Outline

1. Goals
2. Cantor Set
3. Open Sets
4. Compact Sets
Our main aim is to study interesting functions of the kind

\[ X \rightarrow f \rightarrow Y \]

where \( X \) and \( Y \) are subsets of \( \mathbb{R} \).

If \( f \) is a function and the sequence

\[ a_1, a_2, a_3, \ldots, a_n, \ldots \]

lies in the domain of \( f \), then the sequence

\[ f(a_1), f(a_2), f(a_3), \ldots, f(a_n), \ldots \]

is contained in \( Y \).
We want $f$ to have the following property:
- If $(a_n)$ is convergent then $(f(a_n))$ convergent.

This requires us to examine some sets of subsets of $\mathbb{R}$:
- Open Sets
- Closed Sets
- Compact Sets
- Connected Sets
- Strange Sets

These subsets have properties that will explain why continuous functions act as they do.
Cantor Set

Rule: From each subinterval of $C_n$ remove the inner third, to obtain $C_{n+1}$

Cantor Set: $C = \bigcap C_{n \geq 0}$
Building the Cantor set in detail

1. $C_0 = [0, 1]$, $C_1 = C_0 \setminus (1/3, 2/3)$, that is $C_1$ is obtained by removing from the interval $C_0$ its mid third (leaving the endpoints):

   $$C_1 = [0, 1/3] \cup [2/3, 1]$$

2. Iterate by removing from each closed subinterval above its mid third (and so on)

   $$C_2 = ([0, 1/9] \cup [2/9, 1/3]) \cup ([2/3, 7/9] \cup [8/9, 1])$$

3. This leads to a nested sequence of sets

   $$C_0 \supset C_1 \supset C_2 \supset \cdots C_n \supset \cdots$$

4. $C = \bigcap_{n \geq 0} C_n$ is called the **Cantor** set.
Note that \( C \) is obtained from \([0, 1]\) by repeatedly carving out the heart. At least, the endpoints of the various subintervals belong to \( C \). What else?

We are going to argue \( C \) is very thin by adding the lengths of the intervals that were removed:

\[
\frac{1}{3} + 2\frac{1}{3^2} + 2^2\frac{1}{3^3} + \cdots,
\]

a geometric series whose first term is \(1/3\) and whose ratio is \(2/3\), so it has for sum

\[
\frac{1/3}{1 - 2/3} = 1!
\]

So from \([0, 1]\) we took away a subset of measure 1!
Exercise Given $\epsilon > 0$, argue that any countable set $A$ is contained in a countable union $\bigcup_{n \geq 1} [a_n, b_n]$, such that

$$\sum_{n \geq 1} |b_n - a_n| < \epsilon.$$
Cardinality of $C$

If $C$ only contained the endpoints [all rational points] of the subintervals of its construction, it would be countable. Let us show otherwise:

1. We are going to code the elements of $C$ by infinite strings of $\{0, 1\}$ as follows: If $a \in C$, we set $a_1 = 0$ if $a$ belongs to the leftmost subinterval of $C_1$, otherwise we set $a_1 = 1$.

2. Once $a_1$ is assigned, we consider the subinterval of $C_2$ that contains $x$, and apply the same rule. In this we get a unique address for $x$ as the string $(a_1, a_2, a_3, \ldots)$.

3. Conversely, given any such string we build a nested sequence of closed intervals $I_1 \supset I_2 \supset I_3 \supset \cdots$: By NIP there is a point in the intersection. Actually unique why?
We observed two contrasting things about $C$: (i) it is very thin, since $[0, 1] \setminus C$ has length 1. (ii) it is uncountable. Can one compare it in other ways to the unit interval $U = [0, 1]$?

Observe that if we expand $[0, 1]$ by multiplying each number in it by 3, we obtain the interval $[0, 3]$, that is we get 3 copies of $U$. However, if we do the same operation on $C$, we only get 2 copies of $C$! Care to visualize?
One way to define dimension of subset $S$ of $\mathbb{R}^n$ is to compare $S$ with the set obtained by expanding all points in it by a scale, say 3.

For example, the dimension of $[0, 1]$ is 1, because we got $3U = [0, 3]$, while the dimension of a unit square is 2 [9 new squares], of the unit cube is 3 [27 new cubes].

In all of these examples, we say that the dimension is $d$ if $3^d$ is the size relative of the new set obtained by scaling the set by 3: $3 = 3^1$ for the unit interval, $9 = 3^2$ for the unit square, and $27 = 3^3$ for the unit cube. So they have dimensions 1, 2, 3 respectively.
For the Cantor set $C$, if we scale the set by 3 we get the union of two Cantor sets.

This means that

$$2 = 3^d,$$

so

$$\dim C = \frac{\ln 2}{\ln 3}.$$
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Neighborhoods

\[ a - \epsilon \ < x < a + \epsilon \]

Definition

Given a real number \( a \in \mathbb{R} \) and a positive number \( \epsilon > 0 \), the set

\[ V_\epsilon(a) = \{ x \in \mathbb{R} : |x - a| < \epsilon \} \]

is called the \( \epsilon \)-neighborhood of \( a \).

Thus a neighborhood of a point \( a \in \mathbb{R} \) is just an open interval centered at \( a \).
Open Sets

Definition (Open Set)
A set $O$ of $\mathbb{R}$ is open if for all points $a \in O$ there exists an $\epsilon$-neighborhood $V_\epsilon(a) \subset O$.

1. The entire $\mathbb{R}$ is an open set. The definition also fits the empty subset $\emptyset$ of $\mathbb{R}$.

2. Any interval
   
   $$(c, d) = \{ x \in \mathbb{R} \mid c < x < d \}$$

   is open. For any $a \in (c, d)$, if we pick $\epsilon = \min\{a - c, d - a\}$, then the interval $V_\epsilon(a) \subset (c, d)$.

3. The subsets $(c, d]$, $[c, d)$ or $[c, d]$ are NOT open: at least one of the endpoints do not pass the neighborhood test.
Theorem (Template for a Topology)

1. The union of an arbitrary collection of open sets is open.
2. The intersection of a finite collection of open sets is open.

Proof.

1. Let \( \{O_\lambda \mid \lambda \in \Lambda\} \) be a collection of open sets of \( \mathbb{R} \), and \( O \) its union. If \( a \in O \), \( a \in O_\lambda \) for some \( \lambda \). Since \( O_\lambda \) is open, there exists an \( \epsilon \)-neighborhood \( V_\epsilon(a) \subset O_\lambda \subset O \).

2. Let \( \{O_1, O_2, \ldots, O_n\} \) be a finite collection of open subsets of \( \mathbb{R} \). If \( a \in O = \bigcap O_i \), for every open set \( O_i \) pick an \( \epsilon_i \)-neighborhoods \( V_{\epsilon_i}(a) \subset O_i \). Choosing \( \epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\} \), we get \( V_\epsilon(a) \subset O_i \) for each \( O_i \), and therefore \( V_\epsilon(a) \subset O \). \( \square \)
Limit Point of a Set

**Definition**

A point $x$ is a **limit** point of a set $A$ if every $\epsilon$-neighborhood $V_{\epsilon}(x)$ of $x$ intersects $A$ in some point other than $x$.

Other terminology for **limit** point: **accumulation** point, or **cluster** point. It is important to note that a limit point of $A$ does not have to be a point of $A$.

**Theorem**

A point $x$ is a limit point of a set $A$ iff $x = \lim a_n$ for some sequence $(a_n)$ contained in $A$ satisfying $a_n \neq x$ for all $n \in \mathbb{N}$.

**Proof.** By considering a values $\epsilon = 1/n$, to a limit point $x$ of $A$, we select $a_n \in A \cap V_{1/n}(x)$, $a_n \neq x$. Note that $a_n \in V_{1/N}(x)$, for $n \geq N$. This means $(a_n) \rightarrow x$. The converse is clear. □
Isolated Point

Definition

A point $x \in A$ is an isolated point of $A$ if it is not a limit point of $A$.

This essentially means that we have an $\epsilon$-neighborhood $V_\epsilon(x)$ that contains no other point of $A$. For example, let

$$A = \{1/n \mid n \in \mathbb{N}\}$$

The sequence of points of $A$, $(1/n) \to 0$, so $0$ is a limit point.

Any point of $A$ is isolated: For example, if $x = 1/3$, the closest other point in $A$ is $1/4$, so if we choose $\epsilon < 1/3 - 1/4 = 1/12$, $V_\epsilon(1/3) \cap A = \{1/3\}$. 
Let $A = \{1/n \mid n \in \mathbb{N}\}$. Note that the closest point to $1/n$ is $1/(n+1)$: So if $\epsilon < 1/n - 1/(n+1)$

$$V_\epsilon(1/n) \cap A = \{1/n\}$$
A set $F \subset \mathbb{R}$ is **closed** if it contains (all) its limit points.

In other words, for any convergent sequence $(a_n) \to x$ of distinct points $a_n \in F$, $x \in F$ also.

Closed sets are ubiquitous.
Plenty of Closed Sets

Theorem

Let $A$ be a subset of $\mathbb{R}$. The set $L$ of limit points of $A$ is closed.

Proof.

1. Let $x$ be a limit point of $L$. To show that $x \in L$ we must show that $x$ is a limit point of $A$.
2. Let $V_\epsilon(x)$ be a neighborhood of $x$. It contains some $y \in L$. Pick a (possibly) smaller neighborhood $V_\epsilon'(y) \subset V_\epsilon(x)$.
3. Since $y \in L$, $V_\epsilon'(y)$ contains some $z \in A$, as desired.
Examples

1. The interval $A = [c, d]$ is a closed set: If $x$ is a limit point of $A$ there is a sequence $(x_n)$ of points of $A$ with $(x_n) \to x$. Applying Order Theorem to

$$c \leq x_n \leq d,$$

we get $c \leq \lim x_n \leq d$, so $x \in A$.

2. Consider the rational numbers: $\mathbb{Q} \subset \mathbb{R}$. The set of limit points of $\mathbb{Q}$ is $\mathbb{R}$: Given any element $y \in \mathbb{R}$, by the Density Theorem there exists a rational number $r \neq y$ in $V_{\epsilon}(y)$. This can be reformulated as:

**Theorem (Density of $\mathbb{Q}$ in $\mathbb{R}$)**

*Given any $y \in \mathbb{R}$, there is a sequence of rational numbers that converges to $y$.***
Definition

Given a set $A \subset \mathbb{R}$, let $L$ be the set of all limit points of $A$. The closure of $A$ is the set $\overline{A} = L \cup A$.

1. $\overline{A}$ consists of $A$ plus its accumulation points.
2. If $A = (0, 1)$, its closure $\overline{A}$ is $[0, 1]$.
3. If $A = \{1/n \mid n \in \mathbb{N}\}$, its limit set is $L = \{0\}$, so

$$\overline{A} = A \cup \{0\}.$$

4. $\overline{\mathbb{Q}} = \mathbb{R}$
Theorem

For any $A \subset \mathbb{R}$, the closure $\bar{A}$ is a closed set and is the smallest closed set containing $A$.

Proof.

1. Let $x$ be a limit point of $\bar{A}$, which we assume does not lie in $\bar{A}$. Note that any neighborhood of $x$ must contain an element $x \neq y \in \bar{A}$.

2. We will show that $x$ is a limit point of $L$, and since we have already proved that $L$ is closed this would imply $x \in L$.

3. Let $V_{\epsilon}(x)$ be a neighborhood of $x$. We want to argue that it contains some element of $A$. If not, it would have to contain an element $y \in L$.

4. Let $V_{\epsilon'}(y) \subset V_{\epsilon}(x)$. With $y \in L$, $V_{\epsilon'}(y)$ contains an element of $A$, as desired.

\[ \square \]
A set $O$ is open if and only if its complement $O^c$ is closed. Likewise, a set $F$ is closed if and only if $F^c$ is open.

**Proof.** Let $O$ be an open subset of $\mathbb{R}$. To show that $O^c$ is closed, we must show that it contains all of its limit points. If $x$ is a limit point of $O^c$, then every neighborhood of $x$ contains some point of $O^c$. If $x \notin O^c$, $x \in O$ and since $O$ is open there is a neighborhood of $x$ contained in $O$. This contradiction shows that $x \in O^c$. 
For the converse, assume $O^c$ is closed and we argue that $O$ is open. This means that for every point $x \in O$ there must be a neighborhood $V_\epsilon(x) \subset O$. If not, each such neighborhood would intersect $O^c$, which is closed. In this case, $x$ would be a limit point of $O^c$, and thus $x \in O^c$, which is a contradiction.

For the second part, just note that for any subset $E \subset \mathbb{R}$, $(E^c)^c = E$. □
Theorem (Template for a Topology)

1. The intersection of an arbitrary collection of closed sets is closed.
2. The union of a finite collection of closed sets is closed.

Corollary

The Cantor set $C$ is closed.
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**Definition**

A set $K \subset \mathbb{R}$ is **compact** if every sequence in $K$ has a subsequence that converges to a limit that is also in $K$.

**Example:** A closed interval $[a, b]$. The Bolzano-Weirstrass theorem guarantees that any sequence $(a_n) \subset [a, b]$ admits a convergent subsequence. Because $[a, b]$ is closed, the limit of this subsequence is also in $[a, b]$.
Heine-Borel Theorem

**Definition**

A set $K \subset \mathbb{R}$ is **bounded** if there exists $M > 0$ such that $|x| < M$ for all $x \in K$.

**Theorem**

A set $K \subset \mathbb{R}$ is compact if and only if it is closed and bounded.

**Proof.** Let $K$ be compact. We first claim $K$ is bounded. Otherwise, for each $n$ there is $x_n \in K$ such that $|x_n| > n$. Since $K$ is compact:

1. $(x_n)$ has a convergent subsequence $(x_{n_k})$.
2. But convergent sequences are bounded, while $|x_{n_k}| > n_k$, a contradiction as $n_k \to \infty$. 
Next we show that $K$ is closed. Let $x = \lim x_n$ be a limit point of $K$, that is, $x_n \in K$. We must show $x \in K$. From the compactness assumption, $(x_n)$ admits a convergent subsequence $(x_{n_k})$ converging to a point $y \in K$. Since $(x_n)$ is convergent, all of its subsequences have the same limit, so $x = y$ as desired.

The converse is left as an exercise.
**Theorem**

If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ is a nested sequence of nonempty compact sets, then the intersection $\bigcap_{n=1}^{\infty} K_n$ is nonempty.

**Proof.** The strategy is simple: We pick an element $x_n \in K_n$ ($K_n$ is nonempty) and consider the sequence $(x_n)$. Since $x_n \in K_1$, and $K_1$ is compact, it admits a convergent subsequence $(x_{n_k}) \to x \in K_1$. We claim that $x \in K_n$ for every $n$. Given $n_0$, the terms in $(x_n)$ are contained in $K_{n_0}$ as long as $n \geq n_0$. This means that the terms of the subsequence $(x_{n_k})$ are also in $K_{n_0}$ for almost all of them. This implies that its limit lies in $K_{n_0}$, as desired. □