Math 622 **Name (Print):** ____________________
Spring 2014
Midterm 1 - Form A
03/05/2014

This exam contains 12 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note (1 sided), and a scientific calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. For example, in question involved the multi-period binomial model, I would like to see how you derive the no arbitrage price, say by displaying the tree with all the nodes filled out if the situation is appropriate.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	30	
3	25	
4	15	
5	15	
Total:	100	

1. (a) (3 points) Let $W(t)$ be a Brownian motion and $W(0) = 0$. Let $b_1 > 0$ and $b_2 < 0$ be 2 constants. Define

$$\tau_1 := \inf\{t \geq 0 : W(t) = b_1 \text{ or } W(t) = b_2\}.$$

Is τ_1 a stopping time with respect to $\mathcal{F}^W(t)$? Provide a brief explanation for your answer.
Ans: Denote

$$\widetilde{\tau}_1 := \inf\{t : W(t) = b_1\}$$
$$\bar{\tau}_1 := \inf\{t : W(t) = b_2\}$$

Then $\widetilde{\tau}_1$ and $\bar{\tau}_1$ are stopping times, $\tau_1 = \widetilde{\tau}_1 \wedge \bar{\tau}_1$ so τ_1 is a stopping time.

(b) (4 points) Let

$$\begin{aligned} S_t &= rS_t dt + S_t dW_t; \\ S(0) &= 1. \end{aligned}$$

Define

$$\tau_2 := \sup\{t \geq 0 : \int_0^t S(u)du = 2\}.$$

Is τ_2 a stopping time with respect to $\mathcal{F}^S(t)$? Provide a brief explanation for your answer.
Ans:
Note that $S(u) > 0, \forall u$ therefore if we denote

$$Y_t := \int_0^t S(u)du$$

then $Y_0 = 0$, Y_t is differentiable in t, $Y_t' > 0$ so once Y_t hits level 2 it will never hit level 2 again. Thus

$$\tau_2 := \inf\{t : Y_t = 2\}.$$

So we see that τ_2 is a stopping time.

(c) (4 points) Let τ_4 be a stopping time with respect to some filtration $\mathcal{F}(t)$ and $\tilde{\tau}_4$ be a random time such that $\tilde{\tau}_4 > \tau_4$. Can $\tilde{\tau}_4$ be a stopping time with respect to $\mathcal{F}(t)$? If yes provide an example, if no give an explanation.
Ans: It is possible that $\tilde{\tau}_4$ is a stopping time with respect to $\mathcal{F}(t)$. For example, let

$$\tau_4 := \inf\{t \geq 0 : W_t = 1\}$$
$$\tilde{\tau}_4 := \inf\{t \geq 0 : W_t = 2\},$$

where $W_0 = 0$, W_t is a Brownian motion. Then it is clear that both τ_4 and $\tilde{\tau}_4$ are stopping times, $\tilde{\tau}_4 > \tau_4$.

(d) (4 points) Let τ_5 be a stopping time with respect to some filtration $\mathcal{F}(t)$ and $\tilde{\tau}_5$ be a random time such that $\tilde{\tau}_5 < \tau_5$. Can $\tilde{\tau}_5$ be a stopping time with respect to $\mathcal{F}(t)$? If yes provide an example, if no give an explanation.

Ans: It is also possible that $\tilde{\tau}_5$ is a stopping time. We can use exactly the same example as above, except now we define

$$\tilde{\tau}_5 := \inf\{t \geq 0 : W_t = 1\}$$
$$\tau_5 := \inf\{t \geq 0 : W_t = 2\}.$$

2. In this problem, let the interest rate be r , a positive constant. Consider

$$\begin{aligned} dS_t &= \alpha S_t dt + S(t-)dQ(t) + S_t dW_t. \\ S_0 &= 1. \end{aligned}$$

where $Q(t) = b_1 N_1(t) + b_2 N_2(t)$, $N_1(t), N_2(t)$ are independent Poisson processes with rates λ_1, λ_2, respectively and α, b_1, b_2 are constants, $0 < \alpha < r$, $-1 < b_1 < b_2 < 0$.

(a) (5 points) Does there exist a risk neutral probability for this model of S_t?

Ans: Note that

$$dS_t = rS_t dt + S(t-)d\big(Q_t - (r - \alpha + \theta)t\big) + S_t d\big(W_t + \theta t\big).$$

So we need to find a probability $\mathbb{Q}$ such that Q is a compound Poisson process under $\mathbb{Q}$,

$$E^{\mathbb{Q}}\big[Q_1\big] = r - \alpha + \theta, \tag{1}$$

and $W_t + \theta t$ is a $\mathbb{Q}$-Brownian motion.

Let $\tilde{\lambda}_1, \tilde{\lambda}_2$ be the rates of N_1 and N_2 under $\mathbb{Q}$. Then (1) requires

$$b_1\tilde{\lambda}_1 + b_2\tilde{\lambda}_2 = r - \alpha + \theta. \tag{2}$$

So we need to solve this market price of risk equation for $\tilde{\lambda}_1, \tilde{\lambda}_2, \theta$ under the constraint that $0 < \alpha < r$, $-1 < b_1 < b_2 < 0$. We also require that $\tilde{\lambda}_1 > 0, \tilde{\lambda}_2 > 0$. Note that there is no restriction on θ.

Once we are able to find $\tilde{\lambda}_1, \tilde{\lambda}_2, \theta$, then we can proceed to define the change of measure kernel Z_T as in the lecture note:

$$\begin{aligned} Z_(t) &:= \exp\Big[- \int_0^t \theta(u)dW(u) - \frac{1}{2}\theta^2(u)du\Big] \prod_{i=1}^{2} \left(\frac{\tilde{\lambda}_i}{\lambda_i}\right)^{N_i(T)} e^{(\lambda_i - \tilde{\lambda}_i)T}; \\ d\mathbb{Q} &= Z(T)d\mathbb{P} \text{ on } \mathcal{F}(T). \end{aligned}$$

Note that $r - \alpha > 0$. We choose θ so that $r - \alpha + \theta = b_1 + b_2$. Then (2) has a solution $\tilde{\lambda}_1 = 1, \tilde{\lambda}_2 = 1, \theta = b_1 + b_2 - (r - \alpha)$. So a risk neutral measure exists for this model of S_t.

(b) (5 points) If your answer to part (a) is yes, is the risk neutral probability unique?

Ans: It is clear from the analysis above that the risk neutral probability is not unique.

For example, we could choose θ so that $r - \alpha + \theta = 2(b_1 + b_2)$. Then (2) has a solution $\tilde{\lambda}_1 = 2, \tilde{\lambda}_2 = 2, \theta = 2(b_1 + b_2) - (r - \alpha)$ and so on.

(c) (5 points) Now suppose $\lambda_1 = \lambda_2 = 1$,

$$\begin{aligned} b_1 &= e^{-1} - 1 \\ b_2 &= e^{-2} - 1. \end{aligned}$$

Explicitly identify a constant θ and a compound Poisson process $\widetilde{Q}$ such that

$$S(t) = \exp\{W(t) + \theta t + \widetilde{Q}(t)\}.$$

Ans:
The equation

$$\begin{aligned} dS_t &= \alpha S_t dt + S(t-)dQ(t) + S_t dW_t. \\ S_0 &= 1 \end{aligned}$$

has solution

$$\begin{aligned} S_t &= e^{(\alpha - \frac{1}{2})t + W_t} \prod_{0<u\le t} (1 + \Delta Q(u)) \\ &= e^{(\alpha - \frac{1}{2})t + W_t} \prod_{0<u\le t} (1 + b_1 \Delta N_1(u))(1 + b_2 \Delta N_2(u)) \\ &= e^{(\alpha - \frac{1}{2})t + W_t} \prod_{i=1}^{N_1(t)} (1 + b_1) \prod_{i=1}^{N_2(t)} (1 + b_2) \\ &= e^{(\alpha - \frac{1}{2})t + W_t} e^{-N_1(t) - 2N_2(t)} \\ &= e^{(\alpha - \frac{1}{2})t + W_t + \tilde{Q}(t)}, \end{aligned}$$

where $\tilde{Q}(t) = -N_1(t) - 2N_2(t)$ is a compound Poisson process and $\theta = \alpha - \frac{1}{2}$.

(d) (5 points) Let α be chosen so that the model is risk-neutral. If we were to use this risk-neutral model to price a put option at strike K and expiry T, we would find the price is $V(t) = e^{-r(T-t)} E\Big[\big(K - S(T)\big)^+ \big| \mathcal{F}(t)\Big]$. Show that $V(t) = c(t, S(t))$, where $c(t, x)$ is given in the form

$$c(t, x) = e^{-r(T-t)} E\Big[H(x, Y(T-t))\Big],$$

where $Y(s)$ has the form $Y(s) = W(s) + \theta s + \widetilde{Q}(s)$. Your answer should explicitly define H and θ.

Ans: We have

$$S(T) = S(t) \exp\Big[\theta(T-t) + W(T) - W(t) + \tilde{Q}(T) - \tilde{Q}(t)\Big].$$

Thus by the Independence lemma,

$$\begin{aligned} V(t) &= e^{-r(T-t)} E\Big[\Big(K - S(t)\exp\left[\theta(T-t) + W(T) - W(t) + \tilde{Q}(T) - \tilde{Q}(t)\right]\Big)^{+} \Big| \mathcal{F}(t)\Big] \\ &= c(t, S(t)), \end{aligned}$$

where

$$c(t,x) = e^{-r(T-t)} E\Big[H(x, Y(T-t))\Big],$$

and

$$\begin{aligned} H(x,y) &= (K - xe^{y})^{+}; \\ Y(s) &= W(s) + \theta s + \tilde{Q}(s); \\ \theta &= \alpha - \frac{1}{2} = r - (b_1 + b_2) - \frac{1}{2}. \end{aligned}$$

(e) (10 points) Find the partial differential-difference equation that $c(t,x)$ satisfies. (Use the back of the page if you run out of space)

Ans: Apply Ito's formula to $e^{-rt}c(t,S(t))$ gives

$$\begin{aligned} e^{-rt}c(t,S(t)) &= \int_0^t -re^{-ru}c(u,S(u))du + e^{-ru}\frac{\partial}{\partial t}c(u,S(u))du + e^{-ru}\frac{\partial}{\partial x}c(u,S(u))dS^c(u) \\ &\quad +\frac{1}{2}e^{-ru}\frac{\partial^2}{\partial x^2}c(u,S(u))S^2(u)du + \sum_{0<u\le t} e^{-ru}[c(u,S(u)) - c(u-,S(u-))] \\ &= \int_0^t e^{-ru}\Big[-rc(u,S(u)) + \frac{\partial}{\partial t}c(u,S(u)) + \frac{\partial}{\partial x}c(t,S(u))\alpha S(u) \\ &\quad +\frac{1}{2}\frac{\partial^2}{\partial x^2}c(u,S(u))S^2(u)\Big]du \\ &\quad +\int_0^t e^{-ru}\frac{\partial}{\partial x}c(t,S(u))S(u)dW(u) + \sum_{0<u\le t} e^{-ru}[c(u,S(u)) - c(u,S(u-))]. \end{aligned}$$

Since $\lambda_1 = \lambda_2 = 1$, we have

$$\begin{aligned} e^{-rt}c(t,S(t)) &= \int_0^t e^{-ru}\Big[-rc(u,S(u)) + \frac{\partial}{\partial t}c(u,S(u)) + \frac{\partial}{\partial x}c(t,S(u))\alpha S(u) \\ &\quad +\frac{1}{2}\frac{\partial^2}{\partial x^2}c(u,S(u))S^2(u) + \sum_{i=1}^{2}[c(u,S(u)(1+b_i)) - c(u,S(u))\Big]du \\ &\quad +\int_0^t e^{-ru}\frac{\partial}{\partial x}c(t,S(u))S(u)dW(u) + \int_0^t e^{-ru}[c(u,S(u)) - c(u,S(u-))]dM(u), \end{aligned}$$

where

$$\begin{aligned} M(t) &= \sum_{i=1}^{2} N_i(t) - t \\ b_1 &= e^{-1} - 1 \\ b_2 &= e^{-2} - 1. \end{aligned}$$

Setting the dt part to be 0 gives the following:

$$\begin{aligned} -rc(t,x) &+ \frac{\partial}{\partial t}c(t,x) + \alpha x\frac{\partial}{\partial x}c(t,x) + \frac{1}{2}\frac{\partial^2}{\partial x^2}c(t,x)x^2 \\ &+ \sum_{i=1}^{2}[c(t,xe^{-i}) - c(t,x)] = 0, 0 \le t < t, x > 0; \\ c(T,x) &= (K-x)^+, x > 0. \end{aligned}$$

3. Let $S(t)$ satisfy

$$\begin{aligned} dS_t &= rS_t dt + \sigma dW_t \\ S_0 &= 1. \end{aligned}$$

Define

$$S_{ave}(t) := \frac{1}{t}\int_0^t S(u)du.$$

Consider the following option with expiry at time T and payoff

$$V_T = \mathbf{1}_{\{\max_{u\in[0,T]} S_u < B\}}\big(TS_{ave} - K\big)^+,$$

where B, K are positive constants, $B > 1$.

(a) (5 points) What relationship must B and K satisfy so that $\mathbb{P}(V_T > 0) > 0$?
Ans: On $\{\max_{u\in[0,T]} S_u < B\}$, we have

$$TS_{ave} = \int_0^T S_u du \le BT.$$

Therefore require that $BT > K$ so that $\mathbb{P}(V_T > 0) > 0$.

(b) (5 points) Define

$$\tau = \inf\{t \ge 0 : S(t) = B\}.$$

Let V_t be the risk neutral price of the above option. Show that there exists a function $v(t, x, y)$ such that $V_t = v(t, S(t\wedge\tau), Y(t\wedge\tau))$ where

$$Y(t) = \int_0^t S(u)du.$$

Ans: We have

$$V(t) = \mathbf{1}_{\{\max_{u\in[0,t]} S_u < B\}} E\Big\{e^{-r(T-t)}\mathbf{1}_{\{\max_{u\in[t,T]} S_u < B\}}\Big(Y(t) + \int_t^T S_u du - K\Big)^+\Big|\mathcal{F}(t)\Big\}.$$

From the lecture note, we have discussed that $S(t), Y(t)$ is a Markov process. Therefore,

$$V(t) = \mathbf{1}_{\{\max_{[0,t]} S(u) < B\}} v(t, S_t, Y_t)$$

where

$$v(t, x, y) = E\Big\{e^{-r(T-t)}\mathbf{1}_{\{\max_{[t,T]} S_u < B\}}\Big(y + \int_t^T S_u du - K\Big)^+\Big|S_t = x, Y_t = y\Big\}.$$

Note that

$$\mathbf{1}_{\max_{[0,t]} S(u)<B} = \mathbf{1}_{\{\tau>t\}}.$$

Therefore,

$$\begin{aligned} V(t) &= \mathbf{1}_{\{t<\tau\}} v(t, S_t, Y_t) \\ &= v(t, S_{t\wedge\tau}, Y_{t\wedge\tau}) \end{aligned}$$

Indeed, if $t < \tau$ then

$$\mathbf{1}_{\{t<\tau\}} v(t, S_t, Y_t) = v(t, S_t, Y_t) = v(t, S_t, Y_t).$$

On the other hand, if $t \geq \tau$ then the LHS $= 0$ and $S_{t\wedge\tau} = S_\tau = B$. Then

$$v(t, B, Y_\tau) = E\Big\{e^{-r(T-t)} \mathbf{1}_{\{\max_{[t,T]} S_u < B\}} \Big(Y_\tau + \int_t^T S_u du - K\Big)^+ \Big| S_t = B\Big\} = 0,$$

by the immediate crossing property of S_t over the level B once it hits.

(c) (15 points) Find the PDE that $v(t, x, y)$ satisfies. Identify the domain of t, x, y and the boundary conditions for v.

Apply Ito's formula to $e^{-rt} v(t, S_{t\wedge\tau}, Y_{t\wedge\tau})$, recall that

$$\begin{aligned} dS_{t\wedge\tau} &= \mathbf{1}_{\{t<\tau\}}(rS_t dt + \sigma S_t dW_t) \\ dY_{t\wedge\tau} &= \mathbf{1}_{\{t<\tau\}} S_t dt. \end{aligned}$$

we have

$$\begin{aligned} e^{-rt} v(t, S_{t\wedge\tau}, Y_{t\wedge\tau}) &= v(0, S_0, Y_0) + \int_0^{t\wedge\tau} e^{-ru}\Big[-rv(u, S_u, Y_u) + \mathcal{L}v(u, S_u, Y_u)\Big] du + \\ &\quad \int_0^{t\wedge\tau} e^{-ru} v_x(u, S_u, Y_u) \sigma S_u dWu, \end{aligned}$$

where

$$\mathcal{L}v(t, x, y) = v_t(t, x, y) + v_x(t, x, y) rx + \frac{1}{2} v_{xx}(t, x, y) \sigma^2 x^2 + v_y(t, x, y) x.$$

Since the LHS is a martingale, we need to set the dt term to 0 which gives

$$-rv(t, x, y) + v_t(t, x, y) + v_x(t, x, y) rx + \frac{1}{2} v_{xx}(t, x, y) \sigma^2 x^2 + v_y(t, x, y) x = 0,$$
$$0 \leq t < T, 0 < x < B,$$

and for a fixed t, we only consider $0 \leq y < Bt$.

Note the domain: In the integral $\int_0^{t\wedge\tau} e^{-ru}\Big[-rv(u,S_u,Y_u)+\mathcal{L}v(u,S_u,Y_u)\Big]du$, the process $S(t),Y(t)$ remains in $\{0<x<B, 0\le y<Bt\}$ so we want and only need to assume the existence and continuity of the first and second derivatives of v in this region.

We now specify the boundary conditions:

$$\begin{aligned}
v(T,x,y) &= (y-K)^+, 0\le x<B, 0\le y<BT\\
v(t,B,y) &= 0\\
v(t,0,y) &= e^{-r(T-t)}(y-K)^+.\\
v(t,x,Bt) &= 0.
\end{aligned}$$

Where the last condition follows from the fact that

$$Y(t)\le \max_{u\in[0,t]} S_u t < Bt,$$

if S_u never hits B on $[0,t]$. Therefore $Y(t)=Bt$ implies that $S(u)$ has hit B at some time u in $[0,t]$ and thus $V(t)=0$.

4. Let $W(t)$ be a Brownian motion, $W(0) = 0$ and $B > 0$ a constant. Define

$$\begin{aligned} M(t) &:= \max_{[0,t]} W(t) \\ \tau_B &:= \inf\{t \geq 0 : W(t) = B\}. \end{aligned}$$

Recall the following identity: for $B \geq w$

$$\Big\{M(t) \geq B, W(t) \leq w\Big\} = \Big\{B^{\tau_B}(t) \geq 2B - w\Big\},$$

where B^{τ_B} is $W(t)$ reflected at time τ_B, B^{τ_B} is also a Brownian motion.

(a) (10 points) Show that

$$\mathbb{P}(\tau_B \leq t) = 2\int_B^\infty \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}dx.$$

(Hint: $\{\tau_B \leq t\} = \{M(t) \geq B\}$. Use the above identity with $w = B$).
Ans:

$$\begin{aligned} \{\tau_B \leq t\} &= \{M(t) \geq B\} = \{M(t) \geq B, W_t < B\} \cup \{M(t) \geq B, W_t \geq B\} \\ &= \{B^{\tau_B}(t) \geq B\} \cup \{W_t \geq B\}. \end{aligned}$$

The two events $\{B^{\tau_B}(t) \geq B\}, \{W_t \geq B\}$ are disjoint (following their derivation), therefore

$$\begin{aligned} \mathbb{P}(\tau_B \leq t) &= \mathbb{P}(B^{\tau_B}(t) \geq B) + \mathbb{P}(W_t \geq B) \\ &= 2\mathbb{P}(N(0,t) \geq B) = 2\int_B^\infty \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}}dx. \end{aligned}$$

(b) (5 points) Show that

$$\lim_{t\to\infty} \mathbb{P}(\tau_B \leq t) = 1,$$

Thus Brownian motion eventually hits any level B.
Ans: Also from part a:

$$\begin{aligned} \lim_{t\to\infty} \mathbb{P}(\tau_B \leq t) &= 2\lim_{t\to\infty} \mathbb{P}(N(0,t) \geq B) \\ &= 2\lim_{t\to\infty} \mathbb{P}(N(0,1) \geq \frac{B}{\sqrt{t}}) = 2\lim_{t\to\infty} \int_{\frac{B}{\sqrt{t}}}^\infty \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} = 2\frac{1}{2} = 1. \end{aligned}$$

5. (15 points) Let X_t satisfy

$$\begin{aligned} X_t &= 1 + \int_0^t X_u du + \int_0^t X_u dW_u + J(t), \\ X(0) &= 1. \end{aligned}$$

where $W(t)$ is a Brownian motion and $J(t)$ a pure jump process. Solve for an explicit solution of $X(t)$.

(Hint: Let $\Gamma(t)$ satisfy

$$\begin{aligned} d\Gamma(t) &= -\Gamma(t)dt - \Gamma(t)dW(t) \\ \Gamma(0) &= 1. \end{aligned}$$

Then $\Gamma(t)$ has an explicit solution. Apply Ito's formula to find $d\Gamma X(t)$).

Ans:

Note that $\Gamma(t)$ has the explicit solution

$$\Gamma(t) = e^{-\frac{3}{2}t - W_t}.$$

Apply Ito's formula:

$$\begin{aligned} (X\Gamma)(t) &= X_0\Gamma_0 + \int_0^t X_u d\Gamma_u + \int_0^t \Gamma_u dX^c(u) - \int_0^t X_u \Gamma_u du \\ &\quad + \sum_{0<u\leq t} X(u)\Gamma(u) - X(u-)\Gamma(u-). \end{aligned}$$

Observe that

$$\int_0^t X_u d\Gamma_u + \int_0^t \Gamma_u dX^c(u) = -\int_0^t X_u \Gamma_u du - \int_0^t X_u \Gamma_u dWu + \int_0^t \Gamma_u X_u du + \int_0^t \Gamma_u X_u dW_u = 0.$$

And since $\Gamma(u)$ is continuous:

$$X(u)\Gamma(u) - X(u-)\Gamma(u-) = [X(u) - X(u-)]\Gamma(u) = \Gamma(u)\Delta J(u).$$

Thus

$$\begin{aligned} (X\Gamma)(t) &= X_0\Gamma_0 - \int_0^t X_u \Gamma_u du \\ &\quad + \int_0^t \Gamma(u) dJ(u). \end{aligned}$$

Denote $Y_t := X_t \Gamma_t$ then the above equation says

$$Y_t = Y_0 + \int_0^t -Y_u du + \bar{J}(t),$$

where $\bar{J}(t) = \int_0^t \Gamma(u) dJ(u)$ is a pure jump process. Then

$$e^t Y_t = Y_0 + \int_0^t (e^u Y_u - e^u Y_u) dt + \sum_{0<u\leq t} e^u \Delta \bar{J}(u).$$

Hence,

$$X_t \Gamma_t = X_0 \Gamma_0 + \int_0^t e^{u-t} \Gamma_t dJ_t.$$

Also since $\Gamma(t) > 0$, $\Gamma^{-1}(t)$ exists and equals $e^{\frac{3}{2}t + W_t}$. Recall also $X_0 = 1$. Therefore

$$\begin{aligned} X(t) &= \Gamma^{-1}(t) + \Gamma^{-1}(t) \int_0^t e^{u-t} \Gamma(u) dJ(u) \\ &= e^{\frac{3}{2}t + W_t} + \int_0^t e^{\frac{1}{2}(t-u) + W_t - W_u} dJ(u). \end{aligned}$$