Formula Sheet for Math 357

Discrete Fourier Transform

1. The $N \times N$ Fourier matrix F_N has i,j entry $w^{-(i-1)(j-1)}$, where $w = \mathrm{e}^{2\pi\,\mathrm{i}/N}$. It satisfies $F_N \overline{F}_N = N\,I$, where I is the $N \times N$ identity matrix and the bar denotes complex conjugation.

   Special cases: $$F_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad F_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{bmatrix}.$$

2. The discrete Fourier transform of a column vector $\mathbf{x} \in \mathbf{C}^N$ is the column vector $\mathbf{X} = F_N \mathbf{x}$.

Discrete Wavelet Transform

1. A column vector $\mathbf{x} \in \mathbf{C}^N$ corresponds to an N-periodic function with values $\mathbf{x}[k]$ for $k \in \mathbf{Z}$, Here $\mathbf{x}[0]$ is the first component of $\mathbf{x}$, $\mathbf{x}[N-1]$ the Nth component of $\mathbf{x}$, and $\mathbf{x}[N+k] = \mathbf{x}[k]$ (wrap-around) for all integers k. The shift operator S acts on column vectors by shifting down with wrap-around. It acts on N-periodic functions by $(S\mathbf{x})[k] = \mathbf{x}[k-1]$.

2. Let N be even. The symbol $\boxed{\text{split}}$ means the $N \times N$ permutation matrix that acts by
$$\boxed{\text{split}} \begin{bmatrix} \mathbf{x}[0] \\ \mathbf{x}[1] \\ \vdots \\ \mathbf{x}[N-1] \end{bmatrix} = \begin{bmatrix} \mathbf{x}_{\text{even}} \\ \mathbf{x}_{\text{odd}} \end{bmatrix}, \quad \text{where } \mathbf{x}_{\text{even}} = \begin{bmatrix} \mathbf{x}[0] \\ \mathbf{x}[2] \\ \vdots \\ \mathbf{x}[N-2] \end{bmatrix} \text{ and } \mathbf{x}_{\text{odd}} = \begin{bmatrix} \mathbf{x}[1] \\ \mathbf{x}[3] \\ \vdots \\ \mathbf{x}[N-1] \end{bmatrix}.$$
   The inverse matrix $\boxed{\text{split}}^{-1} = \boxed{\text{split}}^{\mathrm{T}}$ is denoted by $\boxed{\text{merge}}$.

3. Let $N = 2^k$ and let I be the $2^{k-1} \times 2^{k-1}$ identity matrix. The $N \times N$ one-scale Haar analysis and synthesis matrices are $\mathbf{T}_{\mathbf{a}}^{(k)} = \frac{1}{2}\begin{bmatrix} I & I \\ I & -I \end{bmatrix} \boxed{\text{split}}$ and $\mathbf{T}_{\mathbf{s}}^{(k)} = \boxed{\text{merge}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}$. For $\mathbf{x} \in \mathbf{R}^N$ write $\mathbf{T}_{\mathbf{a}}^{(k)}\mathbf{x} = \begin{bmatrix} \mathbf{s}^{(k-1)} \\ \mathbf{d}^{(k-1)} \end{bmatrix}$, where $\mathbf{s}^{(k-1)}$ (trend) and $\mathbf{d}^{(k-1)}$ (detail) are in $\mathbf{R}^{N/2}$.

   Special case $N = 4$ (with $\boxed{\text{split}}$ and $\boxed{\text{merge}}$ already included in the matrices):
$$\mathbf{T}_{\mathbf{a}}^{(2)} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \qquad \mathbf{T}_{\mathbf{s}}^{(2)} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

Signals

1. A FIR *signal* or *filter* is a real linear combination $\mathbf{x} = \sum_{n \in \mathbf{Z}} \mathbf{x}[n]\delta_n$ of a finite number of unit impulses δ_n (so $\mathbf{x}[n] = 0$ when $|n|$ is sufficiently large).

2. The inner product of signals $\mathbf{x}$ and $\mathbf{y}$ is $\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{n \in \mathbf{Z}} \mathbf{x}[n]\mathbf{y}[n]$.

3. The nonperiodic right shift transformation S acts on signal values by $S\mathbf{x}[n] = \mathbf{x}[n-1]$. On unit impulses the action is $S\delta_n = \delta_{n+1}$.

4. The *downsampling* operator acts on unit impulses by $\boxed{2\downarrow}\,\delta_n = \begin{cases} \delta_m & \text{if } n = 2m \text{ is even,} \\ 0 & \text{if } n = 2m+1 \text{ is odd.} \end{cases}$

   The *upsampling* operator acts on unit impulses by $\boxed{2\uparrow}\,\delta_m = \delta_{2m}$.

5. The nonperiodic convolution $\mathbf{u} = \mathbf{x} * \mathbf{y}$ of $\mathbf{x}$ and $\mathbf{y}$ has values $\mathbf{u}[k] = \sum_{n\in\mathbf{Z}} \mathbf{x}[k-n]\mathbf{y}[n]$. When $\mathbf{x} = \delta_n$ and $\mathbf{y} = \delta_k$ are unit impulses, then $\mathbf{u} = \delta_{n+k}$. The convolution satisfies $\mathbf{x} * \mathbf{y} = \mathbf{y} * \mathbf{x}$ for all signals $\mathbf{x}$ and $\mathbf{y}$.

6. The z-transform of $\mathbf{x}$ is $X(z) = \sum_{n\in\mathbf{Z}} \mathbf{x}[n]z^{-n}$, where z is a nonzero complex number. When $z = \mathrm{e}^{\,\mathrm{i}\omega}$ with real frequency variable ω, then $X(\mathrm{e}^{\,\mathrm{i}\omega}) = \sum_{n\in\mathbf{Z}} \mathbf{x}[n]\mathrm{e}^{-\,\mathrm{i}n\omega}$ is a trigonometric polynomial.

7. The z-transform of $\mathbf{x} * \mathbf{y}$ is $X(z)Y(z)$.

Filter Banks

1. A two-channel filter bank has FIR *analysis* filters $\mathbf{h}_0$ and $\mathbf{h}_1$ with z-transforms $H_0(z)$ and $H_1(z)$, and analysis modulation matrix $\mathbf{H}_m(z) = \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = \mathbf{H}_p(z^2) \begin{bmatrix} 1 & 1 \\ z & -z \end{bmatrix}$, where $\mathbf{H}_p(z) = \begin{bmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \end{bmatrix}$ is the *analysis polyphase matrix.*

2. The FIR *synthesis* filters are $\mathbf{g}_0$ and $\mathbf{g}_1$ with z-transforms $G_0(z)$ and $G_1(z)$, and synthesis modulation matrix $\mathbf{G}_m(z) = \begin{bmatrix} G_0(z) & G_1(z) \\ G_0(-z) & G_1(-z) \end{bmatrix}$.

3. The filter bank has *perfect reconstruction* (PR) when $\mathbf{G}_m(z)\mathbf{H}_m(z) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$.

4. The *trend/detail* decomposition of a signal is

$$\mathbf{x} = \mathbf{x_s} + \mathbf{x_d} = \sum_{m\in\mathbf{Z}} \langle S^{2m}\overset{\vee}{\mathbf{h}}_0,\, \mathbf{x}\rangle S^{2m}\mathbf{g}_0 + \sum_{n\in\mathbf{Z}} \langle S^{2n}\overset{\vee}{\mathbf{h}}_1,\, \mathbf{x}\rangle S^{2n}\mathbf{g}_1\,,$$

   where $\overset{\vee}{\mathbf{h}}[n] = \mathbf{h}[-n]$ (time-reversed filter) and S is the shift operator.

5. The filter bank is *orthogonal* if $\mathbf{g}_0 = \overset{\vee}{\mathbf{h}}_0$ and $\mathbf{g}_1 = \overset{\vee}{\mathbf{h}}_1$.

Bezout Polynomials

1. The polynomial $B_n(y)$ of degree $n-1$ satisfies the *Bezout equation*

$$(1-y)^n B_n(y) + y^n B_n(1-y) = 1\,.$$

2. The explicit formula is $B_n(y) = 1 + ny + \dfrac{n(n+1)}{1\cdot 2}y^2 + \cdots + \dfrac{n(n+1)\cdots(2n-2)}{1\cdot 2\cdots(n-1)}y^{n-1}$.

3. The Bezout polynomial has the additional property $B_n(y) \geq 1$ for $y \geq 0$.