[a, b]$ and assume that $(f'_n)$ converges uniformly on $[a, b]$ to a function $g$. If there exists a point $x_0 \in [a, b]$ where $(f_n(x_0))$ is convergent, then $(f_n)$ converges uniformly on $[a, b]$.

Let $(f_n)$ be a sequence of differentiable functions defined on the interval $[a, b]$ and assume that $(f'_n)$ converges uniformly on $[a, b]$ to a function $g$. If there exists a point $x_0 \in [a, b]$ where $(f_n(x_0))$ is convergent, then $(f_n)$ converges uniformly on $[a, b]$. Moreover, the limit function $f = \lim f_n$ is differentiable and satisfies $f' = g$. 

Wolmer Vasconcelos
Advanced Calculus
Outline

1. Main Goal
2. Properties of Infinite Series
3. Workshop #10
4. Uniform Convergence and Differentiability
5. Series of Functions
6. Power Series
7. Taylor Series
8. Workshop #11
9. Old Finals
Series of Functions

**Question:** What do we see in the Infinite Series

\[ \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots = ? \]

**Answer:** At least two things

- The sequence of terms, \((a_n)\) and
- The sequence of partial sums, \((s_n)\),

\[ s_n = a_0 + a_1 + \cdots + a_n \]

- We say the **series converges** to \(S \in \mathbb{R}\) if \(\lim s_n = S\). By abuse of notation, we then replace the \(?\) by \(S\).
**Question:** What do we see in the Infinite Series of Functions \( f_n : A \rightarrow \mathbb{R} \)

\[
\sum_{n=0}^{\infty} f_n = f_0 + f_1 + f_2 + f_3 + \cdots = ?
\]

**Answer:** At least three things

- The sequence of terms, \((f_n)\)
- The sequence of partial sums, \((s_n)\),

\[
s_n = f_0 + f_1 + \cdots + f_n
\]

- We say the series **converges** to \(f(x) \in \mathbb{R}\) if \(\lim f_n(x) = f(x)\).
- Main question: Properties of \(f\)? continuous? differentiable
Reasons Why

Two quick reasons why series of functions are widely (and wildly) used:

1. There are equations for which we do not have explicit (short) formulas of their solutions, e.g.

\[ x^5 + 5x + 6 = 0, \]

yet we are still able to write the solutions as the limits of numerical series

\[ x = \sum_{n \geq 0} a_n. \]

2. Series gives the means to break down some functions into basic blocks:

\[ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \]
Noteworthy Examples

1 Geometric series

\[1 + x + x^2 + \cdots + x^n + \cdots\]

2 Exponential series

\[e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\]

3 Arctangent series

\[x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots\]

It is legitimate to evaluate the last series for \(x = 1\) in order to get

\[\pi/4 = 1 - \frac{1}{3} + \frac{1}{5} + \cdots\]
Continuity of Series of Functions

The guiding theorems:

**Theorem**

Let \( f_n \) be continuous functions on a set \( A \subset \mathbb{R} \), and assume

\[
\sum_{n=1}^{\infty} f_n \text{ converges uniformly to a function } f.
\]

Then, \( f \) is continuous on \( A \).

We need the means to test when the sequence of partial sums

\[
s_n(x) = f_0(x) + f_1(x) + \cdots + f_n(x)
\]

converges uniformly.
Cauchy Criterion

Theorem

A series \( \sum_{n=1}^{\infty} f_n \) converges uniformly on \( A \subseteq \mathbb{R} \) if for every \( \epsilon > 0 \) there is an \( N \in \mathbb{N} \) such that for all \( n > m \geq M \),

\[
|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| < \epsilon
\]

for all \( x \in A \).
Weierstrass M-Test

**Theorem**

For each \( n \in \mathbb{N} \), let \( f_n \) be a function defined on a set \( A \subset \mathbb{R} \), and let \( M_n \) be a real number satisfying

\[
|f_n(x)| \leq M_n
\]

for all \( x \in A \). If \( \sum_{n=1}^{\infty} M_n \) converges, then \( \sum_{n=1}^{\infty} f_n(x) \) converges uniformly on \( A \).

This reduces to Cauchy’s Criterion since

\[
|f_{m+1}(x) + f_{m+2}(x) + \cdots + f_n(x)| \leq M_{m+1} + \cdots + M_n,
\]

for all \( x \in A \). Now we use the Cauchy Criterion for numerical series.
Example

Let

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2},$$

whose terms are bounded by the terms of the convergent series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$ 

It converges uniformly to a continuous function $f(x)$.

The series of derivatives

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n},$$

diverges at $x = 0$ (becomes the harmonic series).
Derivative of a Series

The following gives us a criterion of when we can differentiate a series:

**Theorem**

Let \( f_n \) be differentiable functions defined on the interval \([a, b]\), and assume that \( \sum_{n=1}^{\infty} f'_n \) converges uniformly on \([a, b]\) to a function \( g \) on \([a, b]\). If there exists a point \( x_0 \in [a, b] \) where \( \sum_{n=1}^{\infty} f_n(x_0) \) is convergent, then the series \( \sum_{n=1}^{\infty} f_n \) converges uniformly to a differentiable function \( f(x) \) satisfying \( f' = g \) on \([a, b]\). In other words,

\[
\begin{align*}
  f(x) &= \sum_{n=1}^{\infty} f_n(x), \\
  f'(x) &= \sum_{n=1}^{\infty} f'_n(x)
\end{align*}
\]
Outline

1. Main Goal
2. Properties of Infinite Series
3. Workshop #10
4. Uniform Convergence and Differentiability
5. Series of Functions
6. **Power Series**
7. Taylor Series
8. Workshop #11
9. Old Finals

Wolmer Vasconcelos
Advanced Calculus
A power series is a series of the form

\[ \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots \]

Sometimes instead of \( x^n \) on has \( (x - a)^n \).

These series have, unlike more general series, amenable properties: It will be much simpler to study their convergence, continuity and differentiability.
Basic Theorem

Part of the simplicity is grounded on the following:

**Theorem**

*If a power series* $\sum_{n=0}^{\infty} a_n x^n$ *converges at some point* $x_0 \in \mathbb{R}$, *then it converges absolutely for any* $x$ *satisfying* $|x| < |x_0|$.

**Proof.** If $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then the sequence $a_n x_0^n$ is bounded (in fact, by Cauchy’s, converges to 0). Let $M > 0$ satisfy $|a_n x_0^n| \leq M$ for all $n \in \mathbb{N}$. If $|x| < |x_0|$, then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M \left|\frac{x}{x_0}\right|^n$$
The geometric series

\[ \sum_{n=0}^{\infty} M \left| \frac{x}{x_0} \right|^n \]

converges since its ratio is \(< 1\), so by the Comparison Test, the series \( \sum_{n=0}^{\infty} a_n x^n \) converges absolutely. \( \square \)
Radius of Convergence

Here is a surprising property of power series: If we have a power series

$$\sum_{n=0}^{\infty} a_n x^n,$$

what is like the set of all $x$ (besides $x = 0$) where it converges? Here is part of the answer:

**Corollary**

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. The possible sets of points where it converges are: 0 only; all of $\mathbb{R}$; or an interval $(-R, R)$, possibly with one or both of its boundary points.
$R$: radius of convergence: the largest nonnegative number such that $\sum_{n=0}^{\infty} a_n x^n$ converges for all $|x| < R$.

**Theorem**

The radius of convergence of the series $\sum a_n x^n$ is given by

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided the limit exists or is $+\infty$.

**Proof.** We make use of the Ratio Test: The series converges if the limit

$$\lim_{n \to \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right| = L |x| < 1$$

and diverges if $L |x| > 1$. 

---

Wolmer Vasconcelos
Advanced Calculus
Set 6
From this we conclude: $R = 1/L$ if $L \neq 0$. Also, $R = \infty$ if $L = 0$, and $R = 0$ if $L = \infty$.

1. For the exponential series $\sum \frac{x^n}{n!}$, $R = \lim_{n \to \infty} \frac{(n+1)!}{n!} = \infty$
2. For the geometric series $\sum x^n$, $R = 1$
3. For $\sum n!x^n$, $R = 0$
Radius of Convergence and Differentiation/Integration

Let $f(x) = \sum a_n x^n$, $\sum_{n \geq 1} n a_n x^{n-1}$, and $\sum_{n \geq 1} \frac{1}{n+1} x^{n+1}$

**Theorem**

*The three series have the same radii of convergence.*

**Proof.** Suppose $R$ and $R'$ are the radii of convergence of the first two series. Suppose $|x| < R$, and choose $|x| < |x_0| < R$. Then the first series is convergent with $x = x_0$, and consequently $|a_n x_0^n| \leq A$ for all $n$. 

Wolmer Vasconcelos Set 6

Advanced Calculus
Then

\[ na_n x^{n-1} = \frac{n}{x_0} a_n x_0^n \left( \frac{x}{x_0} \right)^{n-1}, \]

\[ |na_n x^{n-1}| \leq \frac{A}{|x_0|} nr^{n-1}, \]

where

\[ r = \frac{|x|}{|x_0|} < 1. \]

The series

\[ \frac{A}{|x_0|} nr^{n-1} \]

is convergent, for the limit of the ratio of the terms is

\[ \lim_{n \to \infty} \frac{n + 1}{n} r < 1. \]

This proves that the series \( na_n x^{n-1} \) converges and therefore \( R < R' \).
Now we show that $R' > R$ is impossible. Otherwise, pick $x$ so that $R < |x| < R'$. Then the series $\sum na_n x^{n-1}$ is absolutely convergent for this $x$ and the first series is divergent. Now

$$|a_n x^n| = |na_n x^{n-1}| \left| \frac{x}{n} \right| < |na_n x^{n-1}|$$

as soon as $n > |x|$. This comparison shows that the series $\sum |a_n x^n|$ is convergent, a contradiction.
Root Formula

Exercise: Prove that the radius of convergence of the series

$$\sum_{n=0}^{\infty} a_n x^n$$

is given by

$$\frac{1}{R} = \lim_{n \to \infty} \sqrt[n]{a_n}.$$ 

Note: In some early Workshops we had several examples of

$$\lim_{n \to \infty} \sqrt[n]{\text{something}}: \sqrt[n]{n}, \sqrt[n]{a^n + b^n + c^n}$$

Note also the consequence: the series of indefinite integrals will have the same radius of convergence

$$\sum_{n=1}^{\infty} \frac{a_n}{n+1} x^{n+1}$$
Uniform Convergence

**Theorem**

*If a power series* \( \sum_{n=0}^{\infty} a_n x^n \) *converges absolutely at a point* \( |x_0| \), *then it converges uniformly on the closed interval* \([-c, c]\), *where* \( c = |x_0| \).

**Proof.** We use Cauchy Criterion for Uniform Convergence of Series.

By assumption, \( \sum_{n=0}^{\infty} |a_n x^n| < \infty \) so that in particular, for any \( \epsilon > 0 \) there exists \( N \in \mathbb{N} \) such that for \( n > m \geq N \)

\[
|am+1 c^{m+1}| + \cdots |an c^n| < \epsilon
\]

which implies that for all \( x \in [-c, c] \)

\[
|am+1 x^{m+1} + \cdots a_n x^n| \leq |am+1 c^{m+1}| + \cdots |an c^n| < \epsilon
\]
Abel’s Lemma

Lemma

Let $b_n$ satisfy $b_1 \geq b_2 \geq b_3 \geq \cdots \geq 0$, and let $\sum_{n=1}^{\infty} a_n$ be a series for which the partial sums are bounded. In other words, assume there exists $A > 0$ such that

$$|a_1 + a_2 + \cdots + a_n| < A$$

for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$

$$|a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq 2A.$$
The proof uses a technique called **summation by parts**. Let \((x_n)\) and \((y_n)\) be sequences and let \(s_n = x_1 + x_2 + \cdots + x_n\). Note that \(x_j = s_j - s_{j-1}\). Now we verify that

\[
\sum_{j=m+1}^{n} x_j y_j = s_n y_{n+1} - s_m y_{m+1} + \sum_{j=m+1}^{n} s_j (y_j - y_{j+1}).
\]

Note that the two sides as sums \(\sum a_{i,j} x_i y_j\), where \(a_{i,j}\) are integers. To verify this is an identity, it is enough to check that for each \(j\) in the range \(m + 1 \leq i, j \leq n + 1\), taking the partial derivative relative to \(x_i\) followed by that of \(y_j\) we get the same values:

\[
\frac{\partial^2}{\partial x_i \partial y_j} \sum a_{i,j} x_i y_j = a_{i,j}
\]
Abel’s Theorem

Theorem

Let \( g(x) = \sum_{n=1}^{\infty} a_n x^n \) be a power series that converges at the point \( x = R > 0 \). Then the series converges uniformly on the interval \([0, R]\). A similar result holds if the series converges at \( x = -R \).

Proof. We use Cauchy Criterion for Uniform Convergence of Series: Set

\[
g(x) = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} a_n R^n \left( \frac{x}{R} \right)^n.
\]
We must show that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n > m \geq N$

$$|a_{m+1}R^{m+1} \left( \frac{x}{R} \right)^{m+1} + \cdots + a_nR^n \left( \frac{x}{R} \right)^n| < \epsilon$$

Because we are assuming that $\sum_{n=1}^{\infty} a_nR^n$ converges, by Cauchy Criterion for convergent numerical series there exists $N \in \mathbb{N}$ such that

$$|a_{m+1}R^{m+1} + \cdots + a_nR^n| < \epsilon/2$$

for all $n > m \geq N$. By Abel’s Lemma

$$|a_{m+1}R^{m+1}(x/R)^{m+1} + \cdots + a_nR^n(x/R)^n| < 2\epsilon/2 = \epsilon$$
Taylor Series

Let \( f(x) \) be a function defined on a neighborhood of \( x = a \), let us assume its derivatives of all orders exist at \( x = a \), \( f^{(n)}(a) \), \( n \geq 0 \). We can assemble these derivatives into several series, the most important being the **Taylor series** of \( f \) at \( x = a \):

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n.
\]

1. For what values of \( x \), in addition to \( x = a \), does the series converge?
2. When will it converge to \( f(x) \)?
The partial sums of this series are the polynomials

\[ s_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x - a)^i. \]

To see whether \( s_n(x) \to f(x) \), we must examine the difference

\[ f(x) - s_n(x) \]

This is called the remainder of the Taylor series.
Note that the series expresses a relationship between values of \( f \) at different points. We recall a basic result of this kind:

1. If \( f : [a, b] \to \mathbb{R} \) is continuous and \( f'(x) \) exists in \((a, b)\), the MVT says that

\[
f(b) = f(a) + (b - a)f'(c),
\]

for some \( c \in (a, b) \).

2. If we assume more: Suppose \( f'(x) \) is continuous on \([a, b]\) and \( f''(x) \) exists in \((a, b)\):

\[
f(b) = f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2}f''(c),
\]

for some \( c \in (a, b) \).
To prove this, consider the function

\[ g(x) = f(b) - f(x) - (b - x)f'(x) - \frac{(b - x)^2}{(b - a)^2}(f(b) - f(a) - (b - a)f'(a)). \]

Note that it vanishes for \( x = a \) and \( x = b \). Since it is differentiable, by Rolle’s Theorem

\[ g'(c) = 0 \]

for some \( c \in (a, b) \). Since

\[ g'(x) = -(b - x)f''(x) - \frac{2(b - x)}{(b - a)^2}(f(b) - f(a) - (b - a)f'(a)), \]

and we get: \( f(b) - f(a) - (b - a)f'(a) = \frac{f''(c)}{2!}(b - a)^2. \)
Taylor’s Theorem

This can be proved in all degrees:

**Theorem**

Suppose that \( f : [a, b] \to \mathbb{R} \) is \( n \)-times differentiable on \([a, b]\) and \( f^{(n)} \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Assume \( x_0 \in [a, b] \). Then for each \( x \in [a, b] \) with \( x \neq x_0 \), there is \( c \) between \( x \) and \( x_0 \) such that

\[
f(x) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}.
\]
Proof of Taylor’s

Define the function

\[ F(t) = f(t) + \sum_{k=1}^{n} \frac{f^{(k)}(t)}{k!}(x - t)^k + M(x - t)^{n+1}, \]

where \( M \) is chosen so that \( F(x_0) = f(x) \). This is possible because \( x - x_0 \neq 0 \).

\( F \) is continuous on \([a, b]\) and differentiable on \((a, b)\), and

\[ F(x) = f(x) = F(x_0). \]

By Rolle’s Theorem,

\[ F'(c) = 0, \quad \text{for } c \text{ between } x \text{ and } x_0. \]
\[ 0 = F'(c) = \frac{f^{(n+1)}(c)}{n!} (x - c)^n - (n + 1)M(x - c)^n. \]

This gives

\[ M = \frac{f^{(n+1)}(c)}{(n + 1)!} \]

and

\[ f(x) = F(x_0) = f(x_0) + \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(c)}{(n + 1)!} (x - x_0)^{n+1}. \]
\[
f(b) = f(a) + (b - a)f'(a) + \frac{(b - a)^2}{2}f''(a) + \cdots + \frac{(b - a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b - a)^n}{n!}f^{(n)}(c),
\]
for some \(c \in (a, b)\). To prove this, consider the function

\[
g(x) = F_n(x) - \left(\frac{b - x}{b - a}\right)^n F_n(a)
\]

where

\[
F_n(x) = f(b) - f(x) - (b - x)f'(x) - \cdots - \frac{(b - x)^{n-1}}{(n-1)!}f^{(n-1)}(x).
\]

The function \(g(x)\) vanishes at \(x = a\) and \(x = b\).
Its derivative is

\[
\frac{n(b - x)^{n-1}}{(b - a)^n} \left( F_n(a) - \frac{(b - a)^n}{n!} f^{(n)}(x) \right),
\]

which must vanish by Rolle’s Theorem for some \( a < c < b \).

This gives the formula

\[
f(x) = \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{i!} (x - a)^i + R_n(x)
\]

We must control the term (remainder)

\[
R_n(x) = \frac{(b - a)^n}{n!} f^{(n)}(c), \quad a < c < x
\]

to study Taylor’s.
Example

**Problem:** Compute the first 5 decimals of $e$.
The Taylor series of $e^x$ around $x_0 = 0$ is

$$1 + x + \cdots + \frac{x^n}{n!} + \cdots$$

The remainder term is

$$\frac{f^{(n+1)}(c)}{(n+1)!}(x - c)^{n+1}, \quad c \in [0, x].$$

We want to find $n$ so that the remainder (for $x = 1$) is $< 10^{-6}$. We know that the derivatives of $e^x$ are $e^x$, so $e^c \leq e < 4$. As $(1 - c) \leq 1$, the remainder is smaller than

$$\frac{4}{(n+1)!}$$
We pick $n$ so that

$$\frac{4}{(n+1)!} < 10^{-6}$$

That is,

$$(n + 1)! > 4 \times 10^6$$

$$7! = 5040$$
$$10! = 720 \times 5040$$
$$11! > 4 \times 10^6$$
Example

Let $f(x) = \log(1 + x)$, $a = 0$: Then

\[
\begin{align*}
    f'(x) &= \frac{1}{1 + x} \\
    f''(x) &= \frac{-1}{(1 + x)^2} \\
    &\vdots \\
    f^{(n)}(x) &= (-1)^{n-1} \frac{(n - 1)!}{(1 + x)^n}
\end{align*}
\]

Thus

\[
|R_n(x)| = \frac{1}{n} \left| \frac{1}{1 + x^n} \right| \leq \frac{1}{n}, \quad 0 \leq x
\]
Example

Let \( f(x) = \arctan x, \ a = 0: \) Then

\[
\begin{align*}
    f'(x) &= \frac{1}{1 + x^2} \\
    f''(x) &= \frac{-2x}{(1 + x^2)^2} \\
    \vdots \\
    f^{(n)}(x) &= ?
\end{align*}
\]

We will be tricky: Consider the geometric series

\[
\frac{1}{1 - x} = 1 + x + x^2 + \cdots + x^n + \cdots
\]
Exercises

1. Decide whether the series converges or diverges

\[ \sum_{n \geq 1} \frac{\sqrt{n+1} - \sqrt{n}}{n} \]

2. Write the Taylor series of \( \ln x \) using powers of \( x - 1 \)

3. Prove that \( e^x \geq 1 + x \) for all \( x \).

4. Use induction to show that \( 1 + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \geq \sqrt{n} \). Which other way?

5. Chapter 6: 9, 19, 22, 24(a,b), 37, 41b, 42
Outline

1. Main Goal
2. Properties of Infinite Series
3. Workshop #10
4. Uniform Convergence and Differentiability
5. Series of Functions
6. Power Series
7. Taylor Series
8. Workshop #11
9. Old Finals
1. Observe that the series

\[ f(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots \]

converges for on \([0, 1)\) but not when \(x = 1\). For fixed \(x_0 \in (0, 1)\), use the M-test to prove that \(f\) is continuous at \(x_0\).

2. Let

\[ f(x) = \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2} \]

1: Show that \(f\) is a continuous function defined on all of \(\mathbb{R}\).
2: Is \(f\) differentiable? If so, is \(f'\) continuous?
Outline

1. Main Goal
2. Properties of Infinite Series
3. Workshop #10
4. Uniform Convergence and Differentiability
5. Series of Functions
6. Power Series
7. Taylor Series
8. Workshop #11
9. Old Finals
1. (10 pts) State carefully and prove the Mean Value Theorem.

2. (8 pts)
   1. What is a countable set? Show that the set of rational numbers is countable.
   2. Show that the set of irrational numbers is not countable.
3. (8 pts)

1. What is a monotone sequence of real numbers?

2. If \((a_n)\) is a bounded monotone sequence, prove that it converges.

4. (8 pts) Let \(x_1 = 1\) and \(x_{n+1} := 1 + \frac{1}{x_n}\). Show that \((x_n)\) is a convergent sequence and find its limit.
5. (8 pts) If \( f : \mathbb{R} \to \mathbb{R} \) is a nonzero function satisfying
\[
f(x + y) = f(x) + f(y) \quad \text{and} \quad f(xy) = f(x)f(y)
\]
for any \( x, y \in \mathbb{R} \), prove:

1. \( f(m/n) = m/n \) for every \( m/n \in \mathbb{Q} \).
2. For \( a \in \mathbb{R} \), if \( a > 0 \) then \( f(a) > 0 \). (Note that every positive number is a square.)
3. Use (2) to prove that if \( x > y \) then \( f(x) > f(y) \).
4. Use (1), (3), the Density of \( \mathbb{Q} \) and NIP, to prove that \( f(x) = x \) for every \( x \in \mathbb{R} \).

6. (8 pts) Let \( f : [a, b] \to \mathbb{R} \) be continuous and differentiable on \( (a, b) \). If \( f(a) = f(b) = 0 \), show that for any \( k \in \mathbb{R} \) there is \( c \in (a, b) \) such that
\[
f'(c) = kf(c).
\]

\textit{Hint:} Consider \( f(x)e^{-kx} \)
7. (8 pts)

1. Describe the Cantor set $C$.
2. Show that $C$ is uncountable.
3. Show that $1/4 \in C$.

8. (8 pts) [Topology]

1. What is an open set of $\mathbb{R}$?
2. If $A$ and $B$ are subsets of $\mathbb{R}$, $A + B = \{a + b \mid a \in A, \ b \in B\}$. If $A = (1, 3)$ and $B = (2, 5)$, what is $A + B$?
3. If $A$ and $B$ are open, prove that $A + B$ is also open.
4. Prove (3) assuming only that $B$ is open.
9. (8 pts) Find the Taylor series of $\arctan x$ and determine where it converges.

10. (8 pts) What is the **radius of convergence** of a power series $\sum_{n \geq 1} a_n x^n$?

If $f(x) = x^2 + x + 1$, and $a_n = f(n)$ for $n \in \mathbb{N}$, find the radius of convergence of the corresponding series.
11. (8 pts) Let

\[ f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n^3}. \]

1. Show that \( f(x) \) is differentiable and that its derivative \( f'(x) \) is continuous.

2. Can we determine if \( f \) is twice differentiable? [Explain]

12. (10 pts) Explain [as in prove] why the Riemann integral, \( \int_{a}^{b} f \), of a continuous function \( f \) on the closed interval \([a, b]\) exists.