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Some Goals

Understand mathematical objects such as

\[ \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots = ? \]

\[ \prod_{n=0}^{\infty} a_n = a_0 \cdot a_1 \cdot a_2 \cdot a_3 + \cdots = ? \]

The building blocks of these objects are

\( a_1, a_2, a_3, \ldots, a_n, \ldots \)
Sequences of real numbers

**Definition**
A sequence is a function \( f \) whose domain is \( \mathbb{N} \).

It can be represented as

\[
\{ f(1), f(2), f(3), \ldots \}
\]

\[
\{ f(0), f(1), f(2), f(3), \ldots \}
\]

or

\[
\{ f(n), \ldots, \quad n \geq n_0 \}
\]

We will first examine sequences of real numbers, \( f : \mathbb{N} \rightarrow \mathbb{R} \). Later we will study sequences of functions.
It allows us to look at real numbers in a concrete manner: If

\[ x = A.a_1 a_2 \cdots a_n \cdots, \]

where \( a_i \) are the decimal digits, we form the sequence of rational numbers

\[
\begin{align*}
x_0 & = A \\
x_1 & = A.a_1 \\
x_2 & = A.a_1 a_2 \\
x_n & = A.a_1 a_2 \cdots a_n, \quad \text{and so on}
\end{align*}
\]
We will look for features such as clustering

1. \((1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots)\)
2. \((c, c, c, c, \ldots)\)
3. \((1, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \ldots)\)
4. \(\left(\frac{1}{2^n}\right)_{n=1}^{\infty} = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)\)
5. \((a_n), \ a_1 = 1, \text{ and } a_{n+1} = \frac{a_n}{2} + 1\)
6. \((a_n), \ a_n \text{ is the } n\text{th digit in the decimal expansion of } \pi.\)
7. \((a_n), \ a_n = (1 + 1/n)^n\)
Why Sequences?

We use sequences to make sense of:

- $\sum_{n \geq 1} a_n$: Series
  
  $$1 + 1/2^2 + 1/3^2 + \cdots + 1/n^2 + \cdots$$

  Question: How to handle
  
  $$(a_0 + a_1 + \cdots + a_n + \cdots)(b_0 + b_1 + \cdots + b_n + \cdots)$$

- $\sum_{m,n \geq 1} a_{m,n}$: Double [multiple] Series
  
  $$\sum_{m,n} \frac{1}{m^2 + n^2}$$

- $\prod_{n \geq 1} a_n$: Infinite Products
  
  $$\prod_p \left(\frac{1}{1 - p}\right), \quad p \text{ prime number}$$
Sequences are wonderful ways to represent data, but we are mostly interested in one of its aspects:

**Definition**

A sequence \((a_n)\) converges to a real number \(a\) if, for every positive real number \(\epsilon\), there exists an \(N \in \mathbb{N}\) such that whenever \(n \geq N\) it follows that \(|a_n - a| < \epsilon\).

One notation: \(\lim a_n = a\), or \((a_n) \to a\). To understand this we introduce the notion of a **neighborhood** of a real number \(a\).
Consider the sequence \((a_n)\), \(a_n = \frac{n+1}{n}\). It is natural to expect that \(\lim a_n = 1\). Let us follow the template:

- Given \(\epsilon > 0\), to determine \(N\) we solve
  \[
  \left| \frac{n+1}{n} - 1 \right| < \epsilon
  \]

- That is
  \[
  \left| \frac{1}{n} \right| < \epsilon \quad \Rightarrow \quad n > \frac{1}{\epsilon}
  \]

- Thus if \(\epsilon = 1/100\), \(N = 101\) will work.
Definition

Given a real number \( a \in \mathbb{R} \) and a positive number \( \epsilon > 0 \), the set

\[
V_\epsilon(a) = \{ x \in \mathbb{R} : |x - a| < \epsilon \}
\]

is called the \( \epsilon \)-neighborhood of \( a \).
Limit and Neighborhoods

\[ a_1 \text{ } a_2 \text{ } a_3 \text{ } \cdots \text{ } a_N \]

\[
\begin{array}{cccc}
\cdot & \cdot & \cdot & \cdot \\
& a & \cdot & \\
\hline
a - \varepsilon & a & a + \varepsilon & b
\end{array}
\]

\( a \) is the limit of \((a_n)\) if once \(a_N\) enters the neighborhood \(V_\varepsilon(a)\), all \(a_n\) that follow will stay in it. That is, the \(a_n\) cluster around \(a\) in a very specific manner.

Note that this implies that if \((a_n)\) converges, its limit is unique: the \(a_n\) cannot be in both \(V_\varepsilon(a)\) and \(V_\varepsilon(b)\) if \(\varepsilon < 1/2|a - b|\).
Exercise

Let \( a_n = \frac{2n^2 + n + 1}{n^2} \). It can be written as

\[
a_n = 2 + \frac{1}{n} + \frac{1}{n^2}
\]

It is now easy to see that \( \lim a_n = 2 \): Just notice that

\[
|a_n - 2| = \frac{1}{n} + \frac{1}{n^2} \leq 2 \frac{1}{n}
\]

and we can use the argument of the previous Example to finish.

Exercise: For every real number \( x \in \mathbb{R} \), there exists a sequence \( (a_n) \) of rational numbers such that \( (a_n) \rightarrow x \).
Let us summarize the procedure to compute the limit of a sequence: 
\((a_n) \rightarrow a\) involves all the following steps:

1. Let \(\epsilon > 0\) be arbitrary
2. Demonstrate a choice for \(N \in \mathbb{N}\): hard work here often
3. Assume \(n \geq N\)
4. Check that

\[ |a - a_n| < \epsilon \]
Example

Define the sequence

\[ a_1 = \sqrt{2}, \quad a_2 = \sqrt{2 \sqrt{2}}, \quad a_3 = \sqrt{2 \sqrt{2 \sqrt{2}}}, \ldots \]

Question: \((a_n) \rightarrow ? \) Note

\[ a_1 = \sqrt{2}, \quad a_2 = a_1 \sqrt{2}, \quad a_3 = a_2 \sqrt{2}, \ldots \]

\[ a_n = 2^{1/2 + 1/4 + \cdots + 1/2^n} < 2 \]

So this sequence is bounded [and increasing]. Show that its least upper bound is 2.
Infinity as the limit of a sequence

If a sequence \((a_n)\) is not convergent, we say that it is divergent. We also use the following terminology for some divergent sequences:

**Definition**

The sequence \((a_n)\) converges to \(\infty\), \(\lim a_n = \infty\), if given any positive number \(b\), there is an \(N \in \mathbb{N}\) such that \(a_n \geq b\) for \(n \geq N\).

**Example:** \(\{1, 2, 3, \ldots , n, \ldots\}\)

Some sequences don’t make up their minds:

1. \(1, -1, 1, \ldots , \pm 1, \ldots\)

2. one gets a very complicated sequence by glueing two unrelated sequences \((a_n), (b_n)\), as in

\[
a_0, b_0, a_1, b_1, a_2, b_2, \ldots , a_n, b_n, \ldots ,
\]
Boundedness of Convergent Sequences

**Definition**

A sequence \((a_n)\) is bounded if there exists a number \(M > 0\) such that \(|a_n| \leq M\) for all \(n \in \mathbb{N}\).

**Theorem**

*Every convergent sequence is bounded.*

**Proof.** Suppose \((a_n) \to \ell\). For \(\epsilon = 1\) let \(N \in \mathbb{N}\) be such that \(|a_n - \ell| < 1\) for \(n \geq N\).

We claim that \(M = \max\{ |a_1|, |a_2|, \ldots, |a_{N-1}|, |\ell| + 1 \}\) satisfies

\[ |a_n| \leq M \]
The sequence \((1, -1, \ldots, (-1)^n, \ldots)\) is bounded but not convergent.

Many sequences are put together from two or more sequences: Say start with

\[
\{a_1, a_2, a_3, \ldots\} \quad \{b_1, b_2, b_3, \ldots\}
\]

\[
\{a_1, b_1, a_2, b_2, a_3, b_3, \ldots\}
\]
Theorem

Let \( \lim a_n = a \) and \( \lim b_n = b \). Then

(i) \( \lim ca_n = ca \), for all \( c \in \mathbb{R} \);

(ii) \( \lim (a_n + b_n) = a + b \);

(iii) \( \lim (a_nb_n) = ab \);

(iv) \( \lim (a_n/b_n) = a/b \) provided \( b_n \neq 0 \) and \( b \neq 0 \).

Note an important consequence: Since we can view real numbers as limits of rational numbers, we can carry out the desired field operations

\[
\begin{align*}
x &= X.x_1x_2\ldots x_n|\ldots \\
y &= Y.y_1y_2\ldots y_n|\ldots
\end{align*}
\]
Proof. (i) [If \( \lim a_n = a \), then \( \lim ca_n = ca \)] Consider the case \( c \neq 0 \). To prove \( (ca_n) \to ca \), we use the proof template. Let \( \epsilon > 0 \). We want to argue that \( |ca_n - ca| < \epsilon \) from some term of the sequence \( (ca_n) \) on. Since \( (a_n) \to a \), given \( \epsilon/|c| \), there is \( N \in \mathbb{N} \) such that for \( n \geq N \)

\[
|a_n - a| < \frac{\epsilon}{|c|}.
\]

This leads to

\[
|ca_n - ca| = |c| |a_n - a| < \epsilon, \quad n \geq N,
\]

as desired. This proves (i) for \( c \neq 0 \). The case \( c = 0 \) is trivial.
(ii) [If \( \lim a_n = a \), \( \lim b_n = b \), then \( \lim (a_n + b_n) = a + b \)]

Given \( \epsilon > 0 \), pick \( N_1 \) and \( N_2 \) so that

\[
|a_n - a| < \frac{\epsilon}{2}, \quad \& \quad |b_n - b| < \frac{\epsilon}{2}
\]

for \( n \geq N_1 \) and \( n \geq N_2 \), respectively. Thus \( n \geq N = \max\{N_1, N_2\} \)

\[
|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| \\
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]
(iii) [If \( \lim a_n = a \), \( \lim b_n = b \), then \( \lim a_nb_n = ab \)] \( \text{If } \lim a_n = a \), 
\( \lim b_n = b \), we know that \( |a_n| \) and \( |b_n| \) are bounded, that is \( |a_n| < M_1 \) 
and \( |b_n| < M_2 \) for all \( n \). Let \( M = \max\{M_1, M_2\} \). Given \( \epsilon > 0 \), pick \( N_1 \) 
and \( N_2 \) so that

\[
|a_n - a| < \frac{\epsilon}{2M}, \quad \& \quad |b_n - b| < \frac{\epsilon}{2M}
\]

for \( n \geq N_1 \) and \( n \geq N_2 \), respectively.

This leads to: for all \( n \geq N = \max\{N_1, N_2\} \)

\[
|a_nb_n - ab| = |(a_nb_n - a_nb) + (a_nb - ab)|
\]
\[
\leq |(a_nb_n - a_nb)| + |(a_nb - ab)|
\]
\[
= |a_n||b_n - b| + |b||a_n - a| \leq M_1|b_n - b| + M_2|a_n - a|
\]
\[
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

which completes the proof.
(iv) [If $\lim a_n = a$, $\lim b_n = b$, $b_n, b \neq 0$, then $\lim a_n/b_n = a/b$]. In the case of $a_n/b_n$, we are going to apply the product rule to the product $a_n \frac{1}{b_n}$. This requires

**Lemma**

*If the sequence $(b_n) \to b$ and $b_n, b \neq 0$, then $(\frac{1}{b_n}) \to \frac{1}{b}$.***

**Proof.** Let $\epsilon_0 = |b|/2$. Pick $N_1$ large enough so that for $n \geq N_1$

$|b_n - b| < \epsilon_0 = |b|/2$. This shows that in this range $|b_n| > |b|/2$. Next, given $\epsilon > 0$, choose $N_2$ so that for $n \geq N_2$

$$|b_n - b| < \frac{\epsilon b^2}{2}$$

Finally, if we let $N = \max\{N_1, N_2\}$,

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \left| \frac{b - b_n}{bb_n} \right| \leq \frac{\epsilon b^2}{2} \frac{1}{|b||b|/2} = \epsilon$$
We examine in detail this important sequence. Two cases are easy: $x = 1$, when the sequence is constant (so $\lim x^n = 1$), and $x = -1$ (when it alternates between 1 and $-1$) when it does not converge. Let us next examine the case $|x| < 1$, that is $-1 < x < 1$. We make a series of technical observations.
Lemma

For any $p > -1$ and all $n \in \mathbb{N}$, $(1 + p)^n \geq 1 + pn$.

Proof. We prove this by induction. It is true for $n = 1$. Now consider

$$(1 + p)^{n+1} = (1 + p)^n (1 + p) \geq (1 + pn)(1 + p)$$

$$= 1 + p(n + 1) + p^2 n \geq 1 + p(n + 1).$$
Back to our limit. If \(|x| < 1\), \(\frac{1}{|x|} = 1 + p, p > 0\) and thus

\[
\frac{1}{|x^n|} = (1 + p)^n \geq 1 + pn > pn
\]

Therefore

\[
|x^n| < \frac{1}{pn}
\]

Which shows that for \(|x| < 1\) \(\lim |x^n| = 0\) and \(\lim x^n = 0\) as well.

The case \(|x| > 1\). Apply the algebraic limit theorem: By the case above, \(\lim \frac{1}{x^n} = 0\), which shows \((x^n)\) does not converge.
Theorem (Order Limit Theorem)

Assume \( \lim a_n = a \) and \( \lim b_n = b \). Then

1. If \( a_n \geq 0 \) for all \( n \in \mathbb{N} \), then \( a \geq 0 \).
2. If \( a_n \leq b_n \) for all \( n \in \mathbb{N} \), then \( a \leq b \).
3. If there exists \( c \in \mathbb{R} \) for which \( c \leq b_n \) for all \( n \in \mathbb{N} \), then \( c \leq b \).

Similarly, if \( a_n \leq c \) for all \( n \in \mathbb{N} \), then \( a \leq c \).
Proof. (i) Assume, by way of contradiction, that $a < 0$. Let us show that this produces some $a_n < 0$. Let $\epsilon = |a|$. There exists $N$ such that

$$|a_n - a| < \epsilon, \quad n \geq N$$

If $a_n \geq 0$ for $n \geq N$,

$$|a_n - a| = |a_n + (-a)| = a_n + |a| \geq \epsilon,$$

a contradiction.

(ii) The Algebraic Limit Theorem guarantees that the sequence $(b_n - a_n)$ converges to $b - a$. Because $b_n - a_n \geq 0$, by Part (i), $b \geq a$.

(iii) Take $a_n = c$ (or $b_n = c$) for all $n \in \mathbb{N}$ and apply (ii). □
Examples

- The constant sequence \((c, c, c, \ldots)\) converges to \(c\):
  \[ x_n = c \text{ for all } n, \text{ so for } \epsilon > 0, \ |x_n - c| = 0 < \epsilon \]

- Let \(x_n \geq 0\) for all \(n \in \mathbb{N}\).

  1. If \((x_n) \to 0\), show that \((\sqrt{x_n}) \to 0\): Given \(\epsilon > 0\) we can find \(N\) such that \(|x_n| < \epsilon^2\) for \(n \geq N\). It follows that \(|\sqrt{x_n}| < \epsilon\) for \(n \geq N\).

  2. If \((x_n) \to x\), show that \((\sqrt{x_n}) \to \sqrt{x}\): We already know that \(x \geq 0\) and that the sequence is bounded, that is \(L < x_n < U\). In particular \(\sqrt{x_n} \geq \sqrt{L}\) and \(x \geq \sqrt{L}\). Given \(\epsilon > 0\) pick \(N\) so that \(|x_n - x| < \epsilon 2\sqrt{L}\) for \(n \geq N\). Then

  \[
  |\sqrt{x_n} - \sqrt{x}| \leq |\sqrt{x_n} - \sqrt{x}| \frac{|\sqrt{x_n} + \sqrt{x}|}{2\sqrt{L}}
  \]

  \[
  = \frac{|x_n - x|}{2\sqrt{L}} < \epsilon
  \]
(i) Show that if \((b_n) \to b\), then the sequence \(|b_n|\) converges to \(|b|\).

(ii) Converse?

2 Let \((a_n)\) be a bounded (not necessarily convergent) sequence, and assume \((b_n) \to 0\). Show that \((a_nb_n) \to 0\). Why we are not allowed to use the Algebraic Limit theorem?

3 Exercises 32(a,c,e) in page 56 of Textbook.
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2. Sequences
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Monotone Sequences

**Definition**

A sequence \((a_n)\) is **increasing** if \(a_n \leq a_{n+1}\) for all \(n \in \mathbb{N}\), and **decreasing** if \(a_n \geq a_{n+1}\) for all \(n \in \mathbb{N}\). A sequence is **monotone** if it is either increasing or decreasing.

**Theorem (Monotone Convergence Theorem)**

*If the sequence \((a_n)\) is monotone and bounded, then it converges.*

**Proof.** The assumption is that there is a \(B\) such that \(a_n \leq B\) for all \(n \in \mathbb{N}\). We are going to ‘build’ \(\lim a_n\). For that we are going to use the decimal representation of the \(a_n\).
Visual Proof

\[ a_1 = A_1.a_{11}a_{12}a_{13}a_{14} \cdots \]
\[ a_2 = A_2.a_{21}a_{22}a_{23}a_{24} \cdots \]
\[ a_3 = A_3.a_{31}a_{32}a_{33}a_{34} \cdots \]
\[ \vdots \]
\[ a_N = A_N.a_{N1}a_{N2}a_{N3}a_{N4} \cdots \]
\[ \vdots \]
\[ a_n = A_n.a_{n1}a_{n2}a_{n3}a_{n4} \cdots \]

Since the \( a_n \) are bounded, its integral parts \( A_n \) are also bounded and non-increasing. Thus, there is an \( N \) such that \( A_n = A_N \) for all \( n \geq N \).
Let us scan the first decimal digits from $a_N$ on:

$$a_{N1} = A_N.a_{N1}a_{N2}a_{N3}a_{N4} \cdots$$

$$\vdots \quad \vdots$$

$$a_n = A_n.a_{n1}a_{n2}a_{n3}a_{n4} \cdots$$

Since $A_n = A_N$, and $a_n$ are increasing, the digits $a_{n1}$ must be increasing so once it hits its maximal value, say at $n = N_1$, it must stay there, i.e. $a_{n1} = a_{N1}1$ for $n \geq N_1$.

We move over the second decimal place, and so on. In this manner we build the element $a = A_N.b_1b_2b_3b_4 \cdots$ with the property $|a - a_n| < 10^{-N_r}$ for $n \geq N_{r+1}$. This shows that $a = \lim a_n$. Note that $a$ is the least upper bound of the set $\{a_n\}$. 
Let \((a_n)\) be a **bounded monotone increasing sequence**, 

\[ a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \leq B \]

Because the set of terms \(\{a_n, \ n \geq 1\}\) is bounded, by the **Axiom of Completeness** the set has a **least upper bound** \(B_0\). Now we verify that \(a_n \to B_0\). We use the limit template:

- Given \(\epsilon > 0\), \(B_0 - \epsilon\) is not an upper bound so there is \(N\) such that \(a_N > B_0 - \epsilon\). Since \(a_n\) is increasing, we have

\[ B_0 \geq a_n \geq a_N > B_0 - \epsilon, \quad n \geq N. \]

- This means that \(|a_n - B_0| < \epsilon\) for \(n \geq N\), thus proving that \(\lim a_n = B_0\).
A sequence we met already was \((x_n)\), where \(x_1 = 1\) and

\[ x_{n+1} = \frac{x_n}{2} + 1 \]

We proved that \(x_n < x_{n+1} < 2\), so this is a monotone bounded sequence. Let \(a = \lim x_n\). If we delete \(x_1\), we obtain the sequence \((x_{n+1}, n \geq 1)\) which obviously is monotone, and has the same limit. Thus

\[ \lim x_{n+1} = a = \frac{\lim x_n}{2} + 1 = \frac{a}{2} + 1 \]

and therefore

\[ a = 2 \]
Calculating Square Roots

Let $x_1 = 2$, and define

$$x_{n+1} = 1/2 \left( x_n + \frac{2}{x_n} \right)$$

- Show that $x_n^2 \geq 2$, and then prove that $x_n - x_{n+1} \geq 0$. Conclude that $\lim x_n = \sqrt{2}$.

We use induction. Squaring we have $x_{n+1}^2 = 1/4(x_n^2 + 4 + 4/x_n^2)$. To show that $x_{n+1}^2 > 2$, it suffices to show that if $x_n^2 > 2$, then $x_n^2 + 4/x_n^2 > 4$. But

$$x_n^2 + 4/x_n^2 - 4 = (x_n - \frac{2}{x_n})^2 > 0$$

Note also $x_n - x_{n+1} = 1/2(x_n - 2/x_n) > 0$, since $x_n^2 > 2$. Thus the sequence $(x_n)$ is bounded and decreasing. Its limit $a$ satisfies $a = 1/2(a + 2/a)$, i.e. $a = \sqrt{2}$. 
Modify the sequence so that it converges to $\sqrt{c}$:

$$x_{n+1} = 1/2 \left( x_n + \frac{c}{x_n} \right)$$

We again check that the sequence $(x_n)$ is monotone and bounded. When solving for the limit, we get $a = 1/2(a + c/a)$, i.e. $a = \sqrt{c}$.

Many other equations $f(x) = 0$ can be set up as

$$x = \frac{g(x)}{h(x)}$$

which we turn into a dynamical scheme

$$x_{n+1} = \frac{g(x_n)}{h(x_n)}$$

If $(x_n)$ is monotone and bounded, the limit is a root.
**Definition**

Let \((a_n)\) be a sequence of real numbers, and let \(n_1 < n_2 < n_3 < \cdots\) be an increasing sequence of natural numbers. Then the sequence

\[ a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \ldots \]

is called a **subsequence** of \((a_n)\) and is denoted by \((a_{n_j})\), where \(j \in \mathbb{N}\) indexes the subsequence.

**Theorem**

*Subsequences of a convergent sequence converge to the same limit as the original sequence.*
Theorem (Bolzano-Weierstrass Theorem)

Every bounded sequence \((a_n)\) contains a convergent subsequence.

Proof. The assumption is that all \(a_n\) lie in some closed interval \(I_1 = [-M, M]\). (Note that we allow repetitions.) Since the sequence is infinite, an infinite subset of terms lies in either \([-M, 0]\) or in \([0, M]\). We pick one of the subintervals with an infinite number of terms and call it \(I_2\).
We continue the process: bisect $I_2$ pick $I_3$ one of its two halves that contain an infinite number of terms. In this manner we get a decreasing sequence of closed intervals

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

If in each subset $I_k$ we pick an element $a_{n_k}$ of the sequence in it, we obtain a subsequence

$$\{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\}$$

We claim this (sub)sequence converges.
By the Nested Interval Property there exists at least one point \( x \in \mathbb{R} \) contained in every \( I_k \).
We claim \( (a_{n_k}) \rightarrow x \). Note that the length of \( I_k \) is \( M \frac{1}{2^{k-1}} \), which converges to 0 (discussed in Workshop #3).
Choose \( N \) so that \( k \geq N \) implies that the length of \( I_k \) is less than \( \epsilon \).
Because \( x \) and \( a_{n_k} \) are both in \( I_k \), \( |x - a_{n_k}| < \epsilon \). \( \square \)
Let \((a_n)\) be a bounded sequence, and define the set

\[ S = \{ x \in \mathbb{R} \mid x < a_n \text{ for infinitely many } a_n \} \]

Show that there exists a subsequence \((a_{n_k})\) converging to \(s = \sup S\). (This is a direct proof of the BW Theorem using AoC.)
Give an example of each of the following, or argue that such a request is impossible.

1. A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.
2. A monotone sequence that diverges but has a convergent subsequence.
3. A sequence that contains subsequences converging to every point in the infinite set \( \{1, 1/2, 1/3, 1/4, \ldots \} \).
4. An unbounded sequence with a convergent subsequence.
5. A sequence that has a subsequence that is bounded but contains no subsequence that converges.
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Thus far we have two basic results about convergence of sequences:

**Theorem (Monotone Convergence Theorem)**

*If the sequence \((a_n)\) is monotone and bounded, then it converges.*

Essentially, if

\[
a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq B,
\]

then \(a_n \to B_0\), least upper bound of the \(a_n\)

**Theorem (Bolzano-Weierstrass Theorem)**

*Every bounded sequence \((a_n)\) contains a convergent subsequence.*

Essentially, if the sequence \((a_n)\) is bounded, that is there is \(M > 0\) such that 

\[-M \leq a_n \leq M\]

for all \(n\), then there is a subsequence

\[a_{n_1}, a_{n_2}, a_{n_3}, \ldots\]

that is convergent.
The notion of convergence of a sequence that we are using is:

**Definition (Convergence of a Sequence)**

A sequence \((a_n)\) converges to the real number \(a\) if, for every \(\epsilon > 0\), there exists an \(N \in \mathbb{N}\) such that whenever \(n \geq N\) it follows that

\[|a_n - a| < \epsilon.\]

\[\lim a_n = a\] if given \(\epsilon > 0\) find \(N\) for \(n \geq N\) \(|a_n - a| < \epsilon\)
Cèsaro Means

There are other ways of defining convergence of sequences. Today we study a powerful notion, but first we do warm ups.

Let \((a_n)\) be a sequence and define the sequence of its means,

\[ c_n = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad n \geq 1 \]

thus forming the sequence \((c_n)\) of averages. For example, the sequence \((1, 0, 1, 0, 1, 0, \ldots)\) has sequence of means

\[(1, 1/2, 2/3, 1/2, 3/5, 1/2, 5/7, \ldots, 1/2, (n + 2)/(2n + 1), \ldots) \rightarrow 1/2\]
Theorem (Cèsaro Means)

If \((a_n) \to a\), then \((c_n) \to a\) also.

Proof.

- Given \(\epsilon > 0\) we will find \(N\) such that \(|c_n - a| < \epsilon\) for \(n \geq N\). Since \((a_n) \to a\), we know that \((a_n)\) is bounded, say \(|a_n| < M\) for some \(M\), and for \(\epsilon' = \epsilon/2\) there is \(N_0\) such that

\[
|a_n - a| < \epsilon' \quad n \geq N_0
\]

- Now consider \(|c_n - a|\)

\[
|c_n - a| = \left| \frac{a_1 + \cdots + a_n}{n} - a \right| = \left| \frac{(a_1 - a) + \cdots + (a_n - a)}{n} \right|
\]

\[
\leq \frac{|a_1 - a| + \cdots + |a_n - a|}{n}
\]
We are going to split the numerator of

\[
\frac{|a_1 - a| + \cdots + |a_n - a|}{n}
\]

into two summands, up to \(N_0\) and from there to \(n\): Note that \(|a_n - a| \leq |a_n| + |a| \leq 2M\) by the triangle inequality. Choosing

\[
N = \max\{N_0, 4N_0 M/\epsilon\}
\]

\[
\frac{2N_0 M}{n} + \frac{(n - N_0)\epsilon/2}{n} \leq \epsilon/2 + \epsilon/2 = \epsilon
\]

for \(n \geq N\), as desired. \(\square\)
A sequence \((a_n)\) is called a **Cauchy sequence** if, for every \(\epsilon > 0\), there is an \(N \in \mathbb{N}\) such that whenever \(m, n \geq N\) it follows that \(|a_n - a_m| < \epsilon\).

Compare to the standard definition of convergence:

**Definition (Convergence of a Sequence)**

A sequence \((a_n)\) converges to the real number \(a\) if, for every \(\epsilon > 0\), there exists an \(N \in \mathbb{N}\) such that whenever \(n \geq N\) it follows that \(|a_n - a| < \epsilon\).

Comment on the differences!
Prove that \( a_n = \frac{2n+1}{n} \) is Cauchy

1. We estimate \( |a_n - a_m| \): For \( n < m \)

\[
\left| \frac{2n+1}{n} - \frac{2m+1}{m} \right| = \left| \frac{1}{n} - \frac{1}{m} \right| = \left| \frac{m-n}{mn} \right|
\]

2. Note that \( \left| \frac{m-n}{mn} \right| \leq \frac{1}{n} \).

3. If \( \epsilon > 0 \) and \( N \) is chosen so that \( \epsilon > \frac{1}{N} \), we have

\[
|a_n - a_m| < \epsilon, \quad n, m \geq N
\]
Let a sequence be defined as follows: $x_1 = 1$, $x_2 = 2$, $x_3 = 1/2(x_1 + x_2)$ and in general $x_{n+1} = 1/2(x_{n-1} + x_n)$. Show that

$$|x_n - x_m| \leq \frac{1}{2^{N-1}}, \quad \forall n, m \geq N,$$

so Cauchy’s condition is fulfilled.

**Hint:** Note that each term is midway between the two preceding ones.
Theorem

Every convergent sequence is a Cauchy sequence.

Proof. Assume \((x_n)\) converges to \(x\). To prove \((x_n)\) is Cauchy, we must find \(N\) such that \(|x_n - x_m| < \epsilon\) for \(n, m \geq N\). This is easily done: given \(\epsilon/2\) find \(N\) such that

\[
|x - x_n| < \epsilon/2, \quad n \geq N.
\]

By the triangle inequality,

\[
|x_n - x_m| \leq |x_n - x| + |x - x_m| \leq \epsilon/2 + \epsilon/2 = \epsilon, \quad n, m \geq N.
\]
A sequence converges if and only if it is a Cauchy sequence.

While the definition of convergence requires a candidate for the limit, Cauchy’s Criterion is a softer requirement. [Discuss]

Proof. The preceding theorem showed that every convergent sequence is a Cauchy sequence. To prove the converse, we first show that every Cauchy sequence is bounded, apply Bolzano-Weierstrass, and then complete proof.
Lemma

Cauchy sequences are bounded.

Proof. Given $\epsilon = 1$, there exists an $N$ such that $|x_n - x_m| < 1$ for all $m, n \geq N$. Thus, making $m = N$, we must have $|x_n| \leq |x_N| + 1$ for all $n \geq N$. It follows that

$$M = \max\{|x_1|, |x_2|, |x_3|, \ldots, |x_{N-1}|, |x_N| + 1\}$$

is a bound for $(x_n)$. 
Cauchy Criterion

**Theorem**

A sequence converges if and only if it is a Cauchy sequence.

**Proof.** By the Bolzano-Weierstrass theorem, since \((x_n)\) is bounded, it has a convergent subsequence \((x_{n_k})\) of limit, say, \(x\). We want to argue that \(x\) is the limit of \((x_n)\) also.

Let \(\epsilon > 0\). Because \((x_n)\) is Cauchy, there exists \(N\) such that

\[
|x_n - x_m| < \frac{\epsilon}{2}, \quad m, n \geq N.
\]

Because \((x_{n_k}) \rightarrow x\), choose a term \(x_{N_K}\), with \(N_K \geq N\) such that

\[
|x_{N_K} - x| < \frac{\epsilon}{2}.
\]
Now observe: If $n \geq N_K$, 

$$|x_n - x| = |x_n - x_{N_K} + x_{N_K} - x|$$

$$\leq |x_n - x_{N_K}| + |x_{N_K} - x|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

This shows that $(x_n) \rightarrow x$
Review the following concepts/techniques:

- Algebraic and order limit theorems
- Your favorite limit tricks [see two slides down for one useful tool]
Warmups

This uses only the cute lemma and some of the algebraic limits theorems.

1. Let $a_n = q^n$. If $q > 1$, prove that $\lim a_n = \infty$: Set $q = 1 + p$, $p > 0$. By the Lemma, $(1 + p)^n \geq 1 + np$, which clearly converges to $\infty$.

2. Let $a_n = q^n$. If $0 < q < 1$, prove that $\lim a_n = 0$. [Hint: work with $1/q$.] This means $(1/q)^n \to \infty$, hence $q^n \to 0$. 

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If $q > 0$, show that $\lim \sqrt[n]{q} = 1$. [Hint: Use the technique above. First assume $q > 1$. Then set $\sqrt[n]{q} = 1 + p_n$, $p_n > 0$. Now $q = (1 + p_n)^n \geq 1 + np_n$. In case $0 < q < 1$, use $\frac{1}{\sqrt[n]{q}}$.]

Show that $\lim \sqrt[n]{n} = 1$. [Hint: Work with $\sqrt[n]{\sqrt[n]{n}} = 1 + k_n$.] Explain why setting $\sqrt[n]{n} = 1 + a_n$ will not work.

Find the limit of $\sqrt[n]{a^n b^n + b^n c^n + a^n c^n}$ if $a > b > c > 0$.

Find the limit of $\sqrt{n^2 + an + b} - n$. 

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5 Give an example or argue request is impossible.

(i) A Cauchy sequence that is not monotone.
(ii) A monotone sequence that is not Cauchy.
(iii) A Cauchy sequence with a divergent subsequence.
(iv) An unbounded sequence containing a subsequence that is Cauchy.
The following lemma discussed in class is helpful.

**Lemma**

If \( p > -1 \), \((1 + p)^n \geq 1 + pn\) for all \( n \in \mathbb{N} \).

**Proof.** We prove this by induction.

- **Base Case:** It is true for \( n = 1 \).

- **Induction Step:** Now consider

\[
(1 + p)^{n+1} = (1 + p)^n(1 + p) \geq (1 + pn)(1 + p) = 1 + p(n + 1) + p^2n \geq 1 + p(n + 1).
\]
Comment on a Limit

In the Workshop #3 Problem like

$$\lim \sqrt[n]{a^n + b^n + c^n}, \quad a > b > c > 0$$

can [?] be argued as follows

$$\lim \sqrt[n]{a^n + b^n + c^n} = \lim a \sqrt[n]{1 + (b/a)^n + (c/a)^n}$$
$$= a \lim \sqrt[n]{1 + (b/a)^n + (c/a)^n}$$

which is fine but then argued wrongly [why?]

$$\lim \sqrt[n]{1 + (b/a)^n + (c/a)^n} = \sqrt[n]{1 + \lim(b/a)^n + \lim(c/a)^n}$$
$$= \sqrt[n]{1 + 0 + 0} = 1$$
One of the proper ways to argue

\[ a = \sqrt[n]{a^n} \leq \sqrt[n]{a^n + b^n + c^n} \leq \sqrt[n]{3a^n} = a\sqrt[n]{3} \]

and then use Problem #4 that shows

\[ \lim \sqrt[n]{3} = 1 \]
\[ \lim (1 + \frac{1}{n})^n = 1 + n \frac{1}{n} + \frac{n(n-1)}{1 \cdot 2} \frac{1}{n^2} + \cdots + \frac{n(n-1) \cdots (n-n+1)}{1 \cdots n} \frac{1}{n^n} \]

\[ = 1 + 1 + \frac{1}{1 \cdot 2} (1 - \frac{1}{n}) + \frac{1}{1 \cdot 2 \cdot 3} (1 - \frac{1}{n}) (1 - \frac{2}{n}) + \cdots + \frac{1}{1 \cdot 2 \cdots n} (1 - \frac{1}{n}) \cdots (1 - \frac{n-1}{n}) \]

Note that

\[ \frac{1}{1 \cdot 2 \cdots n} (1 - \frac{1}{n}) \cdots (1 - \frac{n-1}{n}) < \frac{1}{n!} \]

This shows that

\[ 2 < (1 + \frac{1}{n})^n < 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots < 3 \]
**Intro to Infinite Series**

**Question:** What do we see in the Infinite Series

\[ \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots = ? \]

**Answer:** At least two things

- The sequence of terms, \((a_n)\) and
- The sequence of partial sums, \((s_n)\),

\[ s_n = a_0 + a_1 + \cdots + a_n \]

We say the **series converges** to \( S \in \mathbb{R} \) if \( \lim s_n = S \). By abuse of notation, we then replace the \( ? \) by \( S \).
The perspective we use is to view a series as the pair of related sequences:

\[ a_n, \quad s_n = a_0 + a_1 + \cdots + a_n \]

with emphasis on the question:
What should the sequence \((a_n)\) be like so that the sequence of partial sums \((s_n)\) converges?

We need to look close at some important series.
The Geometric Series

For \( q \in \mathbb{R} \), the geometric series of ratio \( q \) is

\[
1 + q + q^2 + q^3 + \cdots + q^n + \cdots
\]

Sometimes, all terms are multiplied by a same constant, that instead of the sequence of terms \((q^n)\), one has \((aq^n)\). Let us examine when it converges and find the corresponding limit.

- We need an expression for the partial sum \( s_n = 1 + q + \cdots + q^n \).
- If we multiply \( s_n \) by \( q \) and subtract \( s_n \) we get

\[
qs_n - s_n = q(1 + q + \cdots + q^n) - (1 + q + \cdots + q^n)
\]

\[
= q^{n+1} - 1
\]
We get an explicit expression for $s_n$

$$s_n = \frac{1}{1 - q} - \frac{q^{n+1}}{1 - q}$$

According to the value of $q$, we conclude: If $|q| < 1$, since $q^n \to 0$,

$$1 + q + q^2 + q^3 + \cdots + q^n + \cdots = \frac{1}{1 - q}$$

Otherwise the series diverges. If $q \geq 1$, it converges to infinity. [Note the confusing language.]
The Harmonic Series

This is the series

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots \]

This series diverges: It suffices to organize its partial sums in groups that add to at least 1/2:

\[
1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \cdots \\
\geq 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right) + \cdots \\
= 1 + 1/2 + 1/2 + 1/2 + \cdots
\]
The series

\[ 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots, \]

for \( p > 1 \) will always converge. Its sum is denoted by \( \zeta(p) \).

For example, \( \zeta(2) = \frac{\pi^2}{6} \).

This function is actually defined for all complex numbers \( p \) whose real part is \( > 1 \). It is known as Riemann zeta function. It is probably the most famous function of Mathematics.
Let us show that

\[ 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots, \]

for \( p > 1 \) will always converge.

We are going to bound each term \( 1/n^p \) by the terms of another series, and then argue the new series converges.
Consider the function $f(x) = 1/x^p$, $x \geq 2$. This is a decreasing function (draw the graph).

Observe

$$1/n^p \leq \int_{x=n-1}^{n} 1/x^p \, dx$$

Therefore its partial sums are bounded by

$$s_n \leq 1 + \int_{x=1}^{n} \frac{dx}{x^p} = 1 + \frac{1}{p-1} \left[ 1 - \frac{1}{n^{p-1}} \right] < 1 + \frac{1}{p-1}$$
Alternating the Harmonic Series

This is the series

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + (-1)^{n-1} \frac{1}{n} + \cdots \]

- Its even partial sums, \( s_0 = 1, s_2 = 1 - 1/2 + 1/3, \ldots \) are decreasing
- Its odd partial sums, \( s_1 = 1 - 1/2, s_3 = 1 - 1/2 + 1/3 - 1/4, \ldots \) are increasing
- The nested intervals \([s_1, s_0] \supset [s_3, s_2] \supset [s_5, s_4] \supset \cdots\) will define the limit 0.69... [actually \( \ln 2 \)]
Exponential Series

We claim that the series

\[ 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \]

convergent.

Note that the sequence of its partial sums is monotone but it is bounded by the partial sums of a geometric series

\[ 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} \]

a series that converges to 3. We can refine the comparison.

\[ 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{12!} = 2.71828183 \]

with error
\[ \frac{1}{13!} + \frac{1}{14!} + \cdots < \frac{1}{13!} \left( 1 + \frac{1}{13} + \frac{1}{13^2} + \cdots \right) = \frac{1}{13!} \frac{1}{1 - \frac{1}{13}} = \frac{1}{12} \frac{1}{12!} \]

a number that does not affect the 8th decimal place.
The limit of this famous series is denoted \( e \), after Euler.
We claim that the series
\[ e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \]
is not a rational number. We already know that \( 2 < e < 3 \), in particular \( e \) is not an integer. Suppose \( e = \frac{p}{q} \), with \( q \geq 2 \) since \( e \) is not an integer. Multiplying the equality by \( q! \), we have
\[
eq p(q-1)! = \left[ q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{q!} \right] + \frac{1}{q+1} + \frac{1}{(q+1)(q+2)} + \cdots
\]
Note that $p(q - 1)!$ and

$$\left[ q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{q!} \right]$$

are integers, so that its difference

$$\frac{1}{q + 1} + \frac{1}{(q + 1)(q + 2)} + \cdots$$

must also be an integer. But this series is smaller than the geometric series

$$\frac{1}{q + 1} + \frac{1}{(q + 1)^2} + \frac{1}{(q + 1)^3} + \cdots$$

whose sum is

$$\frac{1}{q + 1} - \frac{1}{q + 1} = \frac{1}{q} < 1$$
Is the series
\[(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots\]
convergent or divergent? Justify answer.

Is the series
\[\frac{1^1}{(101)!} + \frac{2^2}{(100 + 2)!} + \cdots + \frac{n^n}{(100 + n)!} + \cdots\]
convergent or divergent? Justify answer.
Outline

1. Some Goals
2. Sequences
3. Limit Theorems
4. Monotone Sequences
5. Bolzano-Weierstrass
6. Cauchy Criterion
7. Workshop #3
8. Series
9. Properties of Infinite Series
10. Convergence Tests for Series
11. Workshop #4
12. Typical E-Questions
13. Hourly #1 Review
Convergence of Series

Given the series

\[ \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots \]

there are two sequences associated to it

- The sequence of terms, \((a_n)\) and
- The sequence of partial sums, \((s_n)\),

\[ s_n = a_0 + a_1 + \cdots + a_n \]

- We say the series converges to \(A \in \mathbb{R}\) if \(\lim s_n = A\). We write this as

\[ \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + a_3 + \cdots = A \]
We pick the alternating harmonic series—which we know to be convergent—and carry out arithmetic operations: See what happens

\[
S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots
\]

\[
\frac{1}{2}S = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots
\]

\[
S + \frac{1}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{4} + \cdots
\]

Thus \( S + \frac{1}{2}S = \frac{3}{2}S \) is just a rearrangement of \( S \)! The arithmetic is saying instead that

\[
\frac{3}{2}S = S!
\]
Algebraic Limit Theorem for Series

**Theorem**

If \( \sum_{k=1}^{\infty} a_k = A \) and \( \sum_{k=1}^{\infty} b_k = B \), then:

1. \( \sum_{k=1}^{\infty} ca_k = cA \) for all \( c \in \mathbb{R} \) and
2. \( \sum_{k=1}^{\infty} (a_k + b_k) = A + B \).

**Proof.** (i) To show \( \sum_{k=1}^{\infty} ca_k = cA \), we consider the sequence of partial sums

\[
t_n = ca_1 + ca_2 + \cdots + ca_n.
\]

Since \( \sum_{k=1}^{\infty} a_k = A \), its sequence of partial sums

\[
s_n = a_1 + a_2 + \cdots + a_n
\]

converges to \( A \). By the Algebraic Limit Theorem for Sequences, \( \lim t_n = c \lim s_n = cA \).
(ii) To show that $\sum_{k=1}^{\infty}(a_k + b_k) = A + B$, let $r_n = a_1 + \cdots + a_n$, $s_n = b_1 + \cdots + b_n$ be the partial sum terms of the series. The partial sum term of the addition of the two series is

$$t_n = (a_1 + b_1) + \cdots + (a_n + b_n) = (a_1 + \cdots + a_n) + (b_1 + \cdots + b_n) = r_n + s_n.$$ 

By the Algebraic Limit Theorem for Sequences,

$$\lim t_n = \lim r_n + \lim s_n = A + B.$$
Other operations are harder: 

**Question:** Given two series, $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$ and $b_0 + b_1 + b_2 + \cdots + b_n + \cdots$, what is

$$(a_0 + a_1 + a_2 + \cdots + a_n + \cdots)(b_0 + b_1 + b_2 + \cdots + b_n + \cdots) = ?$$

Part of the issue arises from the **distributive rule**. We will offer a partial fix later.
Cauchy Criterion for Series

**Definition**
A sequence \((a_n)\) is called a **Cauchy sequence** if, for every \(\epsilon > 0\), there is an \(N \in \mathbb{N}\) such that whenever \(m, n \geq N\) it follows that \(|a_n - a_m| < \epsilon\).

Recall:

**Theorem**
A sequence converges if and only if it is a Cauchy sequence.

We apply this criterion to the sequence \((s_n)\) of partial sums of a series \(\sum_{k=1}^{\infty} a_k\). Note that

\[
|s_m - s_n| = |a_{m+1} + \cdots + a_n|
\]
Cauchy Test for Series

**Theorem**

The series \( \sum_{k=1}^{\infty} a_k \) converges if and only if given \( \epsilon > 0 \), there exists an \( N \in \mathbb{N} \) such that whenever \( n > m \geq N \) it follows that

\[
|a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon.
\]

**Proof.** Just observe

\[
|s_n - s_m| = |a_{m+1} + a_{m+2} + \cdots + a_n| < \epsilon,
\]

and apply the Cauchy’s Criterion for sequences. \( \square \)

**Corollary**

If the series \( \sum_{k=1}^{\infty} a_k \) converges, then \( (a_k) \to 0 \).

**Proof.** Set \( n = m + 1 \), then \( |s_n - s_m| = |a_n| \).
**Question:** Is a series whose sequence of terms $a_n$ converges to 0 convergent? This one is easy:

**Answer:** No. The (harmonic) series

$$1 + 1/2 + 1/3 + \cdots + 1/n + \cdots$$

has $1/n \to 0$ but it is divergent.
Comparisons

Given two series $\sum_{k \geq 1} a_k$ and $\sum_{k \geq 1} b_k$ that loosely connected we seek to link their convergence/divergence:

**Theorem (Comparison Test)**

Assume $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series satisfying $0 \leq a_k \leq b_k$ for all $k \in \mathbb{N}$.

1. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

**Proof.** Both follow from Cauchy’s Criterion applied to the partial sums

$$|a_{m+1} + a_{m+2} + \cdots + a_n| \leq |b_{m+1} + b_{m+2} + \cdots + b_n|$$

If, for instance, given $\epsilon > 0$ we can find $N$ so that for $n, m > N$

$$|b_{m+1} + a_{m+2} + \cdots + b_n| < \epsilon,$$

then the same condition will apply to the $a_n$. 
1. We know that the harmonic series, \( \sum_{n=1}^{\infty} \frac{1}{n} \), diverges. It is clear that the same happens if we form the series \( \sum_{n=N}^{\infty} \frac{1}{n} \) where \( N \) is some fixed number \( N \geq 1 \).

2. If \( a \) and \( b \) are positive numbers, consider the series [called generalized harmonic series] whose terms are given by the rule:

\[
\frac{1}{a}, \frac{1}{a+b}, \frac{1}{a+2b}, \ldots, \frac{1}{a+nb}, \ldots
\]

3. We claim that this series is also divergent: We compare the terms to a multiple of the harmonic series

\[
\frac{1}{a+bn} \geq \frac{1}{n+bn} = \frac{1}{b+1} \frac{1}{n}, \quad n \geq a
\]
Absolute Convergence Test

If \( \sum_{n=1}^{\infty} a_n \) is a series of non-negative terms, its partial sums

\[
s_n = a_1 + a_2 + \cdots + a_n, \quad s_{n+1} = s_n + a_n
\]

is a monotone sequence. Therefore, by the criterion, the series converges exactly when the sequence \((s_n)\) is bounded.

We make use of this:

**Theorem (Absolute Convergence Test)**

*If the series \( \sum_{k=1}^{\infty} |a_k| \) converges, then \( \sum_{k=1}^{\infty} a_k \) converges as well.*
Proof of the Absolute Convergence Test

1 We make use of Cauchy criterion for series: Let $\epsilon > 0$. Since the series $\sum_{k=1}^{\infty} |a_k|$ converges, there exists $N$ so that

$$|a_{n+1}| + |a_{n+1}| + \cdots + |a_m| < \epsilon \quad m \geq n > N$$

2 By the triangle inequality (one that say $|a + b| \leq |a| + |b|$), we get

$$|a_{n+1} + a_{n+1} + \cdots + a_m| < \epsilon \quad m \geq n > N$$

3 Therefore the series $\sum_{k=1}^{\infty} a_k$ satisfies the Cauchy condition and therefore converges.
The series

\[ 1 - \frac{1}{2} + \frac{1}{3} - \cdots (-1)^{n-1} \frac{1}{n} \cdots \]

is convergent (alternating harmonic series) (the one that won a Grammy’s Award), but the series of the absolute values is

\[ 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \cdots , \]

is divergent.
An alternating series is one with consecutive terms have opposite signs. One group of them is easy to study:

**Theorem (Alternating Series Test)**

Let \((a_n)\) be a sequence satisfying

1. \[ a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots, \text{ and} \]
2. \((a_n) \to 0.\)

Then the alternating series \(\sum_{n=1}^{\infty} (-1)^{n+1} a_n\) converges.

In other words: If \((a_n)\) is a decreasing sequence of positive terms then

\[
\sum_{n=1}^{\infty} (-1)^{n+1} a_n \quad \text{converges if and only if} \quad \lim_{n \to \infty} a_n = 0
\]
Proof. Observe the odd and even sequences of partial sums

\[ s_1 = a_1 \geq s_3 = a_1 - (a_2 - a_3) \geq s_5 = s_3 - (a_4 - a_5), \ldots \]

\[ s_2 = a_1 - a_2 \leq s_4 = s_2 + (a_3 - a_4) \leq s_5 = s_3 + (a_5 - a_6), \ldots \]

They are monotone and bounded: Since \((a_n) \to 0\), there exists \(a_n \leq K\), \(s_{2n} = s_{2n-1} + a_{2n} \leq s_{2n-1} + K \leq a_1 + K\), therefore the even sequence is increasing and bounded. Thus it has a limit \(l_1\). Similarly, the other sequence is decreasing and with a lower bound, so it has a limit \(l_2\). Since \(\pm a_n = s_n - s_{n-1}\) converges to 0, \(l_1 = l_2\).
Rearrangements

Definition

Let \( \sum_{k \geq 1} a_k \) be a series. A series \( \sum_{k \geq 1} b_k \) is said to be a **rearrangement** of \( \sum_{k \geq 1} a_k \) if there exists a 1–1, onto function \( f : \mathbb{N} \to \mathbb{N} \) such that \( b_{f(k)} = a_k \) for all \( k \in \mathbb{N} \).

Consider the geometric series of ratio \( q \)

\[
1 + q + q^2 + q^3 + \cdots + q^n + \cdots
\]

Now we shuffle the terms

\[
q + 1 + q^3 + q^2 + q^5 + q^4 + \cdots
\]

This is not a geometric series, but we should expect its fate linked to the first series. The next result says this.
A cautionary tale

Thus \( S + \frac{1}{2} S = \frac{3}{2} S \) is just a rearrangement of \( S \)! The arithmetic is saying instead that

\[
\frac{3}{2} S = S!
\]
Theorem (Dirichlet)

The sum of a series of positive terms [convergence/divergence] is the same in whatever order [rearrangement] the terms are taken.

Proof. Let $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$ be a series of positive terms of sum $s$. Then any partial sum of rearrangement $b_0 + b_1 + b_2 + \cdots + b_n + \cdots$ is bounded by $s$. Thus the second is convergent and its sum $t$ is bound by $s$. We reverse the roles to obtain $s \leq t$. □
Question: Given two series, $a_0 + a_1 + a_2 + \cdots + a_n + \cdots$ and $b_0 + b_1 + b_2 + \cdots + b_n + \cdots$, what is

$$(a_0 + a_1 + a_2 + \cdots + a_n + \cdots)(b_0 + b_1 + b_2 + \cdots + b_n + \cdots) = ?$$
The issue is: we have all the products $a_m b_n$ that can be organized into many different series, and then grouped. For instance, if we list the $a_m b_n$ as the double array, we
We could try the following: **Define** the product as the series

\[ a_0 b_0 + (a_0 b_1 + a_1 b_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0) + \cdots \]

Makes sense? [Discuss] Will see another rearrangement soon.

\[
\begin{align*}
a_0 b_0 & \quad a_1 b_0 & \quad a_2 b_0 & \quad a_3 b_0 & \quad \cdots \\
a_0 b_1 & \quad a_1 b_1 & \quad a_2 b_1 & \quad a_3 b_1 & \quad \cdots \\
a_0 b_2 & \quad a_1 b_2 & \quad a_2 b_2 & \quad a_3 b_2 & \quad \cdots \\
a_0 b_3 & \quad a_1 b_3 & \quad a_2 b_3 & \quad a_3 b_3 & \quad \cdots \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots
\end{align*}
\]
The partial sums remind us how polynomials are multiplied

\[(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)(b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m)\]

\[= \sum_{k=0}^{m+n} \left( \sum_{0 \leq i \leq k} a_i b_{k-i} \right) x^k\]

\[a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_2, \ldots\]

Another aspect of this definition is:

**Theorem**

If \(\sum_{n \geq 0} a_n\) and \(\sum_{n \geq 0} b_n\) are two convergent series of positive terms, and \(s\) and \(t\) are their respective sums, then the third series is convergent and has the sum \(st\).
Out of all products $a_m b_n$, the ‘product’ above is given in terms of the diagonals

\[
\begin{align*}
a_0 b_0 & \quad a_1 b_0 & \quad a_2 b_0 & \quad a_3 b_0 & \quad \ldots \\
a_0 b_1 & \quad a_1 b_1 & \quad a_2 b_1 & \quad a_3 b_1 & \quad \ldots \\
a_0 b_2 & \quad a_1 b_2 & \quad a_2 b_2 & \quad a_3 b_2 & \quad \ldots \\
a_0 b_3 & \quad a_1 b_3 & \quad a_2 b_3 & \quad a_3 b_3 & \quad \ldots \\
\ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots \\
\end{align*}
\]

\[a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_2, \ldots\] whose partial sums don’t write conveniently.
We want to re-write the terms of the product series differently:

\[
\begin{align*}
  a_0 b_0 & \quad a_1 b_0 & \quad a_2 b_0 & \quad a_3 b_0 & \quad \ldots \\
  a_0 b_1 & \quad a_1 b_1 & \quad a_2 b_1 & \quad a_3 b_1 & \quad \ldots \\
  a_0 b_2 & \quad a_1 b_2 & \quad a_2 b_2 & \quad a_3 b_2 & \quad \ldots \\
  a_0 b_3 & \quad a_1 b_3 & \quad a_2 b_3 & \quad a_3 b_3 & \quad \ldots \\
  \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots \\
\end{align*}
\]

\[
a_0 b_0, (a_0 + a_1)(a_0 + a_1) - a_0 b_0, \\
(a_0 + a_1 + a_2)(b_0 + b_1 + b_2) - (a_0 + a_1)(b_0 + b_1), \ldots \text{ whose } n\text{th partial sum is} \\
(a_0 + a_1 + \cdots + a_n)(b_0 + b_1 + \cdots + b_n),
\]

a sequence that converges to \( st \) by the Algebraic Limit Theorem.
Theorem

If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then any rearrangement of this series converges to the same limit.

Proof. Assume $\sum_{k \geq 1} a_k$ converges absolutely to $A$, and let $\sum_{k \geq 1} b_k$ be an rearrangement of $\sum_{k \geq 1} a_k$. Let

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

and

$$t_n = \sum_{k=1}^{n} b_k = b_1 + b_2 + \cdots + b_n$$

be the corresponding partial sums.

Let $\varepsilon > 0$. Since $(s_n) \rightarrow A$, choose $N_1$ such that

$$|s_n - A| < \frac{\varepsilon}{2}$$

for all $n \geq N_1$. 
Because the convergence is absolute, we can choose $N_2$ so that

$$\sum_{k=m+1}^{n} |b_k| < \epsilon/2$$

for all $n > m \geq N_2$. Take $N = \max\{N_1, N_2\}$. We know that the terms \( \{a_1, a_2, \ldots, a_N\} \) must all appear in the rearranged series, and we move far out enough in the series $\sum_{k \geq 1} b_k$ that these terms are all included. Thus, choose $M = \max\{f(k) \mid 1 \leq k \leq N\}$.

It is clear that if $m \geq M$, then $(t_m - s_N)$ consists of a finite number of terms, the absolute values of which appear in the tail of $\sum_{k=N+1}^{\infty} |a_k|$. The earlier choice of $N_2$ guarantees $|t_m - s_N| < \epsilon/2$, and so

$$|t_m - A| = |t_m - s_N + s_N - A| \leq |t_m - s_N| + |s_N - A| \leq \epsilon/2 + \epsilon/2 = \epsilon$$
Outline

1. Some Goals
2. Sequences
3. Limit Theorems
4. Monotone Sequences
5. Bolzano-Weierstrass
6. Cauchy Criterion
7. Workshop #3
8. Series
9. Properties of Infinite Series
10. **Convergence Tests for Series**
11. Workshop #4
12. Typical E-Questions
13. Hourly #1 Review
Convergence Tests for Series

3 elementary tests of convergence

- Integral Test
- Ratio Test
- Root Test
Theorem (Integral Test)

Let \( \sum_{n \geq 0} a_n \) be a series of positive terms. If there is a decreasing function \( f(x) \) such that \( a_n \leq f(n) \) for large \( n \) and

\[
\int_{x=1}^{\infty} f(x) \, dx < \infty,
\]

then \( \sum_{n \geq 0} a_n \) converges.

Proof. If \( a_n \leq f(n) \) for \( n \geq n_0 \), since \( f(x) \) is decreasing,

\[
a_n \leq \int_{n-1}^{n} f(x) \, dx, \quad n > n_0.
\]

From this, and the assumption that \( \int_{1}^{\infty} f(x) \, dx < \infty \), we get that the partial sums of the series \( \sum_{n \geq 0} a_n \) are bounded, and therefore converge by the theorem on bounded monotone sequences.
Consider the function \( f(x) = 1/x^p, \ x \geq 2 \). This is a decreasing function (draw the graph).

Observe

\[
\frac{1}{n^p} \leq \int_{x=n-1}^{n} \frac{1}{x^p} \, dx
\]

Therefore its partial sums are bounded by

\[
s_n \leq 1 + \int_{x=1}^{n} \frac{dx}{x^p} = 1 + \frac{1}{p-1} \left[ 1 - \frac{1}{n^{p-1}} \right] < 1 + \frac{1}{p-1}
\]
Let us show that

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots,$$

for $p > 1$ will always converge.

We are going to bound each term $1/n^p$ by the terms of another series, and then argue the new series converges.
Comparison gives

\[
\sum_{n \geq 1} \frac{1}{n(n+1)} \leq \sum_{n \geq 1} \frac{1}{n^2}
\]

which is convergent.

In the same manner, if

\[
\sum_{n \geq 1} \frac{p(n)}{q(n)},
\]

where \( p(n) \) and \( q(n) \) are positive polynomial expressions with \( \deg q \geq 2 + \deg p \), then the series converges by the same reason. Do it!
There are very useful tests involving the ratio $a_{n+1}/a_n$ of two successive terms of a series. Sometimes we compare the ratio $a_{n+1}/a_n$ to another $b_{n+1}/b_n$. In these we suppose that $a_n$ and $b_n$ are strictly positive.

Suppose $a_n, b_n > 0$ and that

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$$

for sufficiently large $n$, that is for $n \geq n_0$.

Then

$$a_n = \frac{a_{n_0+1}}{a_{n_0}} \frac{a_{n_0+2}}{a_{n_0+1}} \cdots \frac{a_n}{a_{n-1}} a_{n_0} \leq \frac{b_{n_0+1}}{b_{n_0}} \frac{b_{n_0+2}}{b_{n_0+1}} \cdots \frac{b_n}{b_{n-1}} a_{n_0} = \frac{a_{n_0}}{b_{n_0}} b_n = Cb_n, \quad C = a_{n_0}/b_{n_0}.$$
Here are some applications:

Theorem

Let $\sum a_n$ and $\sum b_n$ be series of positive terms.

1. If for $n \geq n_0$

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n},$$

and the series $\sum b_n$ converges, then $\sum a_n$ converges also.

2. If for $n \geq n_0$

$$\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n},$$

and the series $\sum a_n$ diverges, then $\sum b_n$ diverges also.

Theorem (d’Alambert Test)

The series $\sum a_n$ is convergent if $a_{n+1}/a_n \leq r$, where $r < 1$, for all sufficiently large $n$. 
Theorem

Given a series \( \sum_{n \geq 1} a_n \) with \( a_n \neq 0 \), if \( (a_n) \) satisfies

\[
\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1,
\]

then the series converges absolutely.

Proof.

1. Let \( r' \) satisfy \( r < r' < 1 \). For \( \epsilon = r' - r \), there is \( N \) such that for \( n \geq N \) \( \left| \frac{a_{n+1}}{a_n} \right| - r < \epsilon \), and therefore

\[
\left| \frac{a_{n+1}}{a_n} \right| - r \leq \left| \frac{a_{n+1}}{a_n} - r \right| < \epsilon = r' - r,
\]

giving \( \left| a_{n+1} \right| \leq r' \left| a_n \right| \) for \( n \geq N \).

2. The above shows that the series \( \sum_{n=N}^{\infty} \left| a_n \right| \) satisfies

\[
\left| a_n \right| \leq \left| a_N \right| (r')^{n-N},
\]

a geometric series of ratio \( r' < 1 \), which converges.
A quick application of the ratio test: We claim that the series
\[ 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \]
converges for all values of \( x \).

For the ratio of consecutive terms
\[ \frac{a_{n+1}}{a_n} = \frac{x^{n+1}/(n+1)!}{x^n/n!} = \frac{x}{n+1} \]
so that for any \( x \), \( \lim a_{n+1}/a_n = 0 \).

This is a well used technique for power series.
Examples

1. For the series \( \sum_{n \geq 1} \frac{n}{2^n} \) we invoke the ratio test:

\[
\frac{a_{n+1}}{a_n} = \frac{n + 1}{2^{n+1}} / \frac{n}{2^n} = \frac{n + 1}{n} \frac{1}{2}
\]

which has limit \( 1/2 < 1 \). So the series converges.

2. Decide [with justification] whether the series

\[
\sum_{n \geq 1} \frac{n!}{n^n},
\]

is convergent or divergent?
1. Show that if $a_n > 0$ and $\lim n a_n = L$, with $L \neq 0$, then the series $\sum a_n$ diverges.

2. Show that if $a_n > 0$ and $\lim n^2 a_n = L$, with $L \neq 0$, then the series $\sum a_n$ converges.

3. Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more difficult, choose examples where $(a_n)$ and $(b_n)$ are positive and decreasing.
Let $\sum_{n \geq 1} a_n$ be a series of positive terms. We are going to examine how the limit

$$\lim_{n \to \infty} \sqrt[n]{a_n}$$

is used to decide convergence. We recall one special calculation of these limits: If $x > 0$

$$\lim_{n \to \infty} \sqrt[n]{x} = 1$$

Recall another limit: $\lim_{n \to \infty} \sqrt[n]{n} = 1$. 
Theorem

If \( \sum_{n \geq 1} a_n \) is a series of positive terms and \( \lim_{n \to \infty} \sqrt[n]{a_n} = r < 1 \), then the series converges.

Proof. Let \( r < r' < 1 \) and pick \( \epsilon = r' - r \). This is the same subtle point we used above.

1. There is \( N \) so that for \( n > N \)

   \[ |\sqrt[n]{a_n} - r| < \epsilon \]

2. This implies that \( \sqrt[n]{a_n} < r + \epsilon = r' < 1 \) for \( n > N \). As a consequence

   \[ a_n < (r')^n \]

3. We now compare the series \( \sum_{n \geq 1} n \geq 1 a_n \) to the geometric series \( \sum_{n \geq 1} (r')^n \) of ratio \( r' < 1 \). Thus both series converge.
Example

Consider the series (for $q > 0$)

$$1 + q + 2q^2 + \cdots + nq^n + \cdots$$

We invoke the root test

$$\lim_{n \to \infty} \sqrt[n]{nq^n} = q \lim_{n \to \infty} \sqrt[n]{n} = q$$

Therefore it converges if $q < 1$

Let us calculate the sum of the series. For that we must have an inkling on how the series arose from the geometric series. At these times we replace $q$ by $x$ and recall:
1. Differentiate the ‘equality’

\[
\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots
\]

2. To get almost our series

\[
\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots
\]

3. Now multiply by \(x\) and add 1

\[
1 \cdot \frac{x}{(1-x)^2} = 1 + x + 2x^2 + \cdots + nx^n + \cdots
\]

4. Thus for \(0 < q < 1\) the series sums to

\[
1 + \frac{q}{(1-q)^2}
\]
Exercises

- Show that the series

\[ 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \]

converges. (Hint: Look up one of the class examples)

To find the limit, sum the geometric series

\[ 1 - x^2 + x^4 - x^6 + \cdots , \]

and integrate over \([0, 1]\). Indicate what steps will have to be properly justified.

- Is the series

\[ \frac{1^1}{(101)!} + \frac{2^2}{(100 + 2)!} + \cdots + \frac{n^n}{(100 + n)!} + \cdots \]

convergent or divergent? Justify answer.
1. Show that

\[ \sum_{n \geq 0} (-1)^n \frac{2n + 3}{(n + 1)(n + 2)} = 1. \]

2. Determine the values of \( q \) for which the series

\[ q + 2q^2 + 3q^3 + \cdots + nq^n + \cdots \]

is convergent.

3. Show that \( \sum_{n \geq 2} \frac{1}{n(\ln n)^p} \) converges if \( p > 1 \), and diverges if \( p \leq 1 \).
Workshop #4

Think/Do next 4 Questions [in 2 frames]

1. Find the sum of the series
\[ \sum_{n \geq 1} \frac{1}{n(n + 4)}. \]

As a warmup, find the sum of the series
\[ \sum_{n \geq 1} \frac{1}{n(n + 1)}. \]

2. Show that if \( a_n > 0 \) and \( \lim_{n \to \infty} n^p a_n = L \), with \( L \neq 0 \) for some integer \( p > 1 \), then the series \( \sum a_n \) converges. An application: If
\[ \sum_{n \geq 1} \frac{p(n)}{q(n)}, \]

where \( p(n) \) and \( q(n) \) are positive polynomial expressions with \( \deg q \geq 2 + \deg p \), then the series converges.
3 Determine the values of \( q > 0 \) for which the following series converges and find its sum

\[
1 + q + \frac{q^2}{2} + \cdots + \frac{q^n}{n} + \cdots.
\]

Calculate the sum of the series.

4 Is the following series

\[
\sum_{n \geq 0} e^{-n^2}
\]

convergent or divergent? Try all [ratio, root, and integral tests]
Exercises

1. Show that the sequence

\[ \frac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+2} - \sqrt{n+1}}, \quad n \in \mathbb{N} \]

converges. As a challenge, find also a bound for it.

2. Let \(0 \leq a, b \in \mathbb{R}\) and define recursively \(a_0 = a, b_0 = b, a_{n+1} = \sqrt{a_n b_n}\) and \(b_{n+1} = (a_n + b_n)/2\). Show that \([a_n, b_n]\) form a nested sequence of intervals. Prove that the intersection of these intervals is a single point.

3. If the series \(\sum_{n \geq 1} a_n^2\) and \(\sum_{n \geq 1} b_n^2\) are convergent, prove that \(\sum_{n \geq 1} a_n b_n\) is convergent.
Write \( n^{\sqrt{n}} = 1 + a_n \), so that \( n^{\frac{\sqrt{n}}{n}} = (1 + a_n)^2 \) and \( \sqrt{n} = (1 + a_n)^n \).

By a Lemma we have used often, \( \sqrt{n} = (1 + a_n)^n \geq 1 + na_n > na_n \),

\[
\frac{1}{\sqrt{n}} = \frac{\sqrt{n}}{n} > a_n
\]

Thus

\[
1 \leq n^{\sqrt{n}} = (1 + a_n)^2 = 1 + 2a_n + a_n^2 < 1 + \frac{2}{\sqrt{n}} + \frac{1}{n}
\]

Therefore, by the Squeeze Theorem, \( \lim_{n \to \infty} n^{\sqrt{n}} = 1 \)
Typical E-Questions

• Prove that bounded monotone sequences are convergent.
• Why the cardinalities of \( \mathbb{N} \) and of \( \mathbb{N}^4 \) are the same?
• If \( (a_n) \to a \) and \( (b_n) \to b \), with \( b_n, b \neq 0 \), prove that \( \lim (a_n/b_n) = a/b \).
• What is the \textbf{nested interval property} of \( \mathbb{R} \)? Give an interesting example and sketch the proof.
• If \( (a_n) \) and \( (b_n) \) are sequences such that \( \lim a_n + b_n = 5 \) and \( \lim a_n = 2 \), must \( (b_n) \) be convergent? Explain or give counter-example.
• If \( (a_n) \to 5, a_n \geq 0 \), prove with full details that \( \lim \sqrt{a_n} = \sqrt{5} \). [You may use \( \epsilon = 1/10 \).]
• Find \( \lim \sqrt{a_{n+1}b^n + b^{n+1}c^n + c^{n+1}a^n} \), with \( a > b > c > 0 \).
• Do all sequences have a convergent subsequence? If not, when? Explain.

• Let \((a_n)\) and \((b_n)\) be two Cauchy sequences. Prove directly that \((a_n b_n)\) is a Cauchy sequence.

• If \(a\) is a positive integer, give a formula for the sum of the series

\[
\sum_{n \geq 1} \frac{1}{n(n + a)}.
\]
A beautiful limit

• Prove that \( \lim_{n \to \infty} n(\sqrt[n]{x} - 1) \), \( x > 0 \), exists. [Not easy, not in exam, just tossed as a challenge.]

The limit defines a function \( f(x) \). Observe the property

\[
n(\sqrt[n]{xy} - 1) = n(\sqrt[n]{x} - 1)\sqrt[n]{y} + n(\sqrt[n]{y} - 1)
\]

Taking into account \( \lim_{n \to \infty} \sqrt[n]{y} = 1 \) from a Workshop, we get

\[
f(xy) = f(x) + f(y),
\]

a defining property of Logs. [? Maybe \( f(x) = e^x \)]
(15 pts)

1. What is a countable set?
2. Why is \( \mathbb{Q} \) countable?
3. Prove that \( \mathbb{N} \) and \( \mathbb{N}^2 \) have the same cardinality.

(10 pts) Prove that the sequence defined by \( x_1 = 3 \) and

\[
x_{n+1} = \frac{1}{4 - x_n}
\]

converges.

(15 pts) Describe very carefully and in full the following terms:

1. **lower bound** of a subset \( A \subset \mathbb{R} \)
2. **Nested Interval Property**
3. give an example for each term.
(15 pts)
1. Define precisely the notion of a **convergent** sequence.
2. What is a **subsequence** of a sequence?
3. Prove that all subsequences of a convergent sequence have the same limit.

(15 pts)
1. What is a **monotone** sequence? Give an example.
   If a monotone sequence \((a_n)\) is bounded, prove that it is convergent.

(15 pts) Find (with proof!) the limit of the sequence

\[ \sqrt[n]{a^n b^n + b^n c^n + c^n a^n}, \quad a > b > c > 0. \]

(15 pts)
1. What is a **Cauchy** sequence?
2. If \((a_n)\) and \((b_n)\) are Cauchy sequences, prove directly that \((a_n b_n)\) is a Cauchy sequence.
The equation $x^3 - 3x + 1 = 0$ has a root $\alpha$ between 0 and 1. To find it, define the sequence

$$x_1 = 0, \quad x_{n+1} = \frac{1}{3 - x_n^2}$$

Show that the sequence is monotone and converges to $\alpha$. 
1. Show that if $a_n > 0$ and $\lim n a_n = L$, with $L \neq 0$, then the series $\sum a_n$ diverges.

2. Show that if $a_n > 0$ and $\lim n^2 a_n = L$, with $L \neq 0$, then the series $\sum a_n$ converges.

3. Find examples of two series $\sum a_n$ and $\sum b_n$ both of which diverge but for which $\sum \min\{a_n, b_n\}$ converges. To make it more difficult, choose examples where $(a_n)$ and $(b_n)$ are positive and decreasing.
Outline

1. Some Goals
2. Sequences
3. Limit Theorems
4. Monotone Sequences
5. Bolzano-Weierstrass
6. Cauchy Criterion
7. Workshop #3
8. Series
9. Properties of Infinite Series
10. Convergence Tests for Series
11. Workshop #4
12. Typical E-Questions
13. Hourly #1 Review
Important Topics

- Least Upper Bound
- Axiom of Completeness
- Cardinality: Countable and Uncountable Sets, Power Sets
- Sequences, Convergence/Divergence
- Monotone Sequences
- Bolzano-Weirstrauss Theorem
- Cauchy Sequences
- Series: Backbone Examples
- Convergence of Series: Meaning
- Tests of Convergence: Integral, Ratio, Root