The Cauchy equation

Definition: A function \( f : \mathbb{R} \to \mathbb{R} \) is additive if it satisfies the Cauchy equation (pron. Ko-'shee):
\[
f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.
\]
In this note, linear function will mean a function of the form \( f(x) = cx \) (zero intercept).

Clearly, linear functions are additive. Are there any other additive functions? After experimenting for a while, you’ll be convinced that there are none. And in a sense there are none (namely among well-behaving functions), but in a sense there are (the existence of some erratically behaving non-linear additive functions follows from the Axiom of Choice).

Our first theorem is about “tame” additive functions:

**Theorem 1.** Let \( f : \mathbb{R} \to \mathbb{R} \) be an additive function, and let \( I = [a, b] \) be an arbitrary interval (with \( a < b \)).

If \( f \) is monotone on \( I \) then \( f \) is linear (that is, \( (\exists c)(\forall x)f(x) = cx \)).

If \( f \) is continuous on \( I \) then \( f \) is linear.

If \( f \) is bounded on \( I \) then \( f \) is linear.

The proof of Theorem 1 is based on the following lemma which says that all additive functions are linear on \( \mathbb{Q} \) (the set of rational numbers) as well as on all “copies” \( a\mathbb{Q} \) of \( \mathbb{Q} \) (but the slope involved might be varying from \( a \) to \( a \)).

**Lemma 2.** Let \( f \) be an arbitrary additive function. Then, \( (\forall x \in \mathbb{Q})f(x) = f(1)x \). In general, for any \( a \in \mathbb{R} \) we have \( (\forall x \in \mathbb{Q})f(ax) = f(a)x \).

**Proof** steps for Lemma 2:

Fix \( a \in \mathbb{R} \). Use induction to show \( (\forall n \in \mathbb{N})f(an) = nf(a) \).

Use induction to show \( (\forall n \in \mathbb{N})f(a/n) = f(a)/n \).

And finally, use induction to show \( (\forall m, n \in \mathbb{N})f(a(m/n)) = f(a)(m/n) \).

And now the “wild” functions:

**Theorem 3 (assuming the Axiom of Choice).** There are additive functions that are not linear.

**Proof.** We describe all additive functions at once (“most” are easily seen not to be linear):

Recall that since both \( \mathbb{Q} \) and \( \mathbb{R} \) are fields and \( \mathbb{Q} \) is a subfield of \( \mathbb{R} \), so \( \mathbb{R} \) can be considered as a vector space over \( \mathbb{Q} \); let \( B \) be a basis in this vector space (a so-called Hamel basis). [The AC guarantees that every vector space has a basis!]

Define \( f \) arbitrarily on \( B \), and extend it to \( \mathbb{R} \) in the obvious way: if \( x = \sum q_i b_i \) with some \( q_i \in \mathbb{Q} \) and \( b_i \in B \) then let \( f(x) := \sum q_i f(b_i) \). Since such a representation of \( x \) is unique (definition of linear basis!), \( f \) is well-defined, and it is easy to see that \( f \) is additive. \( \square \)
The following homework (6.6 in the LBB) shows that the adjective “wild” above is well-deserved: Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function. Show that if $f$ is not linear, then the graph of $f$ is everywhere dense on the plane. (That means that every rectangle in the plane - however small - contains at least one point of the graph of $f$.)

**Subadditive sequences**

The pathological behavior of certain additive functions resulted from the richness of the set of real numbers. For sequences (that is, functions with domain $\mathbb{N}$), no such erratic behavior is possible as the following trivial fact shows (use induction on $n$):

**Fact 4.** Let $(x_n)$ be a sequence of real numbers satisfying the additivity condition

$$x_{m+n} = x_m + x_n \quad \text{for all } m, n \in \mathbb{N}.$$

Then $x_n$ is linear; indeed, $x_n = nx_1$ for all $n \in \mathbb{N}$.

The next theorem says that if we relax the condition of additivity to subadditivity, then the sequence will still asymptotically behave as linear, in that $\lim_{n \to \infty} x_n/n$ exists (possibly $-\infty$).

**Theorem 5 (Subadditivity Lemma - Fekete 1923).** If a sequence of real numbers $(x_n)$ satisfies the subadditivity condition

$$x_{m+n} \leq x_m + x_n \quad \text{for all } m, n \in \mathbb{N},$$

then

$$\lim_{n \to \infty} \frac{x_n}{n} = \inf_{m \geq 1} \frac{x_m}{m}.$$

**Sketchy proof** (for those who are familiar with lim and lim sup):

1. Induction on $k$ shows that $(\forall m \in \mathbb{N})(\forall k \in \mathbb{N})x_{km} \leq kx_m$.
2. Writing $C_m = \max\{x_r : 1 \leq r < m\}$, we get for all $r \in [1, r-1]$, all $k \in \mathbb{N}$, and $n = km + r$: $x_n = x_{km+r} \leq x_{km} + x_r \leq x_{km} + C_m \leq kx_m + C_m$. Hence,

$$\frac{x_n}{n} \leq \frac{km}{n} \cdot \frac{x_m}{m} + \frac{C_m}{n}.$$

3. Letting $k \to \infty$, we get

$$\limsup_{n \to \infty} \frac{x_n}{n} \leq \frac{x_m}{m} \quad \text{for all } m \in \mathbb{N}, \quad \text{whence} \quad \limsup_{n \to \infty} \frac{x_n}{n} \leq \inf_{m \in \mathbb{N}} \frac{x_m}{m}.$$

4. But since $x_n/n \geq \inf_{m \in \mathbb{N}} x_m/m$ for all $n \in \mathbb{N}$, so $\lim_{n \to \infty} x_n/n = \inf_{m \in \mathbb{N}} x_m/m$. 

**Example** (hereditary properties): Let $S_n$ be the set of all strings of English letters of length $n$ which do not contain the substring $\text{hello}$. Then $S_n$ is of exponential size, in that $|S_n|^{1/n}$ exists. Indeed, since the required property is hereditary (to segments of a string), so $S_{m+n} \subset S_m S_n$, where $S_m S_n := \{xy : x \in S_m, y \in S_n\}$ (concatenated strings). Hence $|S_{m+n}| \leq |S_m||S_n|$, and the sequence $x_n := \log |S_n|$ is subadditive. The claim easily follows.