Math 622
Spring 2014
Final Exam - Form A
05/14/2014

Name (Print): ______________________________

This exam contains 8 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	25	
3	25	
4	30	
5	20	
Total:	120	

1. Let S_t be a stock price with dynamics

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}(t),$$

where r, σ are constants, r is the interest rate, $\tilde{W}(t)$ is a Brownian motion under the risk neutral measure $\tilde{P}$.

Consider an up and in put option on S_t with Barrier L, strike price K and exipiry T, where $K > L$. In other words, the pay off of this option at time T is

$$\begin{aligned} V_T^{UI} &= (K - S_T)^+ \text{ if } \sup_{0\le t\le T} S_t \ge L \\ &= 0 \text{ otherwise.} \end{aligned}$$

(a) (5 points) Set up an expression for V_t^{UI}, the value of the above up and in put option at time t.

Ans:

$$V^{UI}(t) = \tilde{E}\Big[e^{-(T-t)}(K - S_T)^+ \mathbf{1}_{\sup_{0\le t\le T} S_t \ge L}\Big|\mathcal{F}(t)\Big].$$

(b) (5 points) Denote V_t^{UO} as the value at time t of the up and out put option on S_t with the same barrier, strike and expiry; and V_t^E as the value at time t of the European put option on S_t with the same strike and expiry. Show that

$$V_t^{UI} + V_t^{UO} = V_t^E.$$

Ans:

$$V^{UO}(t) = \tilde{E}\Big[e^{-(T-t)}(K - S_T)^+ \mathbf{1}_{\sup_{0\le t\le T} S_t < L}\Big|\mathcal{F}(t)\Big].$$

Thus

$$\begin{aligned} V^{UO}(t) + V^{UI}(t) &= \tilde{E}\Big[e^{-(T-t)}(K - S_T)^+ \Big(\mathbf{1}_{\sup_{0\le t\le T} S_t < L} + \mathbf{1}_{\sup_{0\le t\le T} S_t \ge L}\Big)\Big|\mathcal{F}(t)\Big] \\ &= \tilde{E}\Big[e^{-(T-t)}(K - S_T)^+\Big|\mathcal{F}(t)\Big] = V^E(t). \end{aligned}$$

(c) (5 points) Define

$$\tau := \inf\Big\{t \ge 0 : S_t \ge L\Big\}.$$

Show that there exists a function $v(t, S_t)$ such that

$$V_t^{UI} = v(t, S_t) \text{ for } 0 \le t \le \tau.$$

Moreover, we also have

$$V_t = v^E(t, S_t) \text{ for } t \ge \tau,$$

where $v^E(t, S_t) = V^E(t)$ is the value at time t of the European put option on S_t with strike K and expiry T.

Ans: From the lecture note, we know that there exists a function $v^{UO}(t, x)$ such that

$$V^{UO}(t) = v^{UO}(t, S_t)\mathbf{1}_{t\le\tau}.$$

From the Markovian property of S_t, $V^E(t) = v^E(t, S_t)$. Thererfore, since $V^{UI}(t) + V^{UO}(t) = V^E(t)$ we have

$$v^{UO}(t, S_t)\mathbf{1}_{t\le\tau} + V^{UI}(t)\mathbf{1}_{t\le\tau} = v^E(t, S_t)\mathbf{1}_{t\le\tau}.$$

Thus

$$V^{UI}(t)\mathbf{1}_{t\le\tau} = [v^E(t, S_t) - v^{UO}(t, S_t)]\mathbf{1}_{t\le\tau}.$$

Moreover, it is clear that $V^{UO}(t) = 0$ for $t \ge \tau$. Thus $V^{UI}(t) = v^E(t, S_t), t \ge \tau$. Denoting

$$v(t, S_t) = v^E(t, S_t) - v^{UO}(t, S_t),$$

we see that v satisfies the requirement.

(d) (5 points) Derive the PDE, with boundary and terminal conditions for $v(t, S_t)$ in the previous part.

Ans: From the lecture, we see that both $V^E(t, x)$ and $V^{UO}(t, x)$ satisfy the PDE

$$v_t - rv + rxv_x + \frac{1}{2}x^2\sigma^2 v_{xx} = 0, 0 \le t < T, 0 < x < L,$$

therefore, $v(t, x) = V^E(t, x) - V^{UO}(t, x)$ also satisfies the same PDE. The domain for $v(t, x)$ is $t \in [0, T], 0 \le x \le L$ (since $v(t, x) = v^E(t, x)$ for $x > L$ so there is no need to solve for the PDE when $x > L$). The boundary condition that $v(t, x)$ satisfies is

$$\begin{aligned}
v(t, 0) &= 0, 0 \le t \le T \\
v(t, L) &= v^E(t, L), 0 \le t \le T \\
v(T, x) &= 0, 0 \le x < L \\
v(T, L) &= K - L,
\end{aligned}$$

where $v(t, 0) = 0$ because $v(t, 0)$ is the value of the option when $S_t = 0$. But $S_t = 0$ only if $S_0 = 0$, and so the option is never knocked in to begin with. (Alternatively $v(t, 0) = v^E(t, 0) - v^{UO}(t, 0) = 0$ where we also use the fact that $v^E(t, 0) = v^{UO}(t, 0) = Ke^{-r(T-t)}$ since $S_t = 0$ means $S_0 = 0$ so the up and out option is not knocked out.)

2. Let the forward rate $f(t,T)$ have the following dynamics under the physical measure P:

$$df(t,T) = \alpha(t,T)dt + \sigma W(t),$$

where σ is a *deterministic constant.*

(a) (5 points) What is the dynamics of $f(t,T)$ under the risk neutral measure $\tilde{P}$? (You only need to state, and don't have to derive the result.)

Ans: Let $\sigma^*(t,T) = \int_t^T \sigma du = \sigma(T-t)$, we have

$$\begin{aligned} df(t,T) &= \sigma\sigma^*(t,T)dt + \sigma d\tilde{W}(t) \\ &= \sigma^2(T-t)dt + +\sigma d\tilde{W}(t). \end{aligned}$$

(b) (5 points) Suppose $f(0,T) = \frac{1}{T+1}, T \geq 0$. Find an explicit formula for $f(t,T)$ in terms of $\sigma, \tilde{W}, t, T$.

Ans:

$$\begin{aligned} f(t,T) &= f(0,T) + \int_0^t df(u,T) \\ &= \frac{1}{T+1} - \frac{\sigma^2}{2}\big((T-t)^2 - T^2\big) + \sigma\tilde{W}(t). \end{aligned}$$

(c) (10 points) Find the dynamics of $R(t)$, the short rate, under $\tilde{P}$.

Ans:

$$R(t) = f(t,t) = \frac{1}{t+1} + \frac{\sigma^2 t^2}{2} + \sigma\tilde{W}(t).$$

Therefore

$$dR(t) = \Big(-\frac{1}{(t+1)^2} + \sigma^2 t\Big)dt + \sigma d\tilde{W}(t).$$

(d) (5 points) Find $\tilde{E}\big(R(T)\big)$, $\tilde{\text{Var}}\ \big(R(T)\big)$ where $\tilde{\text{Var}}$ means we compute the variance under the risk neutral measure $\tilde{P}$.

Ans: From the above, we have

$$\begin{aligned} \tilde{E}\big(R(T)\big) &= \frac{1}{T+1} + \frac{\sigma^2 T^2}{2} \\ \tilde{\text{Var}}\ \big(R(T)\big) &= \sigma^2 T. \end{aligned}$$

3. For $k = 2, 3$, we specify the dynamics of the bond $B(t, k)$ under the risk neutral measure $\tilde{P}$ as followed:

$$dB(t,k) = R(t)B(t,k)dt + (k-t)B(t,k)d\tilde{W}(t), 0 \le t \le k.$$

Also denote $\tilde{P}^{k+1}$ as the foward measure on the interval $[0, k+1]$ and $\tilde{W}^{k+1}$ as the corresponding Brownian motion under $\tilde{P}^{k+1}, k = 2, 3$.

(a) (5 points) Denote $L(t, k)$ as the foward LIBOR rate for investment on the time interval $[k, k+1]$. Find the dynamics of $L(t, k)$ under $\tilde{P}^{k+1}$.
Ans: From the note on LIBOR rate:
From the dynamics of the bond, we have

$$\sigma^*(t,k) = t - k.$$

Therefore,

$$\begin{aligned} dL(t,k) &= L(t,k)\frac{1+L(t,k)}{\delta L(t,k)}\left[\sigma^*(t,k+1) - \sigma^*(t,k)\right] d\tilde{W}^{k+1}(t) \\ &= -\big\{1+\delta L(t,k)\big\}\, d\tilde{W}^{k+1}(t). \end{aligned}$$

(b) (5 points) Suppose $L(0, k) = 1$ for all k. Compute $E^{\tilde{P}^{k+1}}\big(L(k,k)\big)$.
Ans: It's clear that $L(t, K)$ is a $\tilde{P}^{k+1}$ martingale. So, $E^{\tilde{P}^{k+1}}\big(L(k,k)\big) = L(0,k) = 1$.

(c) (5 points) Consider an interest rate swap initiated at time $T = 2$, paying a fixed rate K and receiving backset LIBOR $L(k, k)$ for $k = 2, 3$ at time $k = 3, k = 4$. Find the value $V^K(t)$ of this swap at time t, for $t \le 2$.
Ans: By risk neutral pricing

$$\begin{aligned} V^K(t) &= \tilde{E}\Big[D(3)[L(2,2)-K] + D(4)[L(3,3)-K]|\mathcal{F}(t)\Big] \\ &= B(t,3)\tilde{E}^3\Big[L(2,2)-K|\mathcal{F}(t)\Big] + B(t,4)\tilde{E}^4\Big[L(3,3)-K|\mathcal{F}(t)\Big] \\ &= B(t,3)L(t,2) + B(t,4)L(t,3) - K\big[B(t,3)+B(t,4)\big]. \end{aligned}$$

(d) (5 points) For $0 \le t \le 2$, find $R_2^4(t)$, the forward swap rate at t, defined as the value of the fixed rate K such that the value of the above swap at time t is 0 (that is $V^{R_2^4(t)}(t) = 0$).
Ans: From the previous part: $V^K(t) = 0$ if

$$K = R_2^4(t) = \frac{B(t,3)L(t,2) + B(t,4)L(t,3)}{B(t,3)+B(t,4)}$$

(e) (5 points) Consider a $2 \times (4-2)$ swaption at strike K, which gives the holder the right, but not the obligation to enter the above swap at time $T = 2$ with a fixed rate K. Show that the pay off of this swaption at time 2 is

$$\Big(B(2,3) + B(2,4)\Big)\Big(R_2^4(2) - K\Big)^+.$$

Ans:
The pay off of this swaption at time 2 is

$$\begin{aligned}
[V^K(2)]^+ &= [V^K(2) - V^{R_2^4(2)}(2)]^+ \\
&= \Big\{B(2,3)L(2,2) + B(2,4)L(2,3) - K\big[B(2,3) + B(2,4)\big] \\
&\quad -\Big(B(2,3)L(2,2) + B(2,4)L(2,3) - R_2^4(2)\big[B(2,3) + B(2,4)\big]\Big)\Big\}^+ \\
&= \Big(B(2,3) + B(2,4)\Big)\Big(R_2^4(2) - K\Big)^+.
\end{aligned}$$

4. Let $N(t)$ be a Poisson process with rate λ under the risk neutral measure $\tilde{P}$. We specify the dynamics of the short rate $R(t)$ as followed:

$$dR(t) = \alpha R(t)dt + \sigma dN(t),$$

where α is *constant* and $R(0) = r$, a constant.

(a) (20 points) Find an explicit solution for $R(t)$.
Ans: From Homework 2 Problem 6 ii:

$$\begin{aligned}
R(t) &= re^{\alpha t} + \sum_{0<u\le t} e^{\alpha(t-u)}\sigma \Delta N(u) \\
&= re^{\alpha t} + \int_0^t e^{\alpha(t-u)}\sigma dN(u)
\end{aligned}$$

(b) (10 points) Compute $\tilde{E}(R(t))$.
Ans:
Since $e^{\alpha(t-u)}$ is a continuous function of u,

$$\tilde{E}\Big[e^{\alpha(t-u)}d(N(u) - \lambda u)\Big] = 0.$$

Therefore,

$$\begin{aligned}
\tilde{E}\Big[\int_0^t e^{\alpha(t-u)}\sigma dN(u)\Big] &= \sigma\tilde{E}\Big[\int_0^t e^{\alpha(t-u)}\lambda du\Big] \\
&= \frac{1}{\alpha}\sigma\lambda e^{\alpha t}(1 - e^{-\alpha t}) = \frac{1}{\alpha}\sigma\lambda(e^{\alpha t} - 1).
\end{aligned}$$

Thus,

$$\tilde{E}(R(t)) = re^{\alpha t} + \frac{1}{\alpha}\sigma\lambda(e^{\alpha t} - 1).$$

5. Let $S(t)$ be the stock price denominated in Euro and $N^f(t)$ be the price of the Euro money market denominated in dollars. Their dynamics under the US domestic risk neutral measure $\tilde{P}$ is given as followed:

$$\begin{aligned} dS(t) &= rS(t)dt + \sigma_1 S(t)d\tilde{W}(t) + S(t-)d(N(t) - \lambda t) \\ dN^f(t) &= rN^f(t)dt + \sigma_2 N^f(t-)d(N(t) - \lambda t), \end{aligned}$$

where under $\tilde{P}$, $\tilde{W}$ is a Brownian motion, $N(t)$ is a Poisson process with rate λ, $N(t)$ and $\tilde{W}$ are independent. r, σ_1, σ_2 are constants, r is the domestic interest rate.

(a) (5 points) Find the explicit solutions for $S(t), N^f(t)$.

Ans: From the lecture note on Chapter 11

$$S(t) = S(0)\exp\Big[(r - \lambda - \frac{1}{2}\sigma_1^2)t + \sigma_1\tilde{W}(t) + N(t)\log 2\Big]$$
$$N^f(t) = N^f(0)\exp\Big[(r - \lambda\sigma_2)t + N(t)\log(1+\sigma_2)\Big].$$

(b) (10 points) Define the foreign risk neutral measure as followed

$$\tilde{P}^{N^f}(A) = \tilde{E}\Big(\mathbf{1}_A \frac{D(T)N^f(T)}{N^f(0)}\Big).$$

Recall also the following: if we define a new measure $\tilde{P}^Z$ by

$$\tilde{P}^Z(A) = \tilde{E}\Big(\mathbf{1}_A Z(T)\Big),$$

where

$$Z(T) = \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(T)} e^{-(\tilde{\lambda}-\lambda)T},$$

then $N(t)$ is a Poisson process with rate $\tilde{\lambda}$ under $\tilde{P}^Z$. Use this fact to show that $N(t)$ is a Possion process with rate $\lambda(1+\sigma_2)$ under $\tilde{P}^{N^f}$.

Ans:

$$\begin{aligned} \frac{D(T)N^f(T)}{N^f(0)} &= \exp\Big[-\lambda\sigma_2 T + N(T)\log(1+\sigma_2)\Big] \\ &= \exp\Big[\big\{\lambda - \lambda(1+\sigma_2)\big\}T + N(T)\big\{\log(\lambda(1+\sigma_2)) - \log\lambda\big\}\Big] \\ &= \left(\frac{\tilde{\lambda}}{\lambda}\right)^{N(T)} e^{-(\tilde{\lambda}-\lambda)T}, \end{aligned}$$

where $\tilde{\lambda} = \lambda(1+\sigma_2)$. Thus by the change of measure result, $N(t)$ has rae $\tilde{\lambda}$ under $\tilde{P}^{N^f}$.

(c) (5 points) Find the dynamics of $S^{N^f}(t) := \frac{S(t)}{N^f(t)}$ under $\tilde{P}^{N^f}$ and show that it is a martingale under $\tilde{P}^{N^f}$.

Ans:

$$\begin{aligned} S^{N^f}(t) &= \frac{S(0)}{N^f(0)} \exp\left[(-\lambda(1-\sigma_2) - \frac{1}{2}\sigma_1^2)t + \sigma_1 \tilde{W}(t) + N(t) \log \frac{2}{1+\sigma_2}\right] \\ &= S^{N^f}(0) \exp\left[(-\tilde{\lambda}\frac{1-\sigma_2}{1+\sigma_2} - \frac{1}{2}\sigma_1^2)t + \sigma_1 \tilde{W}(t) + N(t) \log \left(1 + \frac{1-\sigma_2}{1+\sigma_2}\right)\right]. \end{aligned}$$

Therefore,

$$dS^{N^f}(t) = \frac{1-\sigma_2}{1+\sigma_2} S^{N^f}(t-) d(N(t) - \tilde{\lambda})t + \sigma_1 S^{N^f}(t) d\tilde{W}(t)$$

is a martingale under $\tilde{P}^{N^f}(t)$.