1. Cardinality

2. Homework #12

3. Infinite Sets

4. Cantor’s Universe

5. Homework #13

6. The Ordering of Cardinal Numbers

7. Final Orientation
Let us begin by introducing a method to size sets. If $A$ and $B$ are two sets we will use functions 

$$f : A \to B$$

...to compare their sizes.

**Definition**

For pair of sets $(A, B)$ we write $A \approx B$ if there is a function $f : A \to B$ that is both **one-to-one** and **onto**.
Recall

- **f one-to-one:** If $x \neq y \Rightarrow f(x) \neq f(y)$

  In particular if $f : A \rightarrow B$ and $g : B \rightarrow C$ are one-to-one

  $$x \neq y \Rightarrow f(x) \neq f(y) \Rightarrow g(f(x)) \neq g(f(y)),$$

  so $g \circ f$ is one-to-one.

- **f onto:** $\forall b \in B \quad \exists x \in A : f(x) = b$

  In particular if $f : A \rightarrow B$ and $g : B \rightarrow C$ are onto

  $$\forall c \in C \quad \exists b \in B : g(b) = c \quad \exists a \in A : f(a) = b.$$

  Thus $g(f(a)) = g(b) = c$ and so $g \circ f$ is onto.
Proposition

\( \approx \) is an equivalence relation.

Proof. Let us verify the requirements:

1. (reflexivity) \( A \approx A \): because \( I_A : A \rightarrow A \) is one-to-one onto.

2. (symmetry) \( A \approx B \Rightarrow B \approx A \): because if \( f : A \rightarrow B \) is one-to-one onto then \( f^{-1} : B \rightarrow A \) is one-to-one onto.

3. (transitivity) If \( A \approx B \) and \( B \approx C \) then \( A \approx C \): because if \( f : A \rightarrow B \) is one-to-one onto and \( g : B \rightarrow C \) is one-to-one onto then \( g \circ f : A \rightarrow C \) is one-to-one onto.

Definition

The equivalence class of \( A \) is called the cardinality of \( A \), \( \text{card}(A) \).
Let $E$ be the set of even numbers,

$$E = \{2, 4, \ldots, 2n, \ldots\}$$

The function $f : \mathbb{N} \rightarrow E$, given by $f(n) = 2n$, gives a one-to-one & onto correspondence between the sets $\mathbb{N}$ and $E$.

We write this as $\text{card} \ (E) = \text{card} \ (\mathbb{N})$: There are as many even numbers as natural numbers...
Equivalence of Sets

**Definition**

Two sets $A$ and $B$ are **equivalent** iff there exists a one-to-one function from $A$ onto $B$, and denote $A \approx B$.

**Example:** The set $E$ of even numbers is equivalent to the set $O$ of odd numbers:

$$f : E \rightarrow O, \quad f(2n) = 2n - 1, \quad n \in \mathbb{N}.$$
Example

Theorem

For \( a, b, c, d \in \mathbb{N} \), with \( a < b \) and \( c < d \), the open intervals \((a, b)\) and \((c, d)\) are equivalent.

Proof. Let \( f \) be the linear function

\[
f(x) = \frac{d - b}{c - a} (x - a) + c.
\]

We must show that \( f : (a, b) \rightarrow (c, d) \) is one-to-one and onto.
In some cases, [the case above included], it is possible to build $f^{-1}$ by solving the equation for $x$

$$f(x) = y, \quad x = f^{-1}(y).$$

$$y = \frac{d - b}{c - a}(x - a) + c,$$

gives

$$x - a = \frac{c - a}{d - b}(y - c)$$

$$x = f^{-1}(y) = \frac{c - a}{d - b}(y - c) + a$$
(0, ∞) ≈ [0, ∞)

Split (0, ∞) and [0, ∞) as follows

\[
(0, \infty) = (0, 1) \cup \{1\} \cup (1, 2) \cup \{2\} \cup (2, 3) \cup \{3\} \cup \cdots
\]

\[
\{0\} \cup (0, \infty) = \{0\} \cup (0, 1) \cup \{1\} \cup (1, 2) \cup \{2\} \cup (2, 3) \cup \cdots
\]

Define the function

\[
f(n) = n - 1
\]

\[
f(x) = x
\]

for all other \( x \).
As a challenge, prove

**Theorem**

For \(a, b \in \mathbb{R},\) with \(a < b,\) the intervals \((a, b)\) and \([a, b]\) are equivalent.
Claim: Let $\mathcal{F}$ be the set of functions from $\mathbb{N}$ to the set of two elements $\{0, 1\}$. Then $\mathcal{F} \approx \mathcal{P}(\mathbb{N})$, the power set of $\mathbb{N}$.

Define the correspondence

$$F : \mathcal{F} \rightarrow \mathcal{P}(\mathbb{N}), \quad F(g) = \{x \in \mathbb{N} : g(x) = 1\}.$$  

1. One-to-one: If $f$ and $g$ are different functions, then there is $x \in \mathbb{N}$ so that $f(x) \neq g(x)$. This means one of these values is 1, the other is 0. Thus the sets $F(f)$ and $F(g)$ different.

2. Onto: Let $A$ be a subset of $\mathbb{N}$. Let $\chi_A$ be the characteristic function of $A$ (someone recalls?) $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise. Note that $F(\chi_A) = A$. 

Theorem

Suppose $A, B, C$ and $D$ are sets and $A \approx C$ and $B \approx D$. Then

1. $A \times B \approx C \times D$.

2. If $A$ and $B$ are disjoint and $C$ and $D$ are disjoint, then $A \cup B \approx C \cup D$.

Proof. Let $f : A \rightarrow C$ and $g : B \rightarrow D$ be one-to-one and onto functions.

1. Let $h : A \times B \rightarrow C \times D$ be given by $h(a, b) = (f(a), g(b))$. It is easy to verify that $h$ is one-to-one and onto.

2. We can glue the functions $f$ and $g$: $f \cup g : A \cup B \rightarrow C \cup D$, so that if $a \in A$, $(f \cup g)(a) = f(a)$, while if $b \in B$, $(f \cup g)(b) = g(b)$. Again, it is clear that $f \cup g$ is one-to-one and onto.
Those rules extend to other products and sums:

**Theorem**

Suppose $A_1, A_2, \ldots, A_n$ and $C_1, C_2, \ldots, C_n$ are two families of sets and for all $i$, $A_i \approx C_i$. Then

1. $A_1 \times A_2 \times \cdots \times A_n \approx C_1 \times C_2 \times \cdots \times C_n$.
2. If the $A_i$ are disjoint and the $C_i$ are disjoint, then $A_1 \cup A_2 \cup \cdots \cup A_n \approx C_1 \cup C_2 \cup \cdots \cup C_n$.

The proof earlier will work. Even works for arbitrary collections of sets.
Finite and Countable Sets

We use the following notation:

\[ \mathbb{N}_n = \{1, 2, \ldots, n\} \subset \mathbb{N} \]

and the following terminology

**Definition**

A set \( S \) is **finite** if \( S = \emptyset \) or \( S \approx \mathbb{N}_k \) for some natural number \( k \). A set \( S \) is **infinite** if \( S \) is not finite.

The attending class today is finite, since we can set a correspondence between it and some \( \mathbb{N}_k \) (\( k \leq 18 \)).

**Definition**

Let \( S \) be a finite set. If \( S \approx \mathbb{N}_k, k \in \mathbb{N} \), we say that \( S \) has **cardinal number** \( k \) (or **cardinality** \( k \)), and write \( \overline{S} = k \). If \( S = \emptyset \) we say that \( S \) has **cardinal number** \( 0 \) (or **cardinality** \( 0 \)) and write \( \overline{\emptyset} = 0 \).
Definition

A set $A$ is said to be **countable**, or **denumerable**, if $A \approx \mathbb{N}$:

$$f : \mathbb{N} \to A$$

$$A = \{f(1), f(2), \ldots, \}.$$  

We write that $\text{card} (A) = \text{card} (\mathbb{N}) = \aleph_0$, and say that $A$ has **cardinal number** $\aleph_0$ and write $\bar{A} = \aleph_0$.

**Warning about Terminology:** The correct usage is to call a set **countable** if it is equivalent to $\mathbb{N}$ or finite. We abuse this often by the definition above.

**Exercise:** If $\text{card} (A)$ is countable and $B \subset A$, then $B$ is countable or finite.

It is obviously a tricky thing to determine the cardinality of sets, particularly of infinite sets. Let us get our hands busy!
Question: Why/How can we list a subset $A$ of the natural numbers $\mathbb{N}$?

1. If $A = \emptyset$, there is nothing to do.
2. If $A$ is not empty, let $a_1$ be the smallest element of $A$. (someone: why can we do this?)
3. Let $A_1 = A \setminus \{a_1\}$. If $A_1 = \emptyset$ we are done; otherwise let $a_2$ be its smallest element.
4. Let $A_2 = A \setminus \{a_1, a_2\}$. If $A_2 = \emptyset$ we are done; otherwise let $a_3$ be its smallest element.
5. In this manner we list the elements of $A$:

$$a_1, a_2, a_3, \cdots$$

6. If the list stops at $a_n$, we have a one-to-one correspondence $f : \{1, 2, \ldots, n\} \rightarrow A$, $f(i) = a_i$, $i \leq n$.
7. Otherwise we have a one-to-one correspondence $f : \mathbb{N} \rightarrow A$, $f(i) = a_i$, $i \in \mathbb{N}$. 
The set $\mathbb{N} \times \mathbb{N}$ is countable: Let define a one-one function

$$f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$$

Define

$$f(m, n) = 2^m 3^n$$

By the unique factorization on integers,

$$2^m 3^n = 2^p 3^q \Rightarrow m = p \quad n = q,$$

which proves the claim that $f$ is injective.
Exercise: Use the infinity of prime numbers to show that the set $X$ of all infinite tuples $(x_1, x_2, x_3, \ldots)$, $x_i \in \mathbb{N}$, such that all $x_i = 0$ except for finitely many exceptions is countable.

Let $P$ be the set of primes, $P = \{p_1, p_2, p_3, \ldots, p_n, \ldots\}$.

Now define the function $f : X \rightarrow \mathbb{N}$ by the rule

$$f(x_1, x_2, \ldots, x_n, \ldots) = p_1^{x_1} p_2^{x_2} \cdots p_n^{x_n} \cdots.$$ 

$f$ is well-defined because almost all $x_i$ are 0. $f$ is one-to-one by the unique factorization of integers by primes.
In one of our examples weeks back, we considered the function

\[ f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \]

given by

\[ f(m, n) = 2^{m-1}(2n - 1). \]

We proved that \( f \) is one-to-one & onto.
1. Prove that $[0, 1] \approx (0, 1)$

2. 5.1: 3(a, i, n), 6(b), 17(a,b), 20(b)
Theorem

Let \( n, r \in \mathbb{N} \). If \( f : \mathbb{N}_n \rightarrow \mathbb{N}_r \) and \( n > r \) then \( f \) is not one-to-one.
Proof of the Pigeonhole Principle

We prove this by induction on $n$.

1. If $n = 2$, since $r < n$, $r = 1$. In this case $f$ is a constant function, $f(1) = f(2) = 1$, so $f$ is not one-to-one.

2. Suppose the Pigeonhole Principle holds for all $r < n$. We argue by contradiction. Suppose $r < n+1$ and $h : \mathbb{N}_{n+1} \rightarrow \mathbb{N}_r$ is one-to-one. The restriction $h_0$ of $h$ to $\mathbb{N}_n$ is one-to-one. Furthermore the range of this function does not contain $h(n+1)$.

3. There is a one-to-one function $g : \mathbb{N}_r \setminus \{h(n+1)\} \rightarrow \mathbb{N}_{r-1}$. Let $f = g \circ h_0$. Thus $f : \mathbb{N}_n \rightarrow \mathbb{N}_{r-1}$ is one-to-one since it is the composite of one-to-one functions. Thus is a contradiction of the induction hypothesis.

4. By the PMI, for every $n \in \mathbb{N}$ if $r < n$ there is no one-to-one function from $\mathbb{N}_n$ to $\mathbb{N}_r$. 

Why Someone?
Exercise

5.1, 20(a): Prove that if five points are in or on a square with sides of length 1, then at least two points are no farther apart than $\sqrt{2}/2$.

For instance, if 4 points are chosen at the vertices then the fifth point must be chosen in one of the 4 triangles determined by the center. The distance of that point to one of the corner points is at most $\sqrt{2}/2$.

Solution: To use the Pigeonhole Principle, split the square into 4 squares of sides of length $1/2$. According to the Pigeonhole Principle, we would have to put at least two points in the same little square: they could not be further apart than $\sqrt{2}/2$. 
Corollaries of the Pigeonhole Principle

Let \( A \) be a finite set, that is \( A \approx \mathbb{N}_n \) for some \( n \). If \( A \approx \mathbb{N}_m \) then \( m = n \).

**Proof:** The first hypothesis means: There is \( f : \mathbb{N}_n \rightarrow A \) that is one-to-one. The second hypothesis means: There is \( h : A \rightarrow \mathbb{N}_m \) that is one-to-one. It follows that

\[
h \circ f : \mathbb{N}_n \rightarrow \mathbb{N}_m
\]

is one-to-one. Therefore \( n \geq m \). Reverse the roles of \( m \) and \( n \) to get \( m \geq n \). Thus \( m = n \).
Corollary to Pigeonhole Principle

Corollary

A finite set is not equivalent to any of its proper subsets.

Proof: We first show that the set $\mathbb{N}_k$ is not equivalent to any of its proper subsets.

If $k = 1$, the only proper subset of $\mathbb{N}_k$ is $\emptyset$ and $\{1\}$ is not equivalent to $\emptyset$. Assume $k > 1$ and $A$ is a proper subset of $\mathbb{N}_k$ and $f: \mathbb{N}_k \approx A$ is one-to-one onto.

There are two cases to consider:

- If $k \notin A$, then $A \subset \mathbb{N}_{k-1}$, and the inclusion function $i: A \rightarrow \mathbb{N}_{k-1}$ is one-to-one. But then the composite $i \circ f: \mathbb{N}_k \rightarrow \mathbb{N}_{k-1}$ would be one-to-one, violating the Pigeonhole Principle.
Suppose $k \in A$. Choose $y \in \mathbb{N}_k \setminus A$. Let $A' = (A \setminus \{k\}) \cup \{y\}$. Then $A \approx A'$ as we simply exchanged $k$ by $y$ in $A$. Thus $A' \approx \mathbb{N}_k$. From the previous case we get a contradiction.
Theorem

The sets $\mathbb{Z}$ and $\mathbb{Q}$ are countable.

We must establish one-one & onto correspondences between $\mathbb{N}$ and each of these sets. In other words, we must describe $\mathbb{Z}$ and $\mathbb{Q}$ as long lists

$$\{f(1), f(2), \ldots, \}.$$ 

For $\mathbb{Z}$, this is very easy

$$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots, \pm n, \ldots\}$$

for example, $0 = f(1)$, $23 = f(46)$, $-55 = f(111)$. If we cared, $f$ can even be made explicit.
A list description of \( \mathbb{Q} \) is not much different. Each \( x \in \mathbb{Q} \), can be written uniquely as

\[
x = \pm \frac{p}{q} \quad | \quad p \geq 0, q > 0
\]

\( \gcd(p, q) = 1 \) when \( q \neq 0 \). Define the finite subsets of \( \mathbb{Q} \), \( A_0 = \{0\} \), for \( n \geq 1 \)

\[
A_n = \left\{ \pm \frac{p}{q} \quad | \quad p + q = n \right\}.
\]

\[
A_{10} = \{\pm 1/9, \pm 3/7\}
\]

\[
\mathbb{Q} = A_0 \cup A_1 \cup A_2 \cup \cdots \cup A_n \cup \cdots
\]

is a disjoint union of finite sets. Listing the elements of each \( A_n \) gives a desired listing for \( \mathbb{Q} \).

A more general argument is the following:
Theorem

If the sets $A_i$, $i \geq 1$, are countable, then $A = \bigcup_{i=1}^{\infty} A_i$ is countable.

Proof. Here is a way to list the elements of $A$. Since the $A_i$ are countable, each comes with an injective mapping $f_i : A_i \rightarrow \mathbb{N}$. We are going to define an injective mapping from $A$ into the set $\mathbb{N} \times \mathbb{N}$. (By a previous exercise $\mathbb{N} \times \mathbb{N}$ is countable.) If $x \in A$, $x$ belongs to some $A_i$ and thus there exists an integer $m$ such that

$$x \in A_m, \quad x \not\in A_i, \quad i < m$$

Define $f : A \rightarrow \mathbb{N} \times \mathbb{N}$ by the rule:

$$f(x) = (m, f_m(x)).$$
To verify that $f$ is one-one we check:

$$f(x) = f(y)$$

means

$$(m, f_m(x)) = (n, f_n(y))$$

and thus

$$x \& y \in A_m = A_n$$

and therefore

$$f_m(x) = f_m(y)$$

implies that

$$x = y$$

since $f_m$ is one-one.
**Theorem**

*If the sets $A_i$, $i \geq 1$, are countable, then $A = \bigcup_{i=1}^{\infty} A_i$ is countable.*

**Proof.** Here is a beautiful way to list the elements of $A$:

- **$A_1$:**
  

- **$A_2$:**
  

- **$A_3$:**
  

- **$A_4$:**

- **$A_5$:**
Exercise: Prove that the set $A$ of finite subsets of $\mathbb{N}$ is countable.

Solution: Let $A_n$ be the subset of $A$ made up of subsets of $\mathbb{N}$ with $n$ elements. Note that $A_0 = \{\emptyset\}$ is not the empty set! and that

$$A = \bigcup_{n \geq 0} A_n.$$

To apply the theorem above, we prove that each $A_n$ is countable. There are various ways to do it.
The set of $n$-tuples of natural numbers

\[ \mathbb{N}^n = \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{N}\} \]

is countable, by the theorem.

The set $A_n$ is on a 1-1 correspondence with the $n$-tuples

\[ \{(a_1, \ldots, a_n) \mid a_1 < a_2 < \cdots < a_n\} \]

so $A_n$ is countable.
Uncountable Sets

Definition

A set $S$ is **uncountable** if it is neither finite nor denumerable.

**Question:** Are there such sets?
Outline

1. Cardinality
2. Homework #12
3. Infinite Sets
4. Cantor’s Universe
5. Homework #13
6. The Ordering of Cardinal Numbers
7. Final Orientation
Let us visit, if briefly, the garden universe that Cantor created for us. It was the first great theory of **infinities**, and has had a profound influence on Mathematics. It helped that his constructions and proofs [sometimes the same thing] were often beautiful, if not even great fun. We will touch on two of them.
Theorem (Cantor’s Proof)

*The interval* $(0, 1)$ *is not countable.*

**Proof.** It will suffice to show that the open interval $(0, 1)$ is not countable. We are going to represent its elements as infinite decimals $x = 0.a_1 a_2 a_3 \ldots a_n \ldots$. We are going to assume, by way of contradiction, that we can list them:

\[
\begin{align*}
x_1 & = 0.a_{11} a_{12} a_{13} a_{14} \ldots \\
x_2 & = 0.a_{21} a_{22} a_{23} a_{24} \ldots \\
x_3 & = 0.a_{31} a_{32} a_{33} a_{34} \ldots \\
x_4 & = 0.a_{41} a_{42} a_{43} a_{44} \ldots \\
& \vdots \\
\end{align*}
\]

We are going, by focusing on the diagonal entries $a_{nn}$, give an element $x \in (0, 1)$ that is not listed.
Define the integer

\[ b_n = \begin{cases} 
  2 & \text{if } a_{nn} \neq 2 \\
  3 & \text{if } a_{nn} = 2 
\end{cases} \]

Set \( x = 0.b_1b_2b_3b_4 \cdots b_n \cdots \). Note that \( x \) differs from \( x_n \) at the \( n \) decimal position. So \( x \) is not listed.
**Definition**

A set $S$ has **cardinality** $c$ iff $S$ is equivalent to the open interval $(0, 1)$; we write $\text{card}(S) = c$.

**Theorem**

The set $\mathbb{R}$ is uncountable and has cardinality $c$.

**Proof.**

Define $f : (0, 1) \rightarrow \mathbb{R}$ by $f(x) = \tan(\pi x - \pi/2)$. Look at the graph:
\[\tan(\pi x - \pi/2) : (0, 1) \approx \mathbb{R}\]
**Claim:** \((0, 1) \times (0, 1) \approx (0, 1)\), that is the interior of the unit square is equivalent to \((0, 1)\). Another form; \(\mathbb{R} \times \mathbb{R} \approx \mathbb{R}\).

An element \((a, b) \in (0, 1) \times (0, 1)\) can be described as

\[
\begin{align*}
a &= 0.a_1 a_2 a_3 \ldots a_n \ldots \\
b &= 0.b_1 b_2 b_3 \ldots b_n \ldots
\end{align*}
\]

Define the function \(f(a, b) = c \in (0, 1)\) by

\[
\begin{align*}
c &= 0.a_1 b_1 a_2 b_2 \ldots a_n b_n \ldots
\end{align*}
\]

\(f\) is one-to-one and onto.
If $X$ is a set, the collection of its subsets is called the **power set** of $X$: notation $P(A)$. If $X = \{0, 1\}$, its subsets are

$$P(X) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}.$$ 

One way to represent a subset $A \subset X$ is as a function

$$f_A : X \rightarrow \{0, 1\}$$

$$f_A(x) = \begin{cases} 
    1 & \text{if } x \in A \\
    0 & \text{if } x \notin A 
\end{cases}$$
This leads to the notation $P(X) = 2^X$.

If $X = \{x_1, \ldots, x_n\}$, we can also represent its subsets by ordered strings of 0’s and 1’s as follows:

$$A \leftrightarrow (a_1, a_2, \ldots, a_n)$$

$$a_i = \begin{cases} 
1 & \text{if } x_i \in A \\
0 & \text{if } x_i \notin A 
\end{cases}$$

This shows that

$$\text{card } (P(X)) = 2^{\text{card}(X)} = 2^n$$
Exercise

Prove the following statements:

- All circles of positive radius are equivalent.
- The circle \( (x^2 + (y - 1/2)^2 = 1/4 \) is equivalent to \( \mathbb{R} \).
\( R \approx \text{Circles} \)
Homework #13

5.1: 3(a, i, n), 6(b), 17(a,b), 20(a)
5.2: 1(g), 5(a, d, e), 10
Outline

1. Cardinality
2. Homework #12
3. Infinite Sets
4. Cantor’s Universe
5. Homework #13
6. The Ordering of Cardinal Numbers
7. Final Orientation
Cantor’s Theorem

The following shows how to build larger infinities from given ones.

**Theorem**

*Given a set* \( X \) *there is no function* \( f : X \rightarrow P(X) \) *that is onto.*

**Proof.** Suppose \( f \) is such a function: For each \( a \in X \), \( f(a) \) is a subset of \( X \) and any subset is a target. Let us build a subset that is not a target.
For each $a \in X$, $a \in f(a)$ or $a \notin f(a)$. Define the subset

$$B = \{a \in X \mid a \notin f(a)\}$$

By assumption, $B = f(x)$ for some $x \in X$.

Now look how cool:

$x \in f(x) = B$, contradicts the definition of $B$, while

$x \notin f(x) = B$, would make $x \in B$, by the definition of $B$.  

$\square$
A consequence of Cantor’s Theorem is to provide chains of increasing cardinals:

\[ \aleph_0 = \mathbb{N} < \mathcal{P}(\mathbb{N}) < \mathcal{P}(\mathcal{P}(\mathbb{N})) < \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N}))) < \cdots \]
Are they equal?

The cardinality of \( \mathbb{N} \) is \( \aleph_0 \), while we have just proved that

\[
\aleph_1 = \text{card} (\mathcal{P}(\mathbb{N})) \neq \text{card} (\mathbb{N})
\]

We have two infinite sets with well-understood cardinalities larger that \( \aleph_0 \): \( \mathcal{P}(\mathbb{N}) \) and \( \mathbb{R} \) which has cardinality \( c \). One of the most famous unsolved problems of Mathematics is: True or False

**Continuum Hypothesis:** \( \mathcal{P}(\mathbb{N}) \approx \mathbb{R} \)
Cantor-Schröder-Bernstein Theorem

Theorem

If \( \bar{A} \leq \bar{B} \) and \( \bar{B} \leq \bar{A} \), then \( \bar{A} = \bar{B} \).
Figure 5.11

\[ C = \text{Rng}(G) \]

\[ D = \text{Rng}(F) \]
Figure 5.12  String $f$: $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$, $f(6)$, ...
1. Cardinality
2. Homework #12
3. Infinite Sets
4. Cantor’s Universe
5. Homework #13
6. The Ordering of Cardinal Numbers
7. Final Orientation
Final will be comprehensive but topics will be emphasized according to the following classification:

- **VITs**: Very Important Topics
- **BITs**: Basic Important Topics
- **LITs**: Basic but Less Important Topics
• Propositions, Truth tables
• Basic Methods of Proof
• Mathematical Induction (PMI, PCI, Well-Ordering)
• Relations, Equivalence Relations, Classes of
• Functions: Ingredients and Important Types (1-1, onto)
• Cardinality
• Finite, Countable and Uncountable Sets
Logical connectives, quantifiers
Set Theory/Operations
Principles of Counting
More relations, Partitions
Constructions of Functions
Functions from Calculus
Review homework
Graphs

Names to recall: Venn, Fibonacci, Cantor

Examples in slides