Math 252
Summer 2019
Final exam
8/14/19

Name (Print): ______________________

This exam contains 7 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total:	200	

1. (20 points) Solve the system

$$\mathbf{Y}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \mathbf{Y}.$$

$\lambda = 1, 2$. For $\lambda = 1$,

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

For $\lambda = 2$,

$$A - \lambda I = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus

$$\mathbf{Y}(t) = c_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^t + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}.$$

2. (20 points) Solve the system

$$\mathbf{Y}' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix} \mathbf{Y}.$$

$\lambda = 1, 2$. For $\lambda = 1$,

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Solving for the pseudo-eigen vector associated with $\lambda = 1$:

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

For $\lambda = 2$,

$$A - \lambda I = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus the general solution is

$$\mathbf{Y}(t) = c_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^t + c_2(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} te^t) + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}.$$

3. (20 points) Solve the IVP

$$\begin{aligned} \mathbf{Y}' &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \mathbf{Y} \\ \mathbf{Y}(0) &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

(Note that here $(A - \lambda I)^2 = \mathbf{0}$ so a similar approach to the book's approach in the 2x2 case works).

$\lambda = 1$. For $\lambda = 1$,

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^t = \begin{bmatrix} e^t \\ 0 \\ e^t - te^t \end{bmatrix}.$$

4. Consider the system

$$\mathbf{Y}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \mathbf{Y}.$$

(a) (10 points) Find the general solution to the system.
The characteristic equation is $(\lambda - 1)(\lambda + 3) + 5 = \lambda^2 + 2\lambda + 2 = 0$. Thus $\lambda = -1 \pm i$. For $\lambda = -1 + i$

$$A - \lambda I = \begin{bmatrix} 2 - i & -5 \\ 1 & -2 - i \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 2 + i \\ 1 \end{bmatrix}.$$

We have

$$e^{-t} \begin{bmatrix} 2 + i \\ 1 \end{bmatrix} (\cos t + i \sin t) = e^{-t}(\begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + i \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}).$$

Thus

$$\mathbf{Y}(t) = e^{-t}(c_1 \begin{bmatrix} 2\cos t - \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t + 2\sin t \\ \sin t \end{bmatrix}).$$

(b) (10 points) Sketch the phase portrait of the system.

The phase portrait should be a counter-clockwise spiral sink. The actual graph is skipped for the solution of this part.

5. Consider the non-homogeneous system

$$\mathbf{Y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{Y} + \begin{bmatrix} \cos t \\ e^t \end{bmatrix}.$$

(a) (10 points) Find the corresponding decoupled non-homogeneous system. You do not have to solve the decoupled system.

The characteristic equation is $(1-\lambda)^2 - 4 = 0$. Thus $\lambda = 3, -1$. For $\lambda = 3$,

$$A - \lambda I = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For $\lambda = -1$,

$$A - \lambda I = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Thus $A = PDP^{-1}$ where

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, P^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

The corresponding decoupled system is

$$\begin{aligned} \mathbf{Z}' &= \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{Z} + \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \cos t \\ e^t \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{Z} + \begin{bmatrix} \frac{\cos t + e^t}{2} \\ \frac{\cos t - e^{-t}}{2} \end{bmatrix}. \end{aligned}$$

(b) (10 points) Find e^{At} where $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Your final answer must be a 2x2 matrix in t.

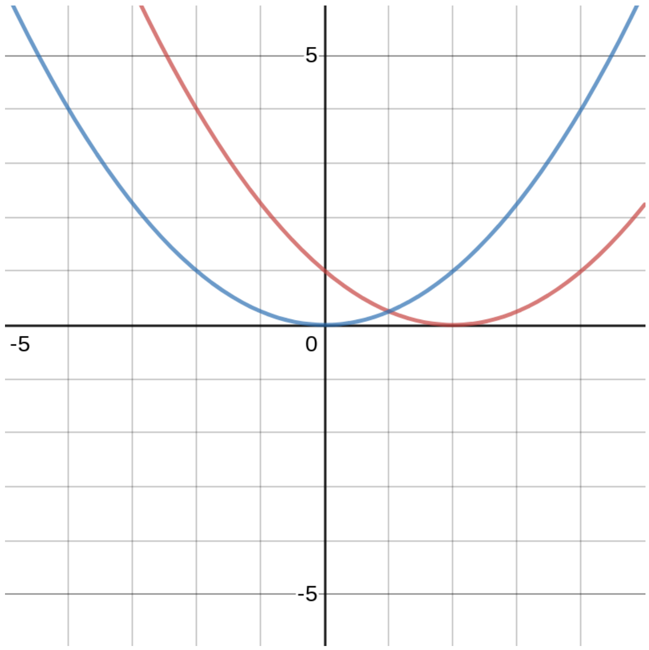
We have

$$\begin{aligned} e^{At} &= Pe^{Dt}P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & e^{3t} \\ e^{-t} & -e^{-t} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{3t} + e^{-t} & e^{3t} - e^{-t} \\ e^{3t} - e^{-t} & e^{3t} + e^{t} \end{bmatrix}. \end{aligned}$$

6. (20 points) Find all bifurcation values of the system

$$\mathbf{Y}' = \begin{bmatrix} a & 1 \\ 2a-1 & a \end{bmatrix} \mathbf{Y}$$

We have $T = 2a, D = a^2 - 2a + 1 = (a-1)^2$. Thus $D = (\frac{T}{2} - 1)^2$. The graph of this parabola together with $D = \frac{T^2}{4}$ gives:



There are two bifurcation points at $T = 0, 1$, corresponding to $a = 0, 1/2$.

7. Consider the spring mass system

$$my'' + \gamma y' + ky = \cos(\omega y).$$

(a) (10 points) Suppose $m = 2, \gamma = 0, k = 4$. Find ω such that resonance occurs in the system. We have $2\lambda^2 + 4 = 0$, thus $\lambda = \pm\sqrt{2}i$ and $\omega = \sqrt{2}$ for resonance to occur.

(b) (10 points) Suppose $m = \gamma = k = \omega = 1$. Find the steady state solution of the system, expressed under the form $A\cos(\omega t + \phi)$. (You must identify A and ϕ to earn full credit for this problem). The steady state solution has the form

$$y_p(t) = a\cos t + b\sin t.$$

It follows that

$$\begin{aligned} y_p'(t) &= b\cos t - a\sin t \\ y_p''(t) &= -a\cos t - b\sin t. \end{aligned}$$

Plug in: $b\cos t - a\sin t = \cos t$. Thus $b = 1, a = 0$. Thus

$$y_p(t) = \sin t = \cos(t - \frac{\pi}{2}).$$

8. Consider the second order ODE

$$y'' - 4y' + 4y = te^{2t} + \cos 2t.$$

   (a) (10 points) Find the homogeneous solution to the equation.

$$y_h(t) = c_1 e^{2t} + c_2 te^{2t}.$$

   (b) (10 points) Find a suitable form for a particular solution to the equation. You do not have to solve for the specific constants.

$$y_p(t) = t^2(At + B)e^{2t} + C\cos 2t + D\sin 2t.$$

9. Consider the non-linear system

$$\begin{array}{rcl} x' & = & x(x^2 - y^2 - 1) \\ y' & = & y(x + y - 1) \end{array}$$

   (a) (10 points) Find all equilbria.
   We have 4 systems:

$$\left\{ \begin{array}{r} x = 0 \\ y = 0 \end{array} \right. ; \left\{ \begin{array}{r} x = 0 \\ x + y - 1 = 0 \end{array} \right. ; \left\{ \begin{array}{r} x^2 - y^2 - 1 = 0 \\ y = 0 \end{array} \right. ; \left\{ \begin{array}{r} x^2 - y^2 - 1 = 0 \\ x + y - 1 = 0 \end{array} \right.$$

   These yield the corresponding equilibria:

$$(0,0); (0,1); (\pm 1, 0); (1,0).$$

   Thus there are 4 equilibria in total.

   (b) (5 points) Find the linearized system at $(0,0)$
   We have $F_x = 3x^2 - y^2 - 1, F_y = -2xy, G_x = y, G_y = 2y - 1$. Thus the linearized system is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

   (c) (5 points) Classify the nature of $(0,0)$ as an equilibrium.
   From the previous part $(0,0)$ is a sink.

10. Consider the recurrence relation

$$x_{n+1} = 3x_n(1 - x_n).$$

(a) (10 points) Find all fixed points.
The fixed points satisfy

$$x = 3x(1 - x).$$

That is $3x^2 - 2x = 0$. Thus $x = 0, x = \frac{2}{3}$ are fixed points.

(b) (10 points) Classify all fixed points as attracting, repelling or neutral (neither attracting nor repelling).
We have $F'(x) = 3 - 6x$. $F'(0) = 3$ thus 0 is repelling. Thus $F'(2/3) = -1$ thus 2/3 is neutral.