Complexity of the Normalization of Algebras

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Abstract

Let $R$ be a normal unmixed integral domain and let $A$ be a semistandard graded $R$-algebra of integral closure $\overline{A}$. Estimating the number of steps that general algorithms must take to build $\overline{A}$ can be viewed as an invariant of $A$. We show how the degree function $jdeg(\cdot)$ can be used to provide bounds that depend on $\overline{A}$. In two major cases, algebras that allow Noether normalizations (affine algebras over fields or $\mathbb{Z}$) and Rees algebras of ideals or modules, such estimates are related to invariants of $A$, in particular they can be said to be known \textit{ab initio}. 
General Goals

- Numerical Indices for $\overline{A}$: e.g. Find $r$ such that

$$\overline{A}_{n+r} = A_n \cdot \overline{A}_r, \quad n \geq 0.$$  

- How many “steps” are there between $A$ and $\overline{A}$,

$$A = A_0 \subset A_1 \subset \cdots \subset A_{s-1} \subset A_s = \overline{A},$$

where the $A_i$ are constructed by an effective process?

- Express $r$ and $s$ in terms of invariants of $A$.

- Generators of $\overline{A}$: Number and distribution of their degrees
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• Generators of $\overline{A}$: Number and distribution of their degrees
A new degree: $j\text{deg}$ (Thuy Pham)

To construct and develop this notion of degree, one uses extensively the notion of $j$–multiplicity introduced and developed by Flenner, O’Carroll and Vogel.

Let $R$ be a Noetherian ring and $A$ be a finitely generated graded $R$-algebra where $A = R[A_1]$. For a finitely generated graded $A$-module $M$, and a prime ideal $p$ of $R$, let

$$H = H^0_{pR_p}(M_p).$$

The $j_p$-multiplicity of $M$ is the integer

$$j_p(M) := \begin{cases} \deg H & \text{if } \dim = \dim M_p \\ 0 & \text{otherwise} \end{cases}$$
Definition

Let $R$ be a Noetherian ring and $A$ be a finitely generated graded $R$-algebra where $A = R[A_1]$. For a finitely generated graded $A$-module $M$,

$$\text{jdeg}(M) := \sum_{p \in \text{Spec} R} j_p(M).$$

Note that this is a finite sum.
For example, if $R$ is an integral domain and $R[lt]$ is the Rees algebra of an ideal,
\[ \text{jdeg} (R[lt]) = 1, \]
while if $R$ is an Artinian local ring and $M$ is a finitely generated graded module over a standard graded $R$-algebra,
\[ \text{jdeg} (M) = \text{deg}(M). \]

\text{jdeg} does not seem to carry much new information! We will argue that this degree is useful in tracking certain processes [especially normalization but also the Nullstellensatz and reductions].
Some Properties

**Theorem (Invariance)**

Let \( R \) be a (bit less than Cohen-Macaulay) ring and let \( I \) be an ideal of \( R \). Let \( B \) be a graded algebra with \( A = R[lt] \subseteq B \subseteq \bar{A} \) and assume that \( B \) satisfies the condition \( S_2 \) of Serre. Then

\[
\text{jdeg} \left( \text{gr} \left( A \right) \right) = \text{jdeg} \left( \text{gr} \left( B \right) \right).
\]
Theorem (Pham-V)

Let $R$ be a Noetherian domain and $A$ a semistandard graded $R$-algebra with finite integral closure $\overline{A}$. Consider a sequence of distinct integral graded extensions

$$A = A_0 \rightarrow A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_s = \overline{A},$$

where the $A_i$ satisfy the $S_2$ condition of Serre. Then

$$s \leq j\text{deg} (\overline{A}/A).$$
The issue is to express bounds for $\jdeg(A/A)$ in terms of invariants of $A$. There are at least two classes of algebras when this is possible: Rees algebras of ideals/modules and algebras finite over polynomial subrings:

$$R[x_1, \ldots, x_p] \subset A \subset R[t_1, \ldots, t_q]$$

The rings of polynomials serve as referential for various constructions. One knows already quite a lot about these issues. In these situations, one connects $\jdeg(A/A)$ to some invariant of $A$. 

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Hilbert functions and \( \text{jdeg} \)

Let \((R, \mathfrak{m})\) be a Noetherian local ring of dimension \( d > 0 \) and let \( I \) be an \( \mathfrak{m} \)-primary ideal. Let \( B = \bigoplus_{n \geq 0} B_n t^n \) be a graded \( R \)-subalgebra of \( R[t] \) with \( R[lt] \subset B \subset R[t] \) and assume that \( B \) is a finite \( R[lt] \)-module. For any such algebra we consider the Hilbert–Samuel function \( \lambda(R/B_n) \). For \( n \gg 0 \) this function is given by the Hilbert–Samuel polynomial

\[
e_0(B) \binom{n + d - 1}{d} - e_1(B) \binom{n + d - 2}{d - 1} + \text{lower terms}.
\]

Notice that the \( e_i(R[lt]) \) coincide with the usual *Hilbert coefficients* \( e_i(I) \) of \( I \). Furthermore \( e_0(B) = e_0(I) \). By \( R[lt] \) we will always denote the integral closure of \( R[lt] \) in \( R[t] \).
We write $\overline{e}_i(I)$ for the normalized Hilbert coefficients $e_i(R[lt])$ of $I$ in case $R[lt]$ is a finite $R[lt]$–module.

**Proposition**

*In this case,*

$$jdeg \left( \overline{R[lt]} / R[lt] \right) = \overline{e}_1(I) - e_1(I).$$
Briançon-Skoda numbers

Definition

If \( I \) is an ideal of a Noetherian ring \( R \), the Briançon-Skoda number \( c(I) \) of \( I \) is the smallest integer \( c \) such that \( I^{n+c} \subseteq J^n \) for every \( n \) and every reduction \( J \) of \( I \).

The motivation for this definition is a result of Briançon-Skoda asserting that for the rings of convergent power series over \( \mathbb{C}^n \) (later extended to regular local rings by Lipman- Sathaye) \( c(I) < \text{dim } R \).
Theorem (PUV)

Let \((R, \mathfrak{m})\) be an analytically unramified Cohen–Macaulay local ring and let \(I\) be an \(\mathfrak{m}\)–primary ideal of Briançon-Skoda number \(c(I)\). Let \(A\) and \(B\) be distinct graded algebras with

\[
R[lt] \subset A \subsetneq B \subset R[lt]
\]

and assume that \(A\) satisfies the condition \(S_2\) of Serre. Then

\[
c(I)e_0(I) \geq e_1(I) \geq e_1(B) > e_1(A) \geq e_1(I) \geq 0.
\]

In particular, any chain of subalgebras that satisfy the condition \(S_2\) of Serre has length at most \(e_1(I)\).
One does not really need such a strict statement as in Briançon-Skoda’s. Through the techniques of [L-S, H-H] one has:

**Proposition**

Let $k$ be a perfect field, let $R$ be a reduced Cohen–Macaulay $k$-algebra essentially of finite type. Then for any ideal $I$ with a reduction $J$ generated by $\ell$ elements, and every integer $n$,

$$\text{Jac}_k(R)[I^{n+\ell(l)-1}] \subset D_n$$

where $D$ is the $S_2$-ification of $R[Jt]$. In particular,

$$\text{Jac}_k(R)[I^{\ell-1}] \subset \text{ann}(R[It]/D).$$
A calculation for an isolated singularity, will give for the bound of distinct subalgebras between $\mathbb{R}[lt]$ and $\mathbb{R}[lt]$, instead of
\[ c(l)e_0(l), \]
where $c(l)$ is imprecise (but probably can be made precise), and can be replaced by
\[ (\lambda(R/J) + \ell(l) - 1)e_0(l). \]
In fact, Briançon-Skoda’s with coefficient ideals seem to lead to even more specific bounds.
Let $(R, m)$ be a local Cohen-Macaulay algebra of type $t$ essentially of finite type over a perfect field $k$. 
Theorem (PUV)

Let $I$ be an $m$–primary ideal.

- If $\delta \in \text{Jac}_k(R)$ is a non zerodivisor, then
  \[
  \overline{e}_1(I) \leq \frac{t}{t+1} \left( (d-1)e_0(I) + e_0(I + \delta R/\delta R) \right).
  \]

- If the assumptions above hold, then
  \[
  \overline{e}_1(I) \leq (d-1)(e_0(I) - \lambda(R/I)) + e_0(I + \delta R/\delta R).
  \]

- If $R/m$ is infinite, then
  \[
  \overline{e}_1(I) \leq c(I) \min \left\{ \frac{t}{t+1} e_0(I), e_0(I) - \lambda(R/I) \right\}.
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  \]
Corollary

If $R$ is a regular local ring of dimension $d$, $\bar{e}_1(I) \leq \frac{1}{2}(d - 1)e_0(I)$. 
The following refines a result of [Polini-Ulrich-V] for equimultiple ideals:

**Theorem**

Let $R$ be a Cohen-Macaulay reduced quasi-unmixed ring and let $I$ be an ideal of Briançon-Skoda number $c(I)$ (or the modified value mentioned above in the case of isolated singularities). Then

$$jdeg \left( \frac{R[lt]}{I[lt]} \right) \leq c(I) \cdot jdeg \left( \text{gr}_I(R) \right).$$
Let $R$ be a Noetherian ring, let $E$ be a finitely generated torsionfree $R$–module having a rank, and choose an embedding $\varphi : E \hookrightarrow R^r$. The Rees algebra $R[lt](E)$ of $E$ is the subalgebra of the polynomial ring $R[t_1, \ldots, t_r]$ generated by all linear forms $a_1 t_1 + \cdots + a_r t_r$, where $(a_1, \ldots, a_r)$ is the image of an element of $E$ in $R^r$ under the embedding $\varphi$. The Rees algebra $R[lt](E)$ is a standard graded algebra whose $n$th component is denoted by $E^n$ and is independent of the embedding $\varphi$ since $E$ is torsionfree and has a rank.

The algebra $R[lt](E)$ is a subring of the polynomial ring $S = R[t_1, \ldots, t_r]$. We consider the ideal $(E)$ of $S$ generated by the forms in $E$. Denote by $G$ the associated graded ring $\text{gr}_E(S)$. Let us list some of its basic properties. This portion of our exposition is dependent on [Hong-Ulrich-V].
Example

Let $R = k[x, y]$, $E$ the submodule of $R^2 = Re_1 + Re_2$ generated by $x^2e_1$ and $y^2e_2$. $E$ is a free module, so $c(E) = 0$. A computation with Normaliz shows that $S[(E)t] = S[Et, xy e_1 e_2 t]$. Note $c((E)) = 1$. 

Proposition

Let $(R, m)$ be a Noetherian integral domain of dimension $d$ and let $E$ be a torsionfree $R$–module of rank $r$ with a fixed embedding $E \hookrightarrow R^r$. Then

- The components of $G = \bigoplus_{n \geq 0} E^n S / E^{n+1}$ have a natural grading

\[ G_n = E^n + E^n S_1 / E^{n+1} + E^n S_2 / E^{n+1} S_1 + \cdots. \]

- There is a decomposition $G = R[\text{lt}](E) + H$, where $R[\text{lt}](E)$ is the Rees algebra of $E$ and $H$ is the $R$-torsion submodule of $G$.

- If $E \subset mR^r$ and $\lambda(R^r / E) < \infty$, $H = H^0_m(G)$ has dimension $d + r$ and multiplicity equal to the Buchsbaum-Rim multiplicity of $E$. 

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2. **There is a decomposition** $G = R[\text{lt}](E) + H$, where $R[\text{lt}](E)$ is the Rees algebra of $E$ and $H$ is the $R$-torsion submodule of $G$.

3. **If** $E \subseteq m R^r$ **and** $\lambda(R^r/E) < \infty$, **$H = H^0_m(G)$ has dimension** $d + r$ **and multiplicity equal to the Buchsbaum-Rim multiplicity of $E$.**
Proposition

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Theorem

Let $E \subset R^r$ be a module as above and let

$$(E) = ER[t_1, \ldots, t_r] \subset S.$$

If $G = \text{gr} (E)(S)$ then

$$\text{jdeg} (G) = \text{br}(E) + 1.$$
Theorem

Let $R$ be a reduced quasi-unmixed ring and let $E$ be a module of Briançon-Skoda number $c((E))$ (or as modified in the case of isolated singularities). Then

$$\text{jdeg}\left(\frac{S[[E]t]}{S[[E]t]}\right) \leq c((E)) \cdot \text{jdeg}\left(\text{gr}_{(E)}(S)\right).$$
The significance here comes when we compare the estimates for $s$ which come from the Corollary together with previous Theorem gives:

$$s \leq c((E)) \cdot jdeg(G) = c((E)) \cdot (br(E) + 1)$$

for the number of subalgebras needed to find $S[(E)t]$. On the other hand, [Hong-Ulrich-V] give

$$s \leq \left( \frac{r + c(E) - 1}{r} \right) \cdot br(E),$$

for the length of chains of subalgebras satisfying the condition $S_2$ in order to find $R(E)$. Since in the case of regular local rings, both $c(E)$ and $c((E))$ are, in general, expected to be comparable to $d + r - 2$, the gain is appreciable. The downside side is that one must do calculations with a larger ring.
Let $R$ be a normal domain and let $A$ be a semistandard graded $R$-algebra of integral closure $\overline{A}$. By a Noether normalization of $A$ we mean a graded polynomial algebra

$$S = R[y_1, \ldots, y_r] \subset A$$

over which $A$ is finite. Unfortunately this does not happen often (e.g. fails for $\mathbb{C}[t]$), although Shimura established it for $R = \mathbb{Z}$, in one of his first papers.
Tracking number of a pair of algebras

In this setting one can define

$$\text{det}_S(A) \cong (\wedge^r A)^{**},$$

where $r$ is the rank of $A$ as an $S$-module. Given a pair of algebras $A \subset B$ of the same rank, one can attach a degree as follows. Fix $\text{det}(B)$ and the image of $\text{det}(A)$ in it, which we still denote by $\text{det}(A)$. Now set

$$I = \text{ann}(\text{det}(B)/\text{det}(A)).$$

This ideal is independent of the choices made. Let

$$I = (\bigcap p_i^{(r_i)}) \cap (\bigcap q_j^{(s_j)}),$$

is its primary decomposition, where we denote by $p_i$ the primes that are extended from $R$, and $q_j$ those that are not.
This means that $q_j \cap R = (0)$ and thus $q_j KS = (f_j)^1 KS$, $\deg(f_j) > 0$. We associate a degree to $I$ by setting

$$\deg(I) = \sum_i r_i + \sum_j s_j \deg(f_j).$$

It depends only on the two algebras and of the embedding $A \subset B$. We will denote it by $tn(A, B)$. If $R$ is a field, it is possible to define an invariant $tn(A)$ directly as the degree of $\det(A)$. It has many positivity properties ([Dalili-V]. One has $tn(B, A) = tn(A) - tn(B)$.

If $A = \mathbb{Z}[x, y, z]/(z^3 + xz^2 + x^2y)$, $S = \mathbb{Z}[x, y]$, then $tn(A, \overline{A}) = 1.$
Theorem

Let $R$ be a normal domain and let $A$ be a semistandard graded $R$-algebra that admits a Noether normalization. Then

$$jdeg \left( \overline{A}/A \right) = tn(A, \overline{A}).$$

This holds when $R$ is an arbitrary field, in [Dalili-V] there are several specific bounds which one expects to extend to $\mathbb{Z}$. 