Math 250–Section #3  Hourly #2  

The exam has 7 questions [check] and one page with formulas. Calculators or personal notes are not allowed. Correct answers with justification required. Good luck.

__________________________________________________________________________

Name: ______________________

1. (15 pts) (a) Evaluate the determinant of the following matrix:

\[ A = \begin{bmatrix}
0 & 0 & 1 & 2 \\
0 & 0 & 3 & 4 \\
5 & 6 & 0 & 0 \\
7 & 8 & 0 & 0
\end{bmatrix} \]

(b) If the 4 \times 4 matrix \( C = [c_1|c_2|c_3|c_4] \) has determinant 1, find the determinant of the matrix

\[ B = [c_2 + c_3|c_3 + c_4|2c_1|c_1 + 2c_2]. \]

Answer: (a) Using co-factors expansion gives quickly that \( \det(A) = 4 \).

(b) The columns of \( B \) are combinations of the columns of \( C \) so we look for a matrix \( D \) such that \( B = CD \). [There were several approaches used by students.] Using the corresponding elementary column operations

\[
D = \begin{bmatrix}
0 & 0 & 2 & 1 \\
1 & 0 & 0 & 2 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]

\[ \det(B) = \det(C) \det(D) = 1 \times 3 = 3. \]

The calculation \( \det(D) = 3 \) used co-factors expansion.

1
2. (15 pts) Let $A$ be the matrix

$$
\begin{bmatrix}
2 & 1 & 1 \\
0 & 1 & 2 \\
0 & 0 & c \\
\end{bmatrix},
$$

where $c$ is some number.

(a) What are the eigenvalues of $A$?

(b) If $c \neq 1, 2$, why is $A$ diagonalizable? What happens when $c = 1$ or $c = 2$?

Answer: (a) The characteristic polynomial is

$$\det(A - tI) = (2 - t)(1 - t)(c - t),$$

whose roots are the eigenvalues: $1, 2, c$.

(b) If $c \neq 1, 2$, there are [automatically] 3 independent eigenvectors and therefore the matrix is diagonalizable.

If $c = 1$ or $c = 2$, it may go either way [diagonalizable or not] so we must check further to see whether the geometric multiplicities are equal or not to the algebraic multiplicities.

For $c = 1$: The nullspace of $A - I$

$$
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 2 \\
0 & 0 & 1 \\
\end{bmatrix}
$$

is generated by

$$
\begin{bmatrix}
-1 \\
1 \\
0 \\
\end{bmatrix}
$$

so it has multiplicity 1—and $A$ is not diagonalizable.

Doing likewise for $c = 2$ will again say that $A$ is not diagonalizable.
3. (15 pts) Given the matrix

\[ A = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 0 \\ 3 & 0 & 5 \end{bmatrix} \]

(a) Find its characteristic polynomial.
(b) Find its eigenvalues.
(c) Explain why \( A \) is diagonalizable. [You do not have to find the eigenvectors to answer.]

Answer: (a) To find \( \det(A - tI) \), we expand along the second column

\[ \det(A - t) = (2 - t)((2 - t)(5 - t) - 9) = (2 - t)(t^2 - 7t + 1). \]

(b) Use the quadratic formula to find the roots of the factor \( t^2 - 7t + 1 \):

\[ t = \frac{7 \pm \sqrt{49 - 4}}{2} = \frac{7 \pm 3\sqrt{5}}{2} \]

Together with 2 these roots are the eigenvalues.

(c) Since the eigenvalues are distinct, we have a basis of eigenvectors for \( \mathbb{R}^3 \) and \( A \) is diagonalizable.
4. (15 pts) Let $V$ be the set of all $2 \times 2$ matrices.
(a) Explain why $V$ is a vector space, describe one of its basis and find the
dimension of $V$.
(b) Let $S$ be the set of all matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with $b - c = 0$ and $a + d = 0$. Show that $S$ is a subspace of $V$ and find a
basis for $S$. Begin by explaining what is a $subspace$ of a vector space.

Answer: (a) According to our class discussion, $V$ is just $\mathbb{R}^4$ in another
dressing. [It is closed under $+$, scaling and has all the properties of vector
spaces.] One observes that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

so $V$ is spanned by these 4 linearly independent matrices—a basis of $V$, so
$\dim V = 4$.

(b) Look the notion of $subspace$. $S$ is indeed a subspace, with basis

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

$\dim S = 2$
5. (10 pts) Let $A$ be a $3 \times 3$ matrix. Explain why the following are not possible:

(a) The eigenvalues of $A$ are: 1, 2, 3 and 4.
(b) $A$ is invertible but one of its eigenvalues is 0.

Answer: (a) The characteristic polynomial $\det(A - tI)$ has degree 3, so it has at most 3 roots.
(b) If $v \neq 0$ is an eigenvector for $A$,

$$Av = \lambda v,$$

with $\lambda = 0$ gives

$$Av = 0.$$

If $A$ is invertible, we would have

$$A^{-1}Av = v = 0,$$

which is a contradiction.
6. (12 pts) (a) Describe the following notions: Eigenvalues of a matrix $A$ and its eigenspaces.

(b) If $A$ is a $2 \times 2$ matrix of eigenvalues 2 and 3, find the eigenvalues of $B = A^2 + A + I_2$.

Answer: (a) Look up in book the precise formulation.

(b) If $Av = 2v$,

$$Bv = (A^2 + A + I_2)v = A^2v + Av + I_2v = 4v + 2v + v = 7v,$$

so 7 is an eigenvalue of $B$.

For $\lambda = 3$, would get 13 as an eigenvalue of $B$. 
7. (18 pts) Given the matrix

\[ A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \]

(a) Verify that

\[ u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \]

is NOT in the nullspace of \( A \). Begin by explaining what is the nullspace of a matrix.

(b) Find a basis for the nullspace \( S \) of \( A \) and use Gram-Schmidt’s to find an orthonormal basis for \( S \). Begin by explaining what is an orthonormal basis of a vector space.

(c) Explain, using the basis of Part (b), how to find the projection of \( u \) onto \( S \).

Answer: (a) The nullspace of \( A \) is the set of all vectors \( v \in \mathbb{R}^4 \) such that \( Av = O \). Checking for the given vector,

\[ Av = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \neq O. \]

(b) Look up orthonormal basis. A basis for the nullspace of \( A \) is

\[ \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \]

Enter this data into the Gram-Schmidt Algorithm to get an orthonormal basis \( w_1, w_2, w_3 \) (do it!). The projection would be

\[ (v \cdot w_1)w_1 + (v \cdot w_2)w_2 + (v \cdot w_3)w_3 \]
The routine to obtain a basis that is orthonormal from another basis [Gram–Schmidt process]:
Input basis $S = \{u_1, \ldots, u_n\}$
Step 1: Set $v_1 = u_1$
Step 2: Compute $v_2, \ldots, v_n$ successively by
$$v_i = u_i - \frac{u_i \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{u_i \cdot v_2}{v_2 \cdot v_2} v_2 - \cdots - \frac{u_i \cdot v_{i-1}}{v_{i-1} \cdot v_{i-1}} v_{i-1}$$
Step 3: Set
$$w_i = \frac{v_i}{||v_i||}$$
Then $T = \{w_1, \ldots, w_n\}$ is an orthonormal basis.

The roots of the quadratic equation
$$ax^2 + bx + c = 0, \quad a \neq 0$$
are
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$