Math 252
Summer 2019
Midterm exam
7/25/19

Name (Print): ____________________

This exam contains 6 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use a scientific calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
8	15	
Total:	120	

1. A tank initially (at time $t = 0$) contains 100 gal of pure water. Water contaning a salt concentration of e^{-t} oz/gal flows into the tank at a rate of 2 gal/min, and the (well mixed) mixture in the tank flows out at the same rate.
   (a) (10 points) Find the amount of salt in the tank at any time.
   Let $y(t)$ be the amount of salt in the tank at time t. Then $y(0) = 0$ and
   $$y' = 2e^{-t} - 2\frac{y}{100} = 2e^{-t} - \frac{y}{50}.$$
   The integrating factor is $e^{\frac{t}{50}}$ and
   $$\begin{aligned} ye^{\frac{t}{50}} &= 2\int e^{-\frac{49}{50}t}dt = -\frac{100}{49}e^{-\frac{49}{50}t} + c \\ y(t) &= -\frac{100}{49}e^{-t} + ce^{-\frac{t}{50}}. \end{aligned}$$
   $y(0) = 0$ implies $c = \frac{100}{49}$.
   (b) (5 points) Find the limiting amount of salt in the tank as $t \to \infty$.
   As $t \to \infty, y(t) \to 0$.

2. (15 points) Solve the ODE
   $$y' = \frac{t}{t^2 y + y}.$$
   Ans:
   $$\begin{aligned} \frac{dy}{dt} &= \frac{t}{y(t^2+1)} \\ \int y dy &= \int \frac{t}{t^2+1}dt \\ \frac{y^2}{2} &= \frac{1}{2}\ln(t^2+1) + c \\ y^2 &= \ln(t^2+1) + c. \end{aligned}$$

3. (15 points) Find the approximate value of $y(1)$ where
   $$y' = t - y^2, y(0) = 1, \Delta t = 0.25.$$

t	$y(t)$	$y'(t)$
0	1	-1
0.25	.75	-0.3125
.5	.6718	.0486
.75	.6839	.2822
1	.7544	

   Thus $y(1) \approx .7544$.

4. (15 points) Consider the IVP
   $$y' = \sqrt{y}, y(0) = 0.$$

Does this problem have a unique solution? If yes, explain why. If no, provide 2 different solutions to the problem.

By separation of variables

$$\begin{aligned} \int \frac{dy}{\sqrt{y}} &= \int dt \\ 2\sqrt{y} &= t + c \\ y(t) &= (\frac{t}{2} + c)^2. \end{aligned}$$

$y(0) = 0$ implies $y(t) = \frac{t^2}{4}, t \geq 0$. On the other hand, $y(t) = 0$ is also a solution so this problem does not have a unique solution.

5. Consider the ODE

$$y' = y(y^2 - 1).$$

   (a) (5 points) Identify the equilibrium solutions and classify them.

   There are 3 equilibrium solutions: $y = 0, y = \pm 1$. $y(t) = 1$ is a source, $y(t) = 0$ is a sink and $y(t) = -1$ is a source.

   (b) (10 points) Sketch the phase portrait of this ODE (note : not the phase line, so you need to exhibit the graphs of y versus t with different choices of initial values in different regions).

6. (15 points) Find all bifurcation values of α for the following family of ODEs :

$$y' = y^2 + 3y + \alpha.$$

(To earn full credit you'll need to point out the change of equilibria before and after the bifurcation values - finding the potential bifurcation points via the formula may not be enough).

We have

$$\begin{aligned} y^2 + 3y + \alpha &= 0 \\ 2y + 3 &= 0. \end{aligned}$$

So $y = -3/2, \alpha = 9/4$. Note that

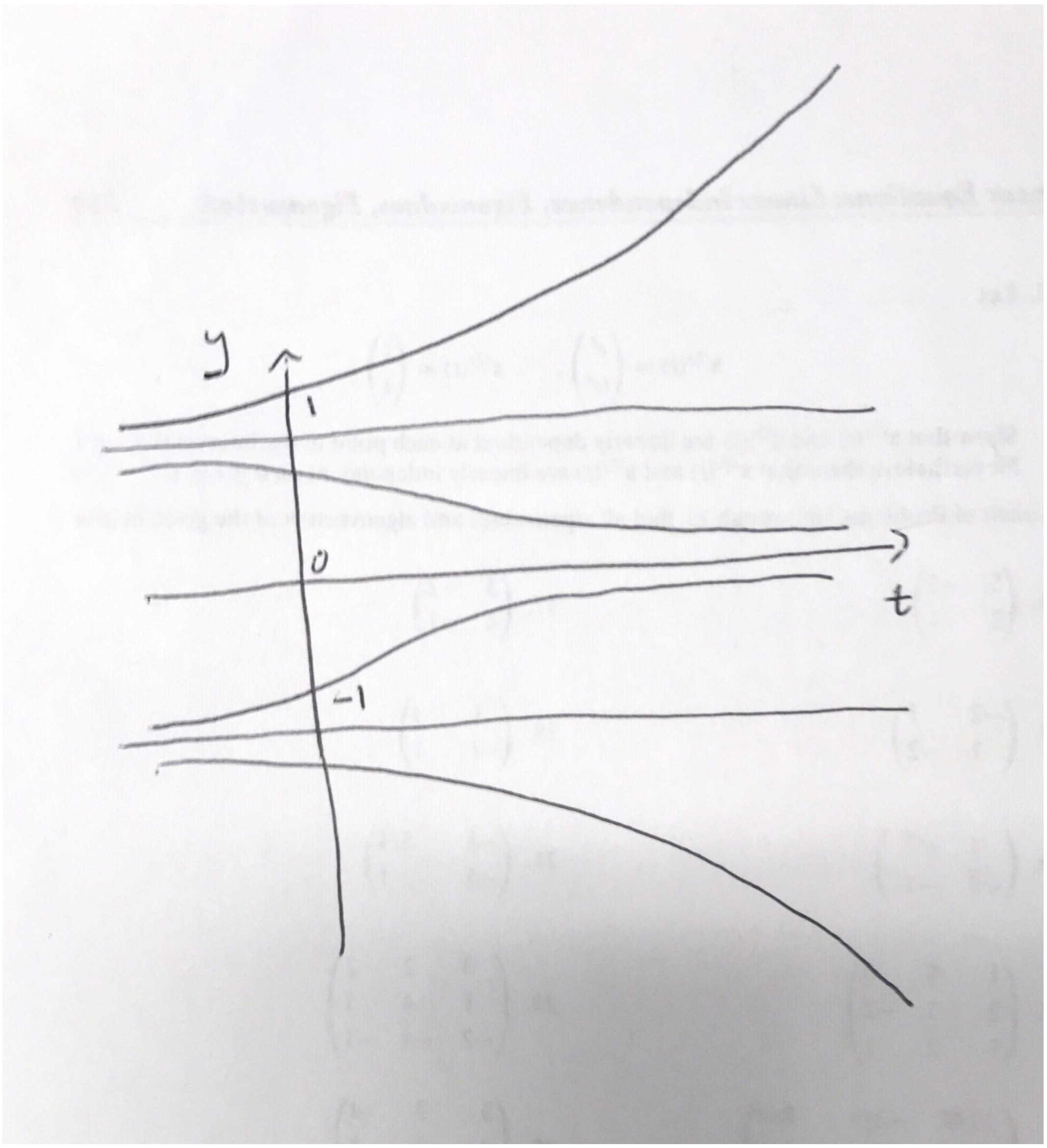
$$y^2 + 3y + 9/4 = (y + 3/2)^2.$$

So for $\alpha < 9/4$, for example $\alpha = 0$ there are 2 equilibrium solutions. When $\alpha = 9/4$, there is 1 equilibrium solution. When $\alpha > 9/4$, for example $\alpha = 4$ there is no equilibrium solution. Thus $\alpha = 9/4$ is the only bifurcation value.

7. Consider the second order linear ODE

$$\begin{aligned} y'' + ty' - y &= t \\ y(0) = 1, y'(0) &= 1. \end{aligned}$$

   (a) (5 points) Convert this equation into a 2x2 first order linear ODE system with initial condition.


y
t
1
0
-1

We have

$$\begin{aligned} y_1' &= y_2 \\ y_2' + ty_2 - y_1 &= t. \end{aligned}$$

That is

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -ty_2 + y_1 + t. \end{aligned}$$

The initial condition is $(y_1(0), y_2(0)) = (1, 1)$.

(b) (10 points) Use the Euler method with $\Delta t = 0.25$ to approximate $y(1)$.

t	$y_1(t)$	$y_2(t)$	$y_1'(t)$	$y_2'(t)$
0	1	1	1	1
0.25	1.25	1.25	1.25	1.1815
.5	1.5625	1.5453	1.5453	1.2898
.75	1.9488	1.8677	1.8677	
1	2.4157			

Thus $y(1) \approx 2.4157$.

8. (15 points) Solve the system

$$\mathbf{Y}' = \begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{bmatrix} \mathbf{Y}.$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & -1 \\ 2 & -\lambda & 0 \\ -1 & 2 & 4-\lambda \end{bmatrix}$$

Expand along the first row:

$$-\lambda(\lambda(\lambda-4)) - (4-\lambda) = (4-\lambda)(\lambda^2 - 1) = 0.$$

Thus $\lambda = \pm 1, 4$. For $\lambda = 1$,

$$A - \lambda I = \begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

For $\lambda = -1$,

$$A - \lambda I = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

For $\lambda = 4$,

$$A - \lambda I = \begin{bmatrix} -4 & 0 & -1 \\ 2 & -4 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix}.$$

Thus the general solution is

$$\mathbf{Y}(t) = c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 2 \\ 1 \\ -8 \end{bmatrix} e^{4t}.$$