Math 632
Fall 2019
Midterm exam
10/23/19

Name (Print): ____________________

This exam contains 8 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.

Problem	Points	Score
1	25	
2	20	
3	10	
4	15	
5	10	
6	10	
7	15	
Total:	105	

1. Suppose that the risk-free rate is 3 % per annum (continuous compouding). Consider a company who issued a 1 year bond with face value \$ 100 and coupon rate 4 % per year (semi-annual coupon). The 1 year bond yield is 5 %.

   (a) (5 points) Find the current price of the 1 year bond.
   Ans: $2 \times e^{-0.05/2} + 102 \times e^{-0.05/2} \approx 98.98$

   (b) (5 points) Suppose the recovery rate is 50 %. Find a rough estimate of the hazard rate of the company during the first year.
   Ans: $\lambda(1) = \bar{\lambda}(1) \approx \frac{s(1)}{1-R} = \frac{.05-.03}{.5} = 4\%$.

   (c) (5 points) What is a rough estimate of the probability that the company will default somewhere within the 3rd and 4th month?
   Ans: $P(\frac{3}{12} \leq \tau \leq \frac{4}{12}) = e^{-\frac{3}{12}\lambda_1} - e^{-\frac{4}{12}\lambda_1} \approx 0.003294$.

   (d) (5 points) Write down a precise equation that allows us to determine the 1st year hazard rate, λ_1. You do not have to solve for λ_1.
   Ans:

$$\begin{aligned} 98.98 &= P(\tau \leq 1/2)RFe^{-r/4} + P(\tau > 1/2)\frac{c}{2}e^{-r/2} \\ &+ P(1/2 < \tau \leq 1)RFe^{-3r/4} + P(\tau > 1)(c/2 + F)e^{-r}, \end{aligned}$$

   where

$$P(\tau \leq x) = 1 - e^{-\lambda_1 x}, 0 \leq x \leq 1.$$

   (e) (5 points) The company also issue a 2 year bond with the same specifications on face value and coupon rate. Its yield is 5.5 %. Estimate the probability that the company defaults somewhere between the 9th month of the first year and the 2nd month of the 2nd year.

Ans: $\bar{\lambda}(2) \approx \frac{s(2)}{1-R} = \frac{.055-.03}{.5} = 5\%$. Thus $\lambda(2) = 2\bar{\lambda}(2) - \lambda_1 = 0.06$. Now

$$\begin{aligned} P(9/12 \leq \tau \leq 14/12) &= P(\tau \leq 14/12) - P(\tau \leq 9/12) \\ &= e^{-9/12\lambda_1} - e^{-\lambda_1 - \frac{2}{12}\lambda_2}. \end{aligned}$$

2. A portfolio consists of 100 bonds of the same rating whose hazard rate in the first year is 0.2 and in the 2nd year is 0.25.

   (a) (5 points) Find the probability that any single bond will default between year 1 and year 2.

   Ans: $P(1 \le \tau < 2) = e^{-0.2} - e^{-0.2-0.25} \approx 0.1811$.

   (b) (5 points) Find the portfolio 99 % - 2 year credit VaR if the bond default times all have correaltions 1.

   Ans: $P(\tau < 2) = 1 - e^{-0.2-0.25} = 0.36$.

   The 99 % - 2 year credit VaR is 100 % since the probability that no bond defaults is $64\% < 99\%$.

   (c) (5 points) Find the portfolio 99% - 2 year credit VaR if the bond default times are independent.

   Ans: Using the CLT approximation: $np + 2.32 \times \sqrt{np(1-p)} \approx 47.164\%$, where $p = 0.36, n = 100$.

   (d) (5 points) Find the approximate portfolio 99% - 1 year credit VaR if the default times all have correlations $\rho = 0.4$.

Ans: Using Vasicek formula:

$$VaR \approx N(\frac{N^{-1}(Q(1)) + \sqrt{\rho}N^{-1}(1-\alpha)}{\sqrt{1-\rho}}),$$

where $Q(1) = 1 - e^{-0.2} \approx 0.18$. Plug in we have

$$VaR \approx N(0.71) \approx 76.11\%.$$

3. (a) (5 points) Fill in the 2nd column of this table

Age	Probability of death within 1 year	Survival probability
65	0.015	0.65
66	0.017	0.64025
67	0.02	0.6294
68	0.023	0.6168

(b) (5 points) Compute the premium that must be paid for a person at age 65 for a two year term life insurance with pay off 100,000 USD. The premium is paid at the beginning of the year, the interest rate is 3 % per annum and compounded semiannually.

$$
\begin{aligned}
\text{Premium present expected payment} &= X + \frac{X}{(1+\frac{r}{2})^2} P(\tau \geq 66 | \tau \geq 65) \\
&= X + \frac{X}{(1.015)^2}(1 - 0.015) \\
\text{Insurance present expected payment} &= \frac{1}{1+r/2} 10^5 P(\tau \leq 66 | \tau \geq 65) \\
&+ \frac{1}{(1+r/2)^3} 10^5 P(66 \leq \tau \leq 67 | \tau \geq 65) \\
&= 10^5 \left(\frac{0.015}{1.015} + \frac{0.64025 - 0.62936}{0.65} \frac{1}{(1.015)^3}\right) \\
&\approx 3080.
\end{aligned}
$$

Finally solving for X gives $X \approx 1574.5$

4. (a) (5 points) Below are the distribution of monthly returns of two companies:

Company A:

Return	-3 %	-2 %	-1 %	0 %	1 %	5 %
Probability	.7 %	.2 %	.1 %	50 %	48 %	1 %

Company B:

Return	-4 %	-1 %	0 %	1 %	5 %
Probability	.4 %	.6 %	50 %	48 %	1 %

Find the 99% monthly VaR and 99% monthly expected short fall of both companies. Which one is "riskier" and why?

Both companies' 99% monthly VaR is - 1 %.

$$\begin{aligned} ES(A) &= \frac{-3\% \times .7\% + -2\% \times .2\% + -1\% \times .1\%}{1\%} = -2.6\% \\ ES(B) &= \frac{-4\% \times .4\% + -1\% \times .6\%}{1\%} = -2.2\%. \end{aligned}$$

Thus company A is riskier.

(b) (5 points) A portfolio consisting of 1000 dollars in Starbucks and 500 dollars in Wendy's stocks. Suppose the correlation between Starbucks and Wendy's returns is 0.15 and the daily volatilities of both assets are 0.5%. Find the daily 95 % VaR of the portfolio.

Ans: We have

$$\begin{aligned} \Delta P &= \frac{1000}{S_1}\Delta S_1 + \frac{500}{S_2}\Delta S_2 \\ \frac{\Delta P}{P} &= \frac{1000}{P}\frac{\Delta S_1}{S_1} + \frac{500}{P}\frac{\Delta S_2}{S_2}. \end{aligned}$$

Thus the volatility of the portfolio is

$$\begin{aligned} \sigma_P &= \sqrt{\frac{4}{9}\sigma_S^2 + 2\rho\frac{2}{3}\sigma_S\frac{1}{3}\sigma_W + \frac{1}{9}\sigma_W^2} \\ &= 0.3944\%. \end{aligned}$$

The distribution of the change of portfolio value ΔP is approximately Normal with mean 0 and standard deviation $P\sigma_P = 5.91$ Thus the 95 % daily VaR is approximately $5.91 \times 1.65 \approx 9.76$.

(c) (5 points) A portfolio of options on Amazon stocks have a delta of 45. The current Amazon stock price is 1,000 USD. If the daily volatility of Amazon is 0.35 %, estimate the 90% VaR of the portfolio.

Ans: We have

$$\Delta P = \delta \Delta S = \delta S \frac{\Delta S}{S}.$$

Thus The distribution of the change of portfolio value ΔP is approximately Normal with mean 0 and standard deviation $\delta S \sigma_A = 45 \times 1000 \times 0.0035 = 157.5$. Thus the 90 % daily VaR is approximately $157.5 \times 1.28 \approx 201.6$.

5. Suppose the diagonalization of the covariance matrix of returns on Apple, Starbucks and Tesla over a period has the form $\Sigma = PDP^T$ where

$$P = \begin{bmatrix} 0.4082 & 0.7071 & 0.5774 \\ 0.4082 & -0.7071 & 0.5774 \\ 0.8165 & 0 & 0.5774 \end{bmatrix}, D = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}.$$

A portfolio worth 100,000 dollars has the following exposure to the stocks with regard to 1% move in the returns:

Name	Apple	Starbucks	Tesla
Change	- 5000	+ 2000	- 2000

(a) (5 points) Find the factor loadings of the first two most important factors in this portfolio. What are their variances? What percentage of variation in the portfolio do they explain?

Ans: They are the first and third eigenvectors:

$$\begin{bmatrix} 0.4082 \\ 0.4082 \\ 0.8165 \end{bmatrix}, \begin{bmatrix} 0.5774 \\ 0.5774 \\ 0.5774 \end{bmatrix}.$$

They explain $(.3 + .5)/(.3 + .2 + .5) = 80\%$ of the variations.

(b) (5 points) Find the portfolio exposure to the first two factors.

Ans: The exposure to the first factor is

$$\begin{bmatrix} -5000 & +2000 & -2000 \end{bmatrix} \begin{bmatrix} 0.4082 \\ 0.4082 \\ 0.8165 \end{bmatrix} = -2857.6 \text{ (per 1 percentage change in returns)}$$

The exposure to the third factor is

$$\begin{bmatrix} -5000 & +2000 & -2000 \end{bmatrix} \begin{bmatrix} 0.5774 \\ 0.5774 \\ 0.5774 \end{bmatrix} = -4949.7 \text{ (per 1 percentage change in returns)} .$$

6. Suppose a 1 year bond that pays coupon with rate 4 % with semiannual coupon and face value 100 dollars is currently selling for 98 dollars. Also suppose the current risk free rate is 6 % per annum, compound continuously.

   (a) (5 points) Find the yield and the yield spread of the bond.

   Ans:

$$98 = 2e^{-y/2} + 102e^{-y}.$$

   Letting $u = e^{-y/2}$ we have $102u^2 + 2u - 98 = 0$, which gives $u \approx= .9704$. Hence the yield $y \approx 6\%$ and the spread $s \approx 0$.

   (b) (5 points) Find the asset swap spread where one sie pays the coupon rate of the bond and the other side pays LIBOR + spread. Verify that this spread is similar to the one you found in part a.

   Ans: The present value of the coupon paying side is

$$2 + 2(e^{-r/2} + e^{-r}).$$

   The present value of the LIBOR + spread paying side is

$$\frac{s}{2}(e^{-r/2} + e^{-r}) + 100(1 - e^{-r}).$$

   Equating these two and sovle for s gives $s \approx 0$.

7. (a) (5 points) Explain the α and β in the CAPM model.

   Ans: α is the constant term and β is the slope when we run the regression of portfolio's excess returns over the market excess returns.

   (b) (5 points) Suppose the risk free rate was 3 % and the return from the market last year was 6 %. A fund manager has a portfolio with beta of 1.4 and a return of 7 % from last year. What is the alpha of this portfolio?

   Ans:

$$R_p - 3 = \alpha + \beta(R_m - 3).$$

   That is

$$4 = \alpha + 1.4 \times 3.$$

   Thus $\alpha = -0.2\%$.

   (c) (5 points) Supposse the volatility of market portfolio is 30 % per annum. What is the volatility per annum of the portfolio?

   Ans: From the equation

$$R_p - R_f = \alpha + \beta(R_m - R_f) + \epsilon$$

   we have

$$Cov(R_p, R_m) = \beta Var(R_m).$$

   Thus

$$\begin{aligned} \rho(R_p, R_m)\sigma_p &= \beta\sigma_m \\ \sigma_p &= \frac{\beta\sigma_m}{\rho(R_p, R_m)}. \end{aligned}$$

   If we know the correlation between the portfolio returns and the market returns we can calculate its volatility.