1 Chapter 1

Exchanged traded markets: Chicago Board of Trade (CBOT), Chicago Mercentile Exchange (CME), Chicago Board Options Exchange (CBOE), New York Mercentile Exchange, the commodity exchange (COMEX).

OTC market : Central counter party (CCP), regulated after 2008. OTC derivatives must be traded on a swap execution facilities (SEFs), CCP used for most transactions, trades reported to central registry.

Market size: OTC much bigger than ETM : 632.6 trillions vs 52.6 trillions by 2012.

Forward contracts: popular on foreign exchange, traded in OTC market.

Futures contracts : largest exchanges are CBOT, CME (merged to become CME group).

Options: largest exchange is CBOE.

Types of traders : hedgers, speculators, arbitrageurs. Hedge fund employs all types.

Hedge funds: relatively free of regulations. Investment strategy often involves speculative or arbitrage position. Some examples : long / short equities, convertible arbitrage, distressed securities, emerging markets, global macro, merger arbitrage.

Stories : SocGen's Big Loss in 2008 : Jerome Kerviel, arbitrageur. Similarly, Nick Leeson in 1990s, Baring Banks, also arbitrageur (on Nikkei 225).

Hedge using forward contracts, hedge using options. Difference: forward contracts are designed to fix price that the hedger will have to pay or receive for the underlying asset. Option provide insurance.

Speculate using futures, speculate using options. Futures : potential loss / gain is very large. Options: limited by the amount paid for the options. Difference between speculate using futures and buying in the spot market: futures requires very small up front investment (margin account).

2 Chapter 2

CME groups : CBOT, CME, NYME

Closing out futures contract: by taking the opposite position.

Price limits: limit up, limit down : trading ceases for the day once the contract is limit up or down. Position limit : max number of contracts a speculator may hold.

Margin account: Change in future price results in change in balance of margin account. Interest rate is not taken into account ? Maintainence margin : to ensure the balance in the margin account never becomes negative. Margin call : top up the margin account to initial level by the end of next day. Extra funds deposited is known as variation margin. Securities can be deposited at a discount of face value : hair cut (the percentage of discount). Treasury bills: 90 % face value, shares: 50 % face value.

Clearing house: an intermediary in futures transactions. OTC markets: requires CCPs. Bilateral clearing : requires collateral, similar to margin accounts. Collateral can be risky when leverage is high : Long-Term Capital Management. Convergence arbitrage : buy less liquid bond and short more liquid bond from the same company, waiting for price convergence. Has to change collateral when interest rates move. Usually the same since both bonds move in the same direction in price. When there's flight to quality, has to post collateral on both since the price of less liquid goes down and the price more liquid (short) goes up.

Futures trades vs OTC trades : futures trades settle daily. So daily variation margin does not earn interest (only initial margin when provided in cash earns interst). OTC trades do not settle daily. So daily variation margin provided by member of a CCP in OTC trades earns interest when in cash.

Settlement price : price used for calculating daily gains and losses and margin requirements. Price at which the contract traded immediately before the end of day's trading session.

Trading volume: number of contracts traded in a day. Open interest : number of contracts outstanding. Trading volume can be higher than beginning or end of day open interst if there is a large amount of trading by day traders.

Normal market: future prices increase as function of maturity. Inverted market: the opposite.

Delivery : Cash settlement - ex: stokc indices. Final settlement price is the spot price of the underlying asset at either open or closed of trading of a predetermined day. Ex: SP 500 predetermined day is third Friday of the delivery month and final settlement is at the opening price.

Types of traders : scalpers - very short term trends, only a few minutes positions. Day traders : less than a day positions. Positio traders : much longer. Hope to make significant profits from major market movements.

Types of orders : Market order : carried out immediately at best price available.

Limit order: executed only at a particular price or one more favorable. Stop order : Order to sell at a particular price (stop loss). Market if touched (MIT) order: Order to sell at a favorable price to gain profit (?) Discretionary order : execution may be delayed at the broker's discretion. Time of day order : specifies a particular period of time during the day for execution. Open order or good till canceled order : in effect until executed or until end of trading in a contract. Fill or kill order : executed immediately or not at all.

Regulations: Futures markets in the US are regulated by the Commodity Futures Trading Commission (CFTC).

Trading irregularities : Corner the market : investor takes huge long futures position. Does not close out position, number of outstanding futures contracts may exceed number of commodity available for delivery. Large rise in both futures and spot prices.

Forward vs Futures contracts : Gain / loss in forward only realized at maturity. Futures : day by day. FX quotes : futures where one currency is in US are always quoted in US currency. Forwards are always quoted in the same ways as spot prices. Ex : CAD. Futures pirce quote 0.95 USD per CAD corresponds to forward price quote 1.0526 CAD per USD.

3 Chapter 3

Hedging using futures : Gain / loss from asset is offset by loss / gain from futures position. Note : this appears as a locked price effect, price is not actually locked (futures position is closed out before expiration, asset purchased / sold with usual suppliers / customers). Futures is usually bought with expiry (immediately) after the delivery month since future price during the delivery month is quite erratic.

Hedging may lead to worse outcomes in case price moves in favorable directions : hedging needs to be annouced to shareholders etc. No hedging leads to constant profit margin !

Basis : Spot price of asset to be hedged - Futures price of contract used.

Basis equation : $S_2 + F_1 - F_2 = F_1 + b_2$ where S_i is spot price and F_i is future price at times $i = 1, 2$. This is the effective price obtained for a short futures position and the effective price paid for a long futures position. In English : effective price equals final spot price plus gain on futures or initial future price plus final basis. Therefore, a basis increase (strengthening) is favorable for short and unfavorable for long position (and vice versa for basis decrease / weakening).

Basis risk : asset to be hedge is not exactly the same as futures underlying (jet fuel vs heating oil) a.k.a cross hedging, delivery date uncertainty, futures closed out before delivery month (liquidity issue in getting the right futures).

Optimal number of contracts in cross hedging : $N^* = \frac{h^* Q_A}{Q_F}$, where Q_A is the size of position (units) being hedged (A for asset, Q_F size of one futures contract (units) and $N*$ is the optimal number of futures contracts for hedging. We also have $h^* = \rho \frac{\sigma_S}{\sigma_F}$ where $\sigma_S = sd(\Delta S), \sigma_F = sd(\Delta F), \rho = cor(\Delta S, \Delta F)$. Choosing $N*$ this way will minimize the variance of the position. Reason:

$$\begin{aligned} Var(position) &= Var(Q_A \Delta S - Q_F \Delta F) \\ &= Var(Q_A \Delta S - h^* Q_A \Delta F) = Q_A^2 Var(\Delta S - h^* \Delta F). \end{aligned}$$

We are done if $Var(\Delta S - h^* \Delta F) = \mathbb{E}(\Delta S - h^* \Delta F)^2$ because this reduces to the least squared solution of a linear regression. This is from the fact that $\Delta S = 0$ usually imples $\Delta F = 0$ and vice versa (that is the intercept of the linear regression line is 0).

Practical issue : $\Delta S, \Delta F$ estimated from historical prices. Length of Δ should be similar to time to delivery (i.e. one month to delivery then we sample 15 monthly change in spot and future prices from history). This applies for forward contracts. For futures, one can also look at day to day percentage change and adjust the optimal positions accordingly. In practice, this day to day change is usually small and ignored.

Hedging an equity portfolio : $N^* = \beta \frac{V_A}{V_F}$, where V_A is the current value of portfolio, V_F is the current value of one futures contract. β (CAPM) is the slope of the best fit line obtained from regressing the excess return of the portfolio over the excess return of the index (both over the risk free rate). For example, a portfolio with a β of 2.0 is twice as sensitive to index price movement. It is therefore necessaray to use twice as many contracts to hedge the portfolio. In a perfect hedge (see Table 3.4) the return on the portfolio is approximately the risk free rate.

Reasons for hedging : Short term protection in uncertain market, confidence in the portfolio outperforming the market (locking in the benefits of stock picking), changing the beta of a portfolio. To change the beta of a portfolio from β to β^*, a position in $(\beta - \beta^*) \frac{V_A}{V_F}$ is required. It is a short position if $beta > \beta^*$ and a long position otherwise. To lock in the benefits of stock picking, it is essentially similar to betting that the beta estimate is not precise (that is the movement is always beneficial to the stock holder. If the stock falls, it won't fall as much as predicted versus market index, if the stock rises, it will rise more than predicted versus market index. See page 64 for more details).

Stacks and rolls : the act of entering a serires of short term futures to hedge a long term delivery date. Liquidity of both the futures and cash flow must be taken into account (Business Snapshot 3.2). Initial entering into a contract is referred as "stacks". After that the investor closes out the contracts and "roll" them into new ones. When future prices are below spot prices, we cannot expect a perfect hedge for price decline (see section 3.6). For example, roll 3 times, dollar gain per barrel of oil from the short futures contracts is

$$(88.20 - 87.40) + (87.00 - 86.50) + (86.30 - 85.90) = 1.70$$

Price decline is 89 to 86, which is 3 dollars. Partially we see the loss in the gap of closing price and rolling in price (87.4, 87) also (86.50, 86.30). Also keeping in mind that spot starts out at 89 and future was at 88.20.

4 Chapter 4

Basis point: 0.01 % per annum.

Types of rates: Treasury rate : rates earned from Treasury bills and Treasury bonds. Risk free. Borrowing period ? LIBOR : an estimate of short term (1 year or less) **unsecured** borrowing rate for a AA-rated financial institution. 15 borrowing periods, including overnight rate. Effective Fed Funds Rate : (Only) Overnight rate for interbank borrowing / lending. Overnight LIBOR is usually 6 % higher than effective fed funds rate : reflection of the borrowing pools and time difference. Repo (repurchase agreement) rates: **secured** borrowing rate by posting collaterals (usually securities). Most common : overnight repo. Also have terms repo for longer terms. Question: risk free rate?

Compounding : Interest rate is given as R percent per annum. If the rate is compounded m times per annum, the terminal value of the investment after n years is

$$V_n = A(1 + \frac{R}{m})^{mn}.$$

Continuous compounding :

$$V_n = Ae^{Rn}.$$

Conversion formula between discrete and continuous compounding :

$$R_m = m(e^{R_c/m} - 1).$$

These are two equivalent rates that give the same terminal value of the investment.

Zero rate : Rate earned from an investment of zero-coupon bond for n years. Denote $R(m, m+n)$ as the risk-free rate for an investment starting at year m and ending after n years (n can be a fraction of a year, such as 0.5) . The structure of $R(0, n)$ as a function of n is referred to as the term structure. It is called upward sloping if it is increasing and downward sloping if it is decreasing (as a function of n). The shape of term structure changes. It may be upward sloping for some n and downward sloping for some other n (see below for more discussion). $R(0, n)$ is deduced by current price of zero-coupon bond expiring in n years.

Pricing coupon paying bond : Most popular bond pays semiannually. Suppose we have a 2 year Treasury bond with a principal of 100 with a semi annual coupon rate of 6 %. Its coupon payment is $100 \times 0.06 \times 0.5 = 3$ dollars. Its current price is

$$V_0 = 3e^{-R(0,0.5)\times 0.5} + 3e^{-R(0,1)\times 1} + 3e^{-R(0,1.5)\times 1.5} + 103e^{-R(0,2)\times 2}.$$

The bond yield y is the single value that we can plug into the above equation in place of $R(0, m)$ to yield the same V_0 :

$$V_0 = 3e^{-y\times 0.5} + 3e^{-y\times 1} + 3e^{-y\times 1.5} + 103e^{-y\times 2}.$$

Bond yield does not have an explicit solution. Can be viewed as a "constant" (with respect to the yield curve) rate earned for the duration of the bond.

The par yield c is the coupon rate that we can plug $\frac{c}{m}$ (m is the coupon frequency per annum) into the above equation in place of the coupon rate 3 to yield the principal value (which is 100) :

$$100 = \frac{c}{2}e^{-R(0,0.5)\times 0.5} + \frac{c}{2}e^{-R(0,1)\times 1} + \frac{c}{2}e^{-R(0,1.5)\times 1.5} + (100 + \frac{c}{2})e^{-R(0,2)\times 2}.$$

Par yield c satisfies

$$100 = A\frac{c}{m} + 100d,$$

where d is the present value of 1 dollar receive at the maturity of the bond (so in effect a discount factor, $P(0, T)$ where P is the zero coupon bond price). And

$$A = e^{-R(0,0.5)\times 0.5} + e^{-R(0,1)\times 1} + e^{-R(0,1.5)\times 1.5} + e^{-R(0,2)\times 2}.$$

This equation shows that the par yield can be viewed as the "compensated income stream" (subject to discount) so that 100 dollars at the future expiry is equivalent to 100 dollars today.

Observation: Bond yield and par yield are close ? One can compare the actual coupon rate with the par yield to determine the credit risk of the bond ? For example the Treasury bill coupon rate in example 4.4 is less than the par yield. This reflects the risk free nature of the Treasury bill ?

Determining treazury zero rates : Done straighforwardly if we know the price of zero coupon bond for the same maturity. If we have a mixed zero coupon and non-zero coupon bond: bootstrapping : use the earlier zero rates to calculate the present value of the income stream as the above equation with the final rate $R(0, n)$ as the unknown to solve for. Interpolating between bond price data is sometimes called for when exact maturity date is not available. Ex: 2.3 year bond with 6% coupon sells for 98 and 2.7 year bond with 6.5 % coupon sells for 99. Then a 2.5 year bond with 6.25 % would sell for 98.5. A chart of zero rate as a function of matuirty is referred to as the zero curve.

Forward interest rate :

$$R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}.$$

This comes from $e^{R_1 T_1} e^{R_F (T_2 - T_1)} = e^{R_2 T_2}$. In the interest rate swap valuation, the future LIBOR $L(T_1, T_2)$ rate can be assumed to equal R_F for the purpose of calculating the present value of the floating leg cash flow. This is actually the present value of the future LIBOR payment, which can be showed rigorously to be the same as using R_F as a payment.

Pushing T_2 toward T_1 we have $R_F = R + T \frac{\partial R}{\partial T}$, where R is the interst rate for maturity T. R_F in this case is known as the instantaneous (over night ?) forward rate for a maturity of T (the rate is available at time T).

Viewing R as a function of T ($R = R(0, T)$) and $P(0, T) = e^{-RT}$ as the zero coupon bond price, we also have

$$R_F = -\frac{\partial}{\partial T} \log P(0, T).$$

Forward rate agreement (FRA) : OTC transaction to fix rate with the underlying being LIBOR and compounding is discrete. If the agreed fixed rate is greater than the actual LIBOR for the period, the borrower pays the lender the difference applied to principle and vice versa. More than one FRA rolled over is called a interest rate swap, see chapter 7.

Concretely, suppose two companies X, Y enter into a FRA in which X agrees to lend money to Y at a fixed rate R_K between T_1 and T_2. The cash flow to X at time

T_2 is

$$L(R_K - R_M)(T_2 - T_1),$$

where L is the principal. The cash flow of Y at time T_2 is the negative of this amount. Typically $FRAs$ are settled at time T_1 (since all rates are known at T_1) and thus the pay off for X at time T_1 is

$$\frac{L(R_K - R_M)(T_2 - T_1)}{1 + R_M(T_2 - T_1)},$$

where R_M is the LIBOR rate available at time T_1 for borrowing during the period $T_1 T_2$. The pay off for Y is the negative of this amount.

Forward LIBOR rate : The current value of a FRA is usually non-zero. The forward LIBOR rate R_F is the FRA fix rate such that the current value of a FRA is zero (the same idea as the forward price). Forward LIBOR rate can be used to calculate the MTM (marked to market) value (a fancy way to refer to market value of a derivative at a particular time) of a FRA. The idea is to use present value evaluation :

$$\begin{aligned} V_0 &= \tilde{\mathbb{E}}\Big(e^{-R_2T_2} L(R_K - R_M)(T_2 - T_1))\Big) \\ &= e^{-R_2T_2}\tilde{\mathbb{E}}\Big(L(R_K - R_F)(T_2 - T_1)) + L(R_F - R_M)(T_2 - T_1))\Big) \\ &= e^{-R_2T_2} L(R_K - R_F)(T_2 - T_1), \end{aligned}$$

where we have used the fact that R_K, R_F are known at time 0 and by definition

$$0 = \tilde{\mathbb{E}}\Big(e^{-R_2T_2} L(R_F - R_M)(T_2 - T_1))\Big).$$

From Hull : this calculation has the assumption that the forward rates are realized (that is $R_M = R_F$) ?

Duration : The duration of a bond is a weighted average of the times when the payments are made, with the weight at time t_i equal to the ratio of the discounted (using the bond yield) cash flow at time t_i and the bond's total present value. In formula:

$$D = \sum_{i=1}^{n} t_i \Big[\frac{c_i e^{-yt_i}}{B}\Big] = \frac{-1}{B}\frac{dB}{dy}.$$

The duration has the property that

$$\frac{\Delta B}{B} = -D\Delta y.$$

Thus it is the first derivative of the percentage bond price change with respect to the yield. If we want just price change, the concept is dollar duration :

$$\Delta B = -D_{\$}\Delta y.$$

Most common is $DV01$ which is the price change from a 1 basis point increase in all rates. Gamma is the change in DV01 from a 1 basis point increase in all rates.

The above formula for continuous compounding. For discrete compounding, we have a modified duration : $D^* = \frac{D}{1+\frac{y}{m}}$ and

$$\frac{\Delta B}{B} = -D^*\Delta y.$$

Duration of a bond portfolio : Weighted average of the duration of individual bonds, weights proportional to bond price. Assumption : parallel shift in zero yield curve (yields of all bonds change by approsimately the same amount). So a zero net duration portfolio is immune to small parallel shift in the yield curve. It is not immune to large or non parallel shift.

Convexity : Second order approximation of percentage bond price change. In formula:

$$C = \frac{\sum_{i=1}^{n} t_i^2 c_i e^{-yt_i}}{B} = \frac{1}{B}\frac{d^2B}{dy^2}.$$

From Taylor series approximation

$$\Delta B = \frac{dB}{dy}\Delta y + \frac{1}{2}\frac{d^2B}{dy^2}(\Delta y)^2.$$

Thus

$$\frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C(\Delta y)^2.$$

For a portfolio with a particular duration, the convexity is largest when it provides payments evenly over a long period, smallest when payments are concentrated around one particular period (this comes from the concept of duration as the average time to receive payments and thus convexity is the change in this average time). See also Figure 4.2, two portfolios having the same duration (since its slope at 0 are the same). Also note that it goes through quadrant II and IV (simply from relation of $\frac{\Delta B}{B}$ and Δy.

Shape of term structure : liquidity preference theory. Investor likes to deposit for short periods of time (more flexibility, shorter fund tie up period). Borrower likes to fix borrowing rate for a long period of time (less risk of rate fluctuation). Thus we usually see a higher deposit rate of long term deposit and higher borrowing rate for long term borrowing as a simple result of supply / demand. In the example of Table 4.7 : the bank uses the deposit to finance the mortgage loan. Thus it pays the deposit rate and receives the mortgage rate. If most people deposit for short term rate and borrow at long term rate, the bank loses money if short rate rises in the future (since it has to pay out more for deposit and receives the same payment for its mortgage loan). Thus the upward slope term structure (provided by the bank of course) also reflects the bank "hedging" this risk. Note that if short rate falls then the bank gains money so it doesn't have to worry about this scenario.

5 Chapter 5:

Assumptions and notations (see 5.2, 5.3)

Forward price on investment asset that provides no income (stock, zero coupon bond) : $F_0 = S_0 e^{rT}$. Derived via no arbitrage argument. The higher forward price can be viewed as the cost of financing the purchase of the asset during the life of the forward contract (for the short seller?). The no arbitrage argument works even if short selling the investment asset is not possible, since one can short sell or long the forward contract instead. This may be the advantage of futures : one can short or long futures more easily than the asset itself.

Forward price on investment asset that provides income (coupon bond) : $F_0 = (S_0 - I)e^{rT}$, where I is the present value of the income stream during the life of the forward contract. The income stream is subtracted from S_0 since the long position does not receive the income stream (the asset is delivered at time T) and the short position receives the income stream. Again can be derived via no arbitrage argument. $S_0 - I$ is the amount borrowed from the bank to finance the short position consisting of a delivery of S at time T and and a long position to provide an income stream worth I at present value at the specified times (thus the total net value is 0 at the beginnign). The income stream closes out at during the life time of the contract. At time T receive $F_0 = (S_0 - I)e^{rT}$ to close out the position with the bank.

Forward price on investment asset that provides known yield (for example stock indices, section 5.9) : The yield is paid as a percentage of the asset price at the time it is paid. If the rate is q and it is compounded m times during the contract life time then each time the payment is $S_{t_i}\frac{q}{m}$. This is assumed to be re-invested into the asset.

Thus, we see that $S_{t_i+} = S_{t_i}(1+\frac{q}{m})$. Under continuous compounding, 1 share of S at time 0 will grow to have value of $e^{qT}S_T$ at time T. (Thus it can also be viewed as dividend by share percentage). Thus we have $F_0 = S_0e^{(r-q)T}$. The arbitrage argument runs as followed: Borrow S_0e^{-qT} from the bank to buy e^{-qT} share of the asset. This e^{-qT} share will grow to S_T at time T, which can be used to purchase 1 share of S_T for delivery and close out the position. Practical issue in stock indices futures: the underlying index may not be the value of an investment asset. Ex : Nikkei 225 Index has a dollar value of 5S, where S is in yen. This is an example of a quanto, where the underlying is measured in one currency and payoff is in another.

Price of forward contract: $V_0 = (F_0 - K)e^{-rT}$. For asset that does not have income : $V_0 = S_0 - Ke^{-rT}$. For asset with income stream with present value $I : V_0 = S_0 - I - Ke^{-rT}$. For asset that pays share dividend with rate $q : V_0 = S_0e^{-qt} - Ke^{-rT}$.

Forward price compared with futures price: Equal when short rate is consant. Futures is subject to daily settlement with the margin account. Suppose S is positively correlated with the short rate. If S increases, the investor gains since the gain from margin is invested with higher interest (compared with the initial rate r). If S decreases, the investor does not lose as much since the loss from margin can be financed with a lower rate. Thus forward price for S is lower than futures price in this case. The reverse is true if$S$ is negatively correlated with short rate. Liquidity is also another factor (futures is more liquid than forward - the market places a value on liquidity thus price of futures is higher than forward in this case?). In most cases, it is reasonable to assume that they are the same.

Futures on currencies (Forward exchange rate): $F_0 = S_0e^{r-r_f}T$, where S_0 is the current spot price in US dollars of a unit of foreign currency and F_0 is the futures price in US dollars of a unit of foreign currency at time T. The arbitrage argument is because of the equation $e^{r_fT}F_0 = S_0e^{rT}$, where the LHS is invested in the foreign market and exchanged at time T, and the RHS is exchanged at time 0 and invested in the US market. Practical observatino: If $r > r_f$ futures price increase with maturities and vice versa. See Table 5.4.

Futures on commodities with cost for storage : $F_0 = (S_0 + U)e^{rT}$, where U is the present value of all the storage costs, net of income, during the life of the contract. The arbitrage argument for the short side is to borrow $S_0 + U$ from the bank to buy a share of S and to buy a "bond" with income stream equals to the times of storage cost payments. If storage cost is incurred as proportional to the price of the commodity, it can be treateed as negative yield and thus $F_0 = S_0e^{(r+u)T}$, where u denotes the storage costs per annum as a proportion of the spot sprice net of any yield earned on asset.

Convenience yield: $F_0 e^{yT} = (S_0 + U)e^{rT}$. In this case $F_0 < (S_0 + U)e^{rT}$. Investment asset has $y = 0$. This reflects the advantage of the ownership of the asset being able to keep a production runnign and perhaps profit from temporary shortages. IT also reflects the market's expectations concerning the future availability of the commodity. The greater the possibility of shortage, the higher the yield. Also where appropriate $F_0 e^{yT} = S_0 e^{(r+u)T}$ or $F_0 = S_0 e^{(r+u-y)T}$ If futures price decreases as a function of maturity, it indicates that $r + u < y$ (Table 2.2).

Cost of carry: measures the storage cost plus interest paid to finance the asset less income earned on the asset. $F_0 = S_0 e^{(c-y)T}$. Thus for non dividend paying stock, $c = r, y = 0$. For stock index, $c = r - q, y = 0$. For a currency, $c = r - r_f, y = 0$. For a commodity that provides income at rate q and requires storage costs at rate u, $c = r - q + u$.

Futures price and expected spot price : Question is $F_0 <=> E(S_T)$? (E is not risk neutral here). Consider a speculator who longs a future contracts. He finance the (future cost of F_0) by borrowing with risk free rate. The cost today is $F_0 e^{-rT}$. The income in the future is S_T. The present value of this investment is $-F_0 e^{-rT} + E(S_T)e^{-kT}$ where k is the investor's required return on the investment, which reflects the systemic risk of the asset. All investment is priced so that the net present value is 0 : $F_0 = E(S_T)e^{(r-k)T}$. If the asset return is uncorrelated with the market, $r = k$ and thus $F_0 = E(S_T)$. If the asset return is positively correlated, $k > r$ and thus $F_0 < E(S_T)$. This is known as **normal backwardation**. If the asset return is negatively correlated, $K < r$ and thus $F_0 > E(S_T)$. This is known as **contango**. (The terms sometimes is used to compare futures price with current spot price, not expected future spot price).

An example of asset with positive systemic risk is a stock index : the expected return of an investor on the index is generally more than r. If stock provides dividend q, the expected return is more than $r - q$. Thus $F_0 = S_0 e^{(r-q)T}$ is consistent with the prediction that futures price understates the expected future stock returns for an index.

6 Chapter 6: Interest rate futures

Remark: For bond futures, there is flexibility to choose the delivery asset (cheapest bond to deliver). Determining the present value of these assets is difficult without knowing the exact term structure. The conversion factor seems to "balance" this term structure effect with the discount rate of 6 % per annum. See also example 6.2. There are also other technicalitites on coupon payments in between start and delivery dates that need to be addressed. In practice the cheapest to deliver is probably hard to know and the quoted price is decided by supply / demand ?

Problem (Ex 6.2) : Suppose that in a Treasury bond futures contract, it is **known** that the **cheapest to deliver** bond will be a 12 % coupon bond (paid semiannually) with a conversion factor of 1.6. The current quoted bond price is 115, The delivery date will take place in 270 days. The last coupon date was 60 days ago, the next coupon date is 122 days (before delivery, relevant for bond cash price) and the coupon date after is 305 days (after delivery, relevant for accrued interest to futures seller to determine futures cash price). Suppose the constant interest rate (i.e. flat term structure) is 10 % per annum. What are the cash price (the price actually paid by the purchaser of the futures) and the quoted price (the price quoted, in the same category as prior settlement price etc.) of the futures?

Remark : Quoted price of futures is affected by the conversion factor, cash price of futures is not. First, cash price of the bond (see below) is obtained by adding to the quoted price the proportion of the next coupon payment accrued to the holder. That is

$$S_0 = 115 + \frac{60}{60+122} \times 6 = 116.978.$$

The present value of the coupon of 6 received after 122 days is $I = 6e^{-0.1\times 122/365} = 5.803$. Thus the cash price of the futures is $(S_0 - I)e^{rT} = (116.978 - 5.803)e^{0.1\times 270/365} = 119.711$.

On the other hand, cash futures price = (quoted futures price $\times$ conversion factor) + accrued interest. The accrued interest (to the futures short seller, different from the accrued interest above - error in example 6.1) is $6 \times \frac{148}{148+35} = 4.85$. Thus the quoted futures price is

$$\frac{119.711 - 6 \times \frac{148}{148+35}}{1.6} = 71.79$$

Remark : Since traders will always choose cheapest bond to deliver (theoretically), any other quoted futures price will result in arbitrage.

Accrued interest : Referred to coupon adjustment method in between period. Also related to day count conventions. There are three day count conventions : Actual / Actual, 30 / 360, Actual / 360. We use Actual / Actual in this discussion as it is used by US treasury bonds. 30 / 360 is used by US corporate and municipal bonds. Note : in 30 / 360 convention, there are 3 days between Feb 28 and March 1 !. Actual / 360 is used for US money market instruments.

We skip price quotations for US treasury Bills. For US treasury bonds, there are quoted price (clean price) and cash price (dirty price, the price actually paid). The relation is : Cash price = Quoted price + Accrued interest since last coupon date. See problem above.

Conversion factor : Equal to the ratio of the quoted price of the bond (over the principal) would have on the first day of the delivery month using semiannual discrete compounding of % 6 interest rate. Example : 8 % coupon bond (semiannual) with 18 years and 4 months to maturity. We assume 18 years and 3 months to maturity. First discount back to 3 months from today gives

$$4 + \sum_{i=1}^{36} \frac{4}{1.03^i} + \frac{100}{1.03^{36}} = 125.83.$$

where .03 comes from semiannual compounding. Next discounting back to today for the 3 months is (NOT) : $125.83/(1 + .015)$. The book calculates interest rate of 3 months as $\sqrt{1.03} - 1 = .014889$, which comes from compounding 3 months twice is 6 months ? This is correct because interest is accrued. That is $(1 + r)^2 = 1.03$ is the equation to solve. Thus the present bond value is $125.83/1.014889 = 123.99$. There is an accrued interest of 2 dollars from the first 3 month period ($4 \times 3 / 6$) . Thus subtracting the accrued interest this becomes 121.99. The conversion factor is 1.2199. (See the last sentence on page 136 for explaination).

Cheapest to deliver bond : At any given time during the delivery bond, there are many bonds that can be delivered. The Treasury bond future allows the short position to choose to deliver any bond that has maturity between 15 and 25 years. The party with short position receives (Most recent futures settlement price $\times$ Conversion factor) + Accrued interest and paid for the cost of purchasing a bond at (Quoted bond price + Accrued interest). So they try to minimize (Quoted bond price + Accrued interest) - (Most recent futures settlement price $\times$ Conversion factor) + Accrued interest. These quantities clearly depend on the underlying bond (even the calculation of the most recent futures price !) so the conversion factor must be taken into account. Several other factors to choose cheapest to deliver is listed on page 138.

The 6% per annum rate in the conversion factor also affects the choice of cheapest to deliver.

Eurodollar futures: futures to lock in a LIBOR rate for a 3 month period at a delivery date (typically March, June, Sept and Dec). (Note: LIBOR rate is still given as rate per annum, but it is calculated as compounded quarterly). Quoted as 100 - R, where R is the prevailing LIBOR rate. 1 contract is 1 million dollars. Designed so that 1 basis point movement (0.01 % or 0.01 change in future price) is equivalent to 25 dollars change in a contract.

The contract price is defined as

$$10,000 \times [100 - 0.25 \times (100 - Q)]$$

where Q is the quoted futures price. For example, the contract price at futures price 99.725 is 999,312.5 and at 99.615 is 999,037.5. This corresponds to a change of 11 basis point (99.615 - 99.725 = -0.11) and 275 change in contract price.

Eurodollar locks in interest rate: Suppose current futures quote is 96.5 or 3.5 % per annum. Suppose at delivery the actual rate is 2.6 % per annum. Lock in the rate of 3.5 % by buying 1 futures contract. The gain from the futures contract is $25 \times (97.40 - 96.50) \times 100 = 2250$. The interest earned in a 3 month period is $10^6 \times 0.25 \times 0.026 = 6500$. The total is 8750, which is the same as a 3.5 % rate earned in a 3 month period on 1 million dollars.

Note : Futures payment is made that time T_1, the beginning of the borrowing period (in contrast to FRA, whose payment is made at time T_2. Even if it is made at time T_1, it is done via discount from T_2 to T_1). On the other hand, interest earned is at time T_2. So there can be an adjustment on the number of contract needed by assuming the prevailing 3 month rate is 3.5 % and buy only $1/(1+0.035 \times 0.25 = 0.9913$ contract (so that the payment from the futures can earn interest during the 3 month period to match the locked in rate at time T_2).

Forward vs Futures interest rates: Rates from FRA tends to be lower than the rates from futures contract, for longer dated contracts. The first (main) reason is the daily settlement of the futures, which leads to the forward rate beling lower than the futures rate, using the same reasoning as the forward price being lower than the futures price. The second (less important) is the payment on the FRA is made at time T_2 instead at time T_1 as the futures contract. Suppose a FRA has a payoff of $R_M - R_F$ at time T_2. If R_M is high which leads to a positive payoff, the cost of receiving this payment at time T_2 rather than at time T_1 is high (sice rate is high).

If R_M is low which leads to a negative payoff, the saving of receiving this payment at time T_2 rather than at time T_1 is small. So overall the desire is for the payment to be made at time T_1 not T_2.

Convexity adjustment: Forward rate = Futures rate - $\frac{1}{2}\sigma^2 T_1 T_2$, where σ is the standard deviation of the change in the short term interest rate in 1 year, both rates expressed in continuous compounding.

Ex: Suppose the current futures rate is 6 % per annum. This corresponds to 1.5 % per 90 days and an annual rate of $e^{\frac{90}{365}R} = 1.015$ under continuous compounding. From here we can use the convexity adjustment to figure the forward rate.

Extending LIBOR zero curve using Eurodollar futures: Usually we can only observe LIBOR rate 12 months out (Hull section 7.6). To extend LIBOR zero curve out to 2 years (sometimes as far as 5 years), we do the following (Further extension is done by swap rates). From the convexity adjustment, we can figure out the forward rates for the period T_i, T_{i+1}. Suppose that F_i is the forward rate calculated from the ith Euro dollar futures contract (for the period T_i, T_{i+1} and R_i is the zero rate for a maturity T_i. We showed

$$F_i = \frac{R_{i+1}T_{i+1} - R_i T_i}{T_{i+1} - T_i}.$$

Or

$$R_{i+1} = \frac{F_i(T_{i+1} - T_i) + R_i T_i}{T_{i+1}}.$$

Duartion based hedging using futures: Suppose a portofolio is interest rate dependent (of bonds, money market security etc.) Let V_F : contract price for one interst rate futures contract, D_F : Duration of the asset underlying the futures contract (at the maturity of the of the contract?), P : Forward value of the portfolio at the maturity of the hedge (usually same as present value) D_P : Duration of the portfolio at the maturity of the hedge. Also assume that Δy change in yield is the same for all maturities. Then it is approximately true that

$$\begin{aligned}\Delta P &= -P D_P \Delta y \\ \Delta V_F &= -V_F D_F \Delta y.\end{aligned}$$

The number of contracts required to hedge against an uncertain Δy is $N^* = \frac{P D_P}{V_F D_F}$. Factors to consider: if Treasury bond futures then need to consider which bond is cheapest to deliver. Since futures price and interest rate move in opposite direction (verify this for the bond futures above), if a company loses money when interest rate drops they should hedge by taking a long future position and conversely. Choose futures contract so that duration of the underlying asset is as close as possible to the duration of the asset being hedge. Ex: Eurodollar futures for short term, ultra T-bond, Treasury bond, Treasury note futures for longer term hedge. (Is the duration of Eurodollar i.e. LIBOR 1?)

7 Chapter 7 : Swaps

Interest rate swap: A contract where one party pays the floating (LIBOR) rate and the other party pays an agreed upon fixed rate. An interest rate swap can be viewed as an exchange of a floating rate bond for a fixed rate bond. It is usually structured so that the initial value of the swap is 0. Thus the value of the fixed rate bond is equal the value of the floating rate bond at the beginning of the swap.

Use: Interest rate swap is used to transform a fixed (floating) rate loan (asset) into a floating (Fixed) rate loan.

Why swap : Comparative advantage argument. If the difference in the spreads in the fixed and floating rate market of the two institutions is not zero, a swap can be structured so that each institution gains an advantage of approximately half of this difference (after the cut from the financial intermediary). This is of course based on the assumption that each institution has the desire to lend / borrow in a compatible market with the other. Criticism of the comparative advantage argument : the difference in spread is due to serveral factors. First the term of borrowing may be different (6 month in float versus 5 years in fixed). Second, this difference reflects the credit risk difference of the two companies. The reason one (lower rated) company may have a comparative advantage in the floating market is because the term is shorter. So by swapping the rate, the higher rated company may bear the risk of the lower rated company defaulting in the long run (5 years).

Financial intermediary, market makers: Usually the two parties deal with a financial institution to structure the swap. Typically the institution earns 3 or 4 basis points (0.03 % or 0.04 %) on "vanilla" LIBOR-for-fixed swaps. This spread is partly to compensate the institution for the risk of default of either the companies on the swap payments. Market makers can enter the swap without having an offsetting swap with a counter party. They post their bid ask fixed rates (ex: bid = 6.03, offer = 6.06) and the swap rate is the average of these two.

Swap rate: not risk-free, but reasonably close to risk-free in normal market conditions. Recall: LIBOR is approximately the rate for AA institutions. A financial institution can earn the 5-year swap rate by : a) lend the principal for the first 6 months to a AA borrowers (at LIBOR), then roll over to another 6 months to other borrowers and b) enter into a swap (which has less chance to default, see below) to exchange LIBOR income for the 5 year rate. This is better than lending a AA institution at a fixed rate for 5 years. Indeed, a 5 year swap rates is less than 5 year AA borrowing rate. The reason is the longer the term, the more likely an institution might default. By rolling over every 6 months to AA borrower whose probability of default in 6 month period is low, this risk is reduced.

LIBOR zero curve: The zero rate curve for continuous compounding discount inferred from the LIBOR rate. This is also referred as the LIBOR / swap zero rate since the swap rate is used to extend the zero rate. In section 4.5 the zero rate determined from Treasury bond prices is referred to as the Treasury zero curve.

Meaning of swap rate : The value of a newly issued floating-rate bond that pays 6-month LIBOR is always equal to its principal value when LIBOR zero rate is used for discounting. (This bond pays semi annual coupon at the LIBOR rate). Since value of fixed rate bond equals value of floating rate bond at the beginning of the swap, this means a fixed rate bond who fixed rate is the swap rate also sells at par.

Using swap rate to extend LIBOR zero curve by bootstrapping : By knowing that the value of the fixed rate bond paying the swap rate is at par, we can bootstrap to get the last zero rate in the bond payment schedule. Ex:

$$2.5e^{-0.04\times 0.5} + 2.5e^{-0.045\times 1.0} + 2.5e^{-0.048\times 1.5} + 2.5e^{-R\times 2} = 100,$$

where 4 %, 4.5 %, 4.8 % are known zero rates. R is the extended zero rate to be found. 2.5 is from the 5% 2 year swap rate (for payments made semi-annually). 100 is the bond value at par.

Interest rate swap pricing: The basic formula is $V_{swap} = B_{fix} - B_{fl}$ (or vice versa depending on the position). B_{fix} can be figured out using the usual method. TO determine B_{fl}, we use the principle that its value is equal to the notional principal L immediately after a (any) coupon payment. We only need to figure out this value at the next immediate coupon payment time t^* after the present time at 0. The coupon payment is k^* which is based on a rate that is known at the present time (actually even known at the previous payment time). Thus its value immediately before the next payment is $L + k^*$ and its present value is $(L + k^*)e^{-r^* t^*}$ where r^* is from the zero LIBOR/swap rate.

Ex: A swap to receive 6 month LIBOR and pay 3 % per annum with semi annual compounding has a remaining life of 1.25 years. 6 month LIBOR at the last payment date was 2.9 %. Suppose $L = 100$. Then $k^* = 0.5 \times 0.029 \times 100 = 1.45$. $t^* = 0.25$ and r^* is given as 3.4 %. Then the floating leg is worth $101.45 \times e^{-0.034 \times 1.25} = 100.7423$. Here $V_{swap} = B_{fl} - B_{fix}$.

A slightly different way to look at this is to compute the floating leg payment, by assuming that the future LIBOR rate is equal to the forward rate (which is true from the present point of view). Then we can find the present value of the future floating cash flow and subtract the present value of the fixed cash flow from it.

Currency swap : Involve the exchange of principals in two different denominations at the beginning and the end of the swap (in contrast with interest rate swap where principals aren't exchanged since they're the same). Can be of three types : fixed for fixed, fixed for float and float for float. Each party receives the rate available to the currency that they exchanged to the other party. Ex: At the beginning, IBM pays 15 million USD and receive 10 million pounds. If fixed for fixed, during the life of the swap, IBM receives 6 % of 15 million USD and pays 5 % of 10 million pound annually. At the end, IBM pays 10 million pounds to receive 15 million USD.

Valuation : $S_0 B_F - B_D$ where B_F is the current price of the foreign bond corresponding to the foreign rate structure and B_D is the current price of the domestic bond corresponding to the domestic rate structure. Can also be valued via the future income streams on both sides and translated into one currency by the future (forward) exchange rate. See Table 7.9.

Comparative advantage : The explanation is similar to the comparative advantage in interest rate swap. One important difference: foreign exchange risk. This is usually taken on by the financial intermediary (who earn the bid ask spread) and structured so that the two counterparties do not bear any exchange rate risk (See Figure 7.11, 7.12, 7.13). The exchanged rate risk of the finanical intermeidary can be hedged by purchasing the foreign currency in the forward market. Note that the bid ask spread here is between two different currencies, so exchange rate should be taken into account (which is reflected in the two different principals denomination).

Credit risk: At any point in time, the value of the swap is positive for the financial institution wrt to one counterparty and negative wrt to one other counterparty. The counterparty with positive value may default. The one with negative value usually does not (even if they default they can sell their position to a third party.) Even if one party defaults the financial institution still has to honor the contract with the other party. Some swap position tends to hav positive value at the beginning and negative values later on due to term structure effect. These positions are less likely to default.

Potential losses : Swap potential losses are much less than the potential losses on a loan with the same principal. This is because the value of the swap is usually a much less than the value of the loan. On the other hand, potential losses on currency swap is higher than interest rate swap. The reason is the exchange of the principal at the end of the swap (exposed to exchange rate risk).

Market risk versus credit risk : Market risks are risks due to interest rate and exchange rate change. Can be hedged be entering into offsetting contracts. Credit risks are due to possible defaults of the counterparties. Harder to hedge. Can be hedged by credit default swap (CDS).

8 Chapter 8 : Securitization

Securitization and ABS: The process of re-organizing a portfolio of income producing assets into cash-flow generating tranches (securities). The product is referred to as asset backed security (ABS). If the asset is mortgage loans, it is referred to as MBS (mortgage backed security). MBS is special because it is guarantted against defaults by borrower by Ginnie Mae. Other ABS does not necessarily have this kind of guarantee against defaults. The process of securitization of loans frees the bank up to make more loans because they don't keep the securitized loans on their balance sheets.

Tranches of ABS: Typical example includes Senior, Mezzanine and Equity tranches with proportion (80 %, 15 %, 5 %) respectively in principal. The return on each tranche is LIBOR plus a spread, with spread increasing as one goes down the tranche levels. The cashflow goes from top tranche down. In terms of recovery of principals, if there is any loss, the loss absorption starts from the lowest tranche up. The Senior tranche typically has AAA rating, Mezzanine BBB, equity is unrated.

ABS CDOs: The ABS of of Mezzanine tranches of ABS. The assorption of the ABS CDOs is different than the original losses on the underlying assets (see Table 8.1).

CDOs: An ABS where the underlying assets are bonds.

9 Chapter 9: OIS discounting

Treasury rate, even though risk free, is artificially low for 3 reasons: a) Treasury Bills and Bonds must be purchased to fulfill certain obligations, this drives demand and price up, hence rate down b) The capital required to support an investment in Treasury bonds and bills is substantially lower than other low risk instruments c) Favorable tax treatment (not taxed at state level) compared with other fixed income instrument.

OIS rate: the swap rate for exchanging a fixed rate with the geometric average of the overnight rate over a 3 month period (or multiples of 3 month period). An investor borrowing at over night rate and roll over is equivalent to borrowing at the geometric average over the 3 month period. At the end of each 3 month period, the fixed rate payment is exchanged with the geometric average payment. The OIS term structure results in the OIS zero curve.

OIS-LIBOR spread: OIS rate is lower than LIBOR rate. A bank can : a) borrow 100 from an over night market for 3 months, roll over each night b) lend 100 for 3 months at LIBOR 3 month rate c) Use OIS to swap the over night rate for LIBOR rate. In doing this, the bank undertakes the risk of 3 month LIBOR loan defaulting. The connterparty to bank A stands less risk because the overnight loan can be reviewed every day. The difference between LIBOR-OIS rate is the spread, used as a measure of stress in the market. In normal condition, is about 10 basis points. In Oct 2008, 364 basis points : banks do not want to lend to each other.

OIS as proxy for risk free rate: good proxy, since the risk of overnight default is low and the risk of OIS swap defaulting is also low (as the swap is usually collateralized).

OIS zero curve : 1 month OIS rate defines 1 month zero rate, 3 month OIS rate defines 3 month zero rate and so on. When there are periodic settlements in the OIS contract, the OIS rate defines a par yield bond. Ex: 5 year OIS rate is 3.5 % with quarterly settlement. Then a 5 ear bond paying a quarterly coupon rate of 3.5 % is sold at par.

OIS discounting is different from LIBOR discounting : Suppose 1 year LIBOR rate is 5 % and 2 year LIBOR for fixed swap with annual payment is 6 %. If R is the 2 year LIBOR / swap zero rate (just a discount rate figured out from the swap, NOT the swap rate), we can calculate R in 2 ways: using the bond sells at par concept

$$\frac{.06}{1.05} + \frac{1.06}{(1+R)^2} = 1.$$

Using the swap value is zero with forward LIBOR F equals $\frac{(1+R)^2}{1.05} - 1$ (since $1.05 \times (1+F) = (1+R)^2$) :

$$\frac{.06 - .05}{1.05} + \frac{0.06 - F}{(1+R)^2} = 0.$$

Either way gives $R = 6.030\%$ which gives $F = 7.0707\%$ On the other hand, if 1 year OIS zero rate is 4.5 % and 2 year OIS zero rate is 5.5 % (so that OIS zero rates is about 50 basis points lower than LIBOR zero rates) then the forward LIBOR F satisfies

$$\frac{.06 - .05}{1.045} + \frac{0.06 - F}{(1 + .055)^2} = 0.$$

From this $F = 7.0651\%$.

This calculation is referred to as calculation of a forward LIBOR curve when OIS rates are used as risk-free discounting. When a swap is valued using OIS discounting, the forward rartes corresponding to the cash flows are obtained from the appropriate LIBOR curve. Cash flows are then calculated assuming these forward rates will realize and discouned using the appropriate OIS zero rates.

OIS vs LIBOR: the industry practice is to use OIS discounting for collateralized derivatives and LIBOR for non collateralized derivatives. The justification is the different costs of funding (in non collateralized derivatives the cost of funding is higher. Collateralized derivatives are funded by the collaterals.) However, finance principle states that the method of funding is irrelevant in valuation of an investment. It is the risk and the expected cash flow of the investment that is important. Hull : OIS rate is as close to risk free as we can get. Thus it should be used in BOTH cases.

Value adjustments : Different ways to adjust the present value of a current position based on factors such as future default events, funding methods (collaterals, funding costs). There are : DVA : debit (debt) value adjustment, CVA : credit value adjustment, CRA : collateral rate ajustment, FVA: funding value adjustment.

CVA : Calculated based on the event the counterparty (from the bank's point of view) default and the present expected loss of the portfolio at the default. $CVA = \sum_{i=1}^{N} q_i v_i$, where q_i is the probability of counterparty default during the time interval i and v_i is the present value of the loss (may not be simple to calculate).

DVA: Calculated based on the event of the bank default and the expected gain of the bank from its own default. $DVA = \sum_{i=1}^{N} q_i^* v_i^*$, where q_i^* is the proability of the bank default during the time interval i and v_i^* is the present value of the gain.

Thus if V_{nd} is the present no default value of the portfolio, the adjusted value is $V_{nd} - CVA + DVA$. If collateral is posted, interest rate paid on the collateral may be different from the risk free rate. Denoted as CRA (can be + or -), this needs to be taken into account by being added to V_{nd}: $V_{nd} - CVA + DVA - CRA$. Lastly, some banks take into account their different funding cost from the risk free rate and this becomes the FVA (See funding cost below).

Funding cost : (This is related to the OIS discussion above) A bank may have an average funding cost different from the risk free rate (probably depending on their credit rating). This is the interest rate they can borrow fund from the money market etc. Some banks use their funding cost rate as the discounting factor for present value valuation. Hull argues that this is not appropriate. If a project is risk free, its present value should be evaluated via risk free discounting. Hull gave an example of a risk free project giving return of 6 % while the risk free rate is 5 % and bank funding cost at 7 %. He argues that this project should be undertaken. Questions not answered : how can the bank fund such project? Why if the project is risk free its return is 6 % while risk free rate is 5 %? On the other hand, Hull made the relevant point that the correct way to account for risk is via CVA and DVA. Maybe it is about separating between the bank's way of generating funding (such as a short position) versus the project's own value (which is a long position). In evaluating the long position it should be risk free discounting. The bank's funding cost should be in the short position and the net value is how the bank should make decision upon.

10 Chapter 10 : Options markets mechanics

Option types : Put or Call, American, European.

Remark: If we want to upperbound the buy price, we long a call, lowerbound the buy price, we short a put. If we want to lowerbound a sell price, we long a put. If we want to upperbound a sell price, we short a put. (Sounds strange and is discussed in range forwards contract, chapter 17.2 in currency options. The point is to make the cost of insurance zero.)

Types of uderlying assets: Stock options, Foreign currency options, Index Options, Futures Options.

Terminology: Option class: all options of the same type (calls or puts) on a stock . Ex: IBM calls are on class, IBM puts are nother class. Option series : all options of a given class with the same strike and expiry. Ex: IBM 200 Oct 2014 is an option series. In the money, At the money, Out of the money. Intrisic value is $\max(S-K,0)$ for call and $\max(K-S,0)$ for put. For American option, there is also time value, which is the difference between the option value (since it's no less than the intrinsic value) and its intrinsic value.

Dividends and stock splits: Early OTC options terms (strikes) are adjusted to accomodate dividends and stock splits. The overall effect of dividend or split is to make the stock price goes down a percentage amount. A 20 % dividend is essentially the same as a 6 for 5 stock plit. Ex: A call option to buy 100 shares at 30 dollars per share. The company makes a 2 for 1 stock split. The options then change to buy 200 shares at 15 dollars per share. Ex: A put option to sell 100 shares for 15 dollars per share. The company declares a 25 % dividend. The optio is changed to sell 125 shares for 12 dollars. This is not true for options in general (?) since in the next chapter Hull dicusses the effect of dividend on option price.

Limits: Position limit: max number of options that an investor can hold on one side of the market. Exercise limit: max number of contracts that can be exercised by an individual (group) in any period of 5 consecutive business days (usually equals position limit).

Market makers: Individual who provides liquidity to the option market by quoting the bid / ask price on the option. Bid is always less than ask, difference is the spread.

Commissions : Money paid for executing an option trade or exercise (in this case same as commission for stock trade). The commission schedule can encourage the investors in a certain direction (i.e. selling the options rather than exercising them).

Margin requirements : Traders who write options are required to matain funds in a margin account. The amount of margin depends on the trader's position. If a trader writes an option without entering an offsetting position in the underlying stock, it is referred to as a naked option. There are specific margin requirements for naked options.

Warrants : Options issued by a financial instittution or nonfinancial corporation. Ex: (Common use of warrant) A corportion issues call warrants on its own stock and attaches them to the bond issue to make it more attractive to the investors.

Employee stock options: Call options issued to employee, at the money of the time of issue.

Convertible bonds: Bonds issued by a company that can be converted to equity at certain times using a predetermined ratio. Essentially bonds with an embedded call option on the company's stock (see above).

11 Chapter 11: Properties of stock options

Factors affecting option prices : See table 11.1 for general information. Some factors to note: Time to expiration for Euro call and put, effect is uncertain. If dividend is expected then the shorter expiry is more desirable than the longer expiry call. What about Euro put? What would make the effect of expiry on Euro put uncertain since dividend payout increases the value of a Euro put? Probably from Black Scholes since if T is increased to infinity the present value of the strike is 0 while the stock price present value is S_0 which would lead to a negative or at least decrease in option price. Effect of increasing volatility is positive for both put and call. It's because with volatility increase the option payout is protected in one direction and gains more from the other direction (compared with holding a stock only then the effect of increase in volatility will cancel out with both directions). Effect of interest rate is positive for call and negative for put. The reason is as interest rate increases the expected return for stock increases (think risk neutral model for S) while the present value of the strike decreases. This makes the stock more likely to be in the money at expiry while cheaper to purchase from the present value point of view. The exact reverse argument applies to put. Note that this is only for increase in interset rate only while keeping all other factors constant. In reality, it usually happens that an increase in r brings a decrease in S_0 so the overall effect may be uncertain. Dividend pays results in a decrease in stock price so the effect is negative for call and positive for put.

Bounds on non-dividend paying stocks: Euro call : $\max(S_0 - Ke^{-rT}, 0) \leq c \leq S_0$. Euro put: $\max(Ke^{-rT} - S_0, 0) \leq p \leq Ke^{-rT}$. Some remarks: A call option can never be worth more than the price of the stock (by definition), plus present value of S_T is S_0 hence the upper bound. A put option is never worth more than K hence the upper bound of Ke^{-rT}. The lower bound can be obtained by risk neutral pricing or no arbitrage argument when compared with a forward contract.

American call is worth the same as European call since its intrisic value is $S_0 - K$ and a European call value is always higher than this intrinsic value (due to $r > 0$). Thus it is not optimal to exercise American call before expiry. Another argument can be viewed by an American call option holder who wants to hold on to the stock until after expiry. Then the option holder does not want to exercise early since the cost he has to pay to buy the stock is more than what he has to pay at expiry (in terms of time value). If the holder does not want to hold on to the stock then he should sell the option rather than exercising because there would be others who would want to hold on to the stock (obviously here the option is deep in the money and thus there are people wanting to hold on to the stock). This would make the option value higher than the exercise or intrinsic value. Thus its bounds are $\max(S_0 - Ke^{-rT}, 0) \leq C \leq S_0$.

American put should be exercised early especially when deep in the money and interest rate is high. It is because then the stock price is very low (and thus can only go up) and the option holder would rather get the strike price earlier than later. In general, the early exercise of a put option is more attractive if S_0 decreases, r increases and volatility decreases. Its bounds are $\max(K - S_0, 0) \leq P \leq K$. (Note the difference here with American call compared to the Euro counterpart due to early exercise possible for put).

Effect of dividends : Change the lower bound of call and put : $\max(S_0 - Ke^{-rT} - D, 0) \leq c$ and $\max(Ke^{-rT} - S_0 + D, 0) \leq p$ where D is the present value of the dividend payments during the life of the option. Also now American call may be optimal to exercise immediately before ex=dividend date. Put call parity becomes $c - p = S_0 - Ke^{-rT} - D$.

Note : Compare with chapter 5: Does the index price go down after dividend yield payment? This seems to be the case per the description at the bottom of page 115. In this case coupon payment (bond) is different from dividend payment exactly where the price changes or not after payment. The argument in chapter 5 should be modified to : $S_{t_i} = S_{t_i-}(1-q)$ and $d = S_{t_i-}q$. Thus $\pi_{t_i-} = \pi_{t_i}$ while $\Delta_{t_i} = \Delta_{t_i-}\frac{1}{1-q}$, where Δ_t is the number of shares held at time t. Note that at any time $\pi_t = \Delta_t S_t$. Replacing q by $q\Delta T$ we have $\Delta_T = \Delta_0 \frac{1}{(1-q\Delta T)^{\frac{T}{\Delta T}}}$. Push ΔT to 0 gives $\Delta_T = \Delta_0 e^{qT}$. If $\Delta_0 = 1$ then $\Delta_T = e^{qT}$ and thus S_0 grows to $e^{qT}S_T$ at time T.

12 Chapter 12: Trading strategies involving options

Principal protected notes: Consists of a zero coupon bond and a call or a put option on an asset (portfolio). The strike of the option is usually the principal of the bond. Thus the investor invests the principal at the beginning of the investment and receives the principal at the end of the investment from the bond plus the option pay off (if any). This allows the investor to take a bet on the price movement of the asset without incurring a risk on the principal. On the other hand, the investor forgoes any interest on the principal and / or dividend payout from the asset. For the bank (the seller) to make a profit, the price of zero coupon bond plus the option price must be less than the principal. This depends on the interest rate (the higher it is the lower the bond price) and the volatiltiy (the higher it is the higher the option price). If this sum exceeds the principal, some adjustments can be made such as increasing strike price, increasing the expiry, capping the investor's return, using barrier option. A critical variable for the bank is the dividend yield (in the example 12.1). The higher it is, the more profitable the product is for the bank (possibly because the call option price is lower?). If not then the lower bound on call option prices shows that the bank cannot make profit in this example.

Option plus underlying asset: Covered call : long a stock and short a Euro call. Reverse covered call (reverse the position of covered call). Protective put : long a stock and a Euro put. Reverse protective put. The profit patterns of all these strategies are similar to their counterpart with Euro put / call (with a shift) because of the put call parity.

Spread : Long / short on different calls (or puts) on the same stock.

Bull spreads (Call): Long a Euro call at K_1 and short a Euro call at K_2 at same expiries where $K_1 < K_2$. Pay off $(S_T - K_1)^+ - (S_T - K_2)^+$. Requires initial investment. Thus positive profit if S_T sufficiently above K_1.

Bull spreads (Put) : Long a Euro put at K_1 and short a Euro put at K_2 at same expiries where $K_1 < K_2$. Pay off $(K_1 - S_T)^+ - (K_2 - S_T)^+$ is non positive. Positive initial cash flow. Thus also profit if S_T sufficiently above K_1. Actually the same profit structure with Bull spreads (Call) (is exactly $K_2 - K_1$ less than the bull call pay off if stock pays no dividend by put call parity).

Bull spread is thus betting on stock price increasing.

Bear spreads (Put): Short a Euro put at K_1 and long a Euro put at K_2 at same expiries where $K_1 < K_2$. Payoff $(K_2 - S_T)^+ - (K_1 - S_T)^+$. Requires initial investment. Positive profit if S_T is sufficiently below K_2.

Bear spreads (Call): Short a Euro call at K_1 and long a Euro call at K_2 at same expiries where $K_1 < K_2$. Payoff $(S_T - K_2)^+ - (S_T - K_1)^+$ is non positive. Positive initial cash flow. Thus also positive profit if S_T is sufficiently below K_2.

Bear spread is thus betting on stock price decreasing.

Box spreads : Combination of a bull call and bear put with the same two strike prices. Pay off is always $K_2 - K_1$. Thus is a way to arbitrage if the market price and the interest rate $(K_2 - K_1)e^{-rT}$ do not match. Only works with European option.

Butterfly spreads: Long a Euro call with strike K_1, short 2 Euro call with strike

K_2 and long a Euro call with strike K_3, with $K_1 < K_2 < K_3$. Pay off $(S_T - K_3)^+ + (S_T - K_1)^+ - 2(S_T - K_2)^+$. Has limited loss (just initial option price) when $S_T < K_1$ or $S_T > K_3$ (assuming $K_2 = 1/2(K_1 + K_3)$) and gain approximately $K_2 - K_1$ when S_T is close to K_2. Thus is a bet that the stock price does not have large movement in the life of the option. A short butterfly spread gives moderate profit when there is significant movement in stock price. A loss is incurred here if the stock remains close to K_2.

The assumption is the initial investment of the butterfly spread is positive. Thus the call option price structure must be in a certain way. This is not obvious just by looking at the relation of the strike prices.

Calendar spreads: Short a call option with expiry T_1 and long a call option with expiry T_2 and same strike, $T_1 < T_2$. Pay off at time T_1 is positive if S_T remains close to K because the option with expiry T_2 is still worth high value. Incurs a loss if S_T is sufficiently far away from K at time T_1. Is similar to butterfly spreads in pay off structure. Neutral calendar spread: Strike is close to current spot. Bullish calendar spread: strike is higher than spot, bearish calendar spread : strike lower than spot.

Combinations: Combining calls and puts on the same stock, usually long only position.

Straddle: Long a Euro call and put with the same strike and expiry. Positive pay off with large swing and loss with moderate move. Also referred to as a bottom straddle or straddle purchase. A top straddle or a straddle write is the reverse position. Highly risky since it has unlimited loss from large movement of stock price.

Strangles: Buy a Euro put at strike K_1 and call at strike K_2 with same expiration, $K_1 < K_2$. Positive pay off with large movement of stock price and moderate loss with moderate move. Requires bigger swing than straddle but also provides less loss.

Strips : Long one Euro call and two Euro put with the same strike and expiry. Straps : Long two Euro call and one Euro put with the same strike and expiry. Also positive pay off with large movement but betting on a specific direction.

Range forwards (a topic of chapter 17, related to foreign currency) : if in position to sell foreign currency : long a put at strike K_1 and short a call at strike K_2, $K_1 < K_2$. If in position to buy foreign currency : short a put at K_1 and long a call at K_2, $K_1 < K_2$. This is an alternative to forward contract and it is assumed that $K_1 < F < K_2$ where F is the forward exchange rate at expiry. The point is to construct the contract so that the cost of the contract is zero, to be an alternative to a forward contract.

13 Chapter 17: Options on stock indices and currencies

How many options to buy for portfolio insurance : If we have a portfolio of stock indices that have the same dividend yield as the index yield. To protect the portfolio from falling below a certain value, we can buy put options on the index. One index option is on 100 times the index. For example, upon exercise, the holder of a call option receives $(S - K) \times 100$ and holder of a put option receives $(K - S) \times 100$ in cash. Suppose that the value of the index today is S_0. If the portfolio's β is not 1.0, β put options must be purchased for each 100 S_0 dollars in the portfolio. To calculate the appropriate strike price (quoted in index value) we need to figure out the corresponding portfolio value at the strike index level (from CAPM model, using excess return (from risk free rate) of the index and excess return of the portfolio from the beta) and reverse the process with the desired protected portfolio value. See Business Snapshot 17.1 for an interesting application.

Options on stock paying known dividend yields: Suppose the yield rate is q. The important argument is the price S_0 changes to S_T at time T when paying dividend at rate q. Thus without paying dividend at rate q, a price S_0e^{-qT} at time T will also change to S_T at time T, all else being equal. In terms of probability distribution, we have the same S_T distribution in two cases: 1. the stock starts at S_0 and pays dividend yield at rate q 2. the stock starts at S_0e^{-qT} and pays no dividend (which is the classical Black-Scholes case). Thus we just have to use the B-S formula with S_0 replaced by S_0e^{-qT}. The formula is

$$\begin{aligned} V_0 &= S_0e^{-qT}N(d+) - Ke^{-rT}N(d-) \\ d\pm &= \frac{(r \pm \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0e^{-qT}})}{\sigma\sqrt{T}} \\ &= \frac{(r - q \pm \frac{1}{2}\sigma^2)T - \log(\frac{K}{S_0})}{\sigma\sqrt{T}}. \end{aligned}$$

Thus in terms of option pricing, the word dividend should be defined as the reduction in the stock price on the ex-dividend date arising from any dividend declared. On the other hand, this is exactly the result we obtain if we compute $\tilde{\mathbb{E}}(e^{-rT}(S_T - K)^+)$ where $dS_t = (r - q)S_tdt + \sigma S_td\tilde{W}_t$ under $\tilde{\mathbb{P}}$. Therefore, the Black-Scholes PDE is

$$\begin{aligned} -rv + v_t + (r - q)xv_x + \frac{1}{2}\sigma^2x^2v_{xx} = 0 \\ v(T, x) = (x - K)^+. \end{aligned}$$

Put-call parity: (From the same principle of replacing S_0 with $S_0 e^{-qT}$) $c - p = S_0 e{-}qT - Ke^{-rT}$. If we also recall that the forward price of S at time T is $F_0 = S_0 e^{r-q} T$ then put-call parity can be expressed as $F_0 = K + (c-p)e^{rT}$. This equation can be used to estimate term structure of forward prices at the expiries that have pairs of put and call actively traded. Other options can be valued using the B-S formula expressed in (estimated) F_0 :

$$\begin{aligned} V_0 &= F_0 e^{-rT} N(d+) - Ke^{-rT} N(d-) \\ d\pm &= \frac{(r \pm \frac{1}{2}\sigma^2)T - \log(\frac{K}{F_0})}{\sigma\sqrt{T}}. \end{aligned}$$

This approach has the advantage of not having to estimate the yield on the index explicitly. On the other hand, if estimates of dividend yield is required (say with American option), the implied yield rate can be used from put call parity:

$$q = \frac{-1}{T} \log(\frac{c - p + Ke^{-rt}}{S_0}).$$

This rate can be used, for example, in binomial tree evaluation of American tree where the risk neutral probability is

$$p = \frac{e^{(r-q)\Delta T} - d}{u - d},$$

where $u = e^{\sigma\sqrt{\Delta T}}$ and $d = \frac{1}{u}$. These values (see 13.7 is so that the volatility (standard deviation of the stock return $\frac{S_{t+\Delta T} - S_t}{S_t}$ in time period ΔT) is $\sigma\sqrt{\Delta T}$. Equivalently the variance of return in time period ΔT) is $\sigma^2 \Delta T$. Is this the discrete version of $\tilde{\mathbb{E}}(e^{-rT}(S_T - K)^+)$, where we discretize the dynamics of the stock price under the risk neutral probability and approximate the Brownian motion part ?

Option on foreign currency: Let S_0 be the value of one unit of foreign currency in US dollars. A foreign currency is analogous to a stock paying a known dividend yield : 1. The rate r_f paid depends on the value of the currency 2. A unit of foreign currecy S_0 would grow to $e^{r_f T} S_T$ at time T 3. Thus even though price wise S_0 changes to S_T we only need $e^{-r_f T} S_0$ to grow to S_T in investment at time T (in other words the fair price for S_T at time 0 is $e^{-r_f T} S_0$ and hence S "loses" value with rate r_f) . So all formula in the above can be reapplied here with q replaced by r_f and the forward exchange rate as $F_0 = S_0 e^{r - r_f} T$.

14 Chapter 18 : Futures, Options and Black's model

Distinguish spot options versus futures options (different underlyings)

Futures call options : right to enter into a long futures contract. If exercised, the holder acquires a long position in the futures contract plus a cash amount equals to the most recent settlement futures minus strike. Effective payoff : $(F_T - K)^+$.

Futures put options : right to enter into a short futures contract. If exercised, the holder acquires a short position in the futures contract plus a cash amount equals to the most recent settlement futures minus strike. Effective payoff : $(K - F_T)^+$.

Usually futures option expiry is before the underlying futures contract expiry. If they are the same, then the futures option is the same as the spot option, which is the basis for Black's formula.

Options on Interest rate futures: Options on Treasury bond, Treasury note and Eurodollar futures. If an investor thinks short term (long term) interest rate will fall (rise), he can buy a call (put) Eurodolalr futures (Treasury bond, Treasury notes) option.

Spot versus futures option: Futures option is more popular. Futures price quote much easier to get (from trading floor) than spot price quote. Further, one can close out a futures contract before its expiry thus excluding the need to obtain the underlying futures asset (commodity, bonds etc.), no delivery hassle. Futures is much cheaper to trade so the funding required is less.

Most futures options are traded American style. We discuss Euro futures options for Black's formula.

Dynamics of futures price in the risk neutral world: If T is the futures expiry then $F_t = S_t e^{r(T-t)}$. Therefore, $dF_t = \sigma F_t d\tilde{W}_t$. That is F_t is a martingale under the risk neutral probability and has no drift. Futures can also be looked at then as asset that pays dividend with rate r. (In fact, futures price is a martingale under the forward measture. That is futures price is stock price denominated by zero coupon price) Thus the fair price of F_T at time 0 is $F_0 e^{-rT}$.

Observation on martingale pricing: Martingale pricing is the mathematical expression of the principle : the present value of a tradable asset (at some future point) is its current price. This extends also to a portfolio of tradable assets. The emphasis is on the asset. Thus the price process is not necessarily a martingale (price of a dividend paying asset, futures price etc). On the other hand, the value of a share of an asset paying dividend yield (as long as it is reinvested) is still a martingale. From this point of view, we get the risk neutral pricing of derivatives of dividend paying asset by looking at the replicating portfolios of such derivatives. On the other hand, the futures price process actually is not a price of any asset. So looking at a portfolio of asset with futures price is not possible. Actually the futures price is the price of S_t denominated under the unit of the zero coupon bond $B(t,T)$. From this point of view, the t_1 value of S_{t_2} denominated under $B(t_2,T)$ should be the value of S_{t_1} denominated under $B(t_1,T)$. This is risk neutral pricing under the T forward measure. This still does not give the pricing for F_T at time 0. The reason is we want F_T itself in dollars, not in the unit of $B(t,T)$. Indeed, F_t is not the price of any traded asset. It is the price of $\frac{S_t}{B(t,T)}$. In this writing it is also strange if we consider it as fraction of prices. We want to look at $B(t,T)$ as a "discounting factor" with no unit. Thus we need to revert to risk neutral pricing formula to price F_T using the replicating portfolio idea (portfolio consisting both of $B(t,T)$ and S_t). The replicating portfolio for F_t may not be straightforward. On the other hand, we know that the dynamics of F_t is like a dividend paying stock. Thus for all pricing purposes it can be treated that way. If there is an asset whose price process is F_t that asset MUST pay dividend with rate r. Otherwise no one would hold on to such asset. F_t can be looked at as a synthetic price in that sense.

Short term dynamics of futures price: Suppose the settlements days are $\Delta T, 2\Delta T \cdots$ for the margin account. At time ΔT the payoff is $F_{\Delta T} - F_0$. The time 0 value of this payoff is $\tilde{\mathbb{E}}[e^{-r\Delta T}(F_{\Delta T} - F_0)]$. But this is the value of the forward contract, which is 0. This argument can be repeated at time i to show that F_t is a martingale udner $\tilde{\mathbb{P}}$. In fact, this shows the difference between futures price and forward price. Futures price is martingale under the choice of numeraire as the money market account. Forward price is a martingale under the choice of numeraire as the zero coupon bond. Forward price is NOT a martingale under the usual risk neutral world of the money market numeraire (see Shreve chapter 9 for example). Also see Hull footnote page 392 and chapter 28. This is also why futures price can be viewed as an asset paying dividend rate r. In general pricing $V_T = F_T$ would be hard if F_T is a forward price of an asset. Lastly to show that the pricing formula is correct, we can use a replicating portfolio consisting of a dividend paying stock whose price matches F_t. Because if we start out with $e{-}rTF_0$ at time 0 the final value of the portfolio is F_T it must be the right price. This shows that as long as two price processes match, their derivatives value are the same, including dividend payment etc. Thus price doesn't have to be of a physical asset !

Futures price part 2: Actually if we hold a futures contract, by the daily settlement, we actually do get the price stream change (minus the initial amount). This price stream change can then be subjected to risk free rate if need be (?).

Thus the Black's formula for Call futures option with expiry T is the pricing formula for dividend paying asset with rate r

$$\begin{aligned} V_0 &= F_0 e^{-rT} N(d+) - K e^{-rT} N(d-) \\ d\pm &= \frac{\pm\frac{1}{2}\sigma^2 T - \log\frac{K}{F_0}}{\sigma\sqrt{T}}. \end{aligned}$$

Black's formula is more popular even for spot options price calculation. We only need to find the futures with the same expiry as the option so that spot and futures option price are equal. This is convenient because then the underlying asset can be commodity, consumption, investment asset and can even provide income to the holder. The relevant variable is just the futures price. Thus we don't even need to estimate the income (convenience yield) of the underlying asset. The futures price incorporated the market's estimate of this income. Black's formula is also true when interest rate is random. See Shreve's 9.4.3

American futures option vs American spot option : American futures option can be exercised early, in both call and put version. If we have a normal market (futures price is higher than spot) then there is reason to exercise American futures call early. In this case American futures call option is worth more and American futures put options is worth less than their spot counterparts. If we have an inverted market with futures price lower than spot then there is reason to exercise American futures put early. In this case American futures call option is worth less and American put options is worth more than their spot counterparts. This holds true even if option expiry is the same as futures expiry.

15 Chapter 19: The Greeks

Stop loss strategy : Buy the underlying when the price is just above K and sell the underlying when the price is just below K. Not practical because not self-financing basically. In reality will need to buy when the price is about $K + \epsilon$ and sell when the price is $K - \epsilon$ and thus incur a cost of 2ϵ plus transaction cost when doing so. If push ϵ to 0 will have to infinitely many transactions.

Delta hedging: Hold $-\Delta$ shares of stock for every share of option in the portfolio. Delta hedging is self-financing (approximately so in practice). See Table 19.2 and 19.3. Performance measure of delta hedging: the ratio of the standard deviation of the cost of hedging the option to the theoretical price of the option (the smaller the better?) Unlike stop-loss strategy, the performance measure gets beter as the hedge is monitored more frequently (as rebalancing takes place more frequently, the variation in the cost of hedging is reduced, see table 19.4). Delta hedging of a short position in call option generally involves selling stock just when price decreases and buying just when price increases (a buy high sell low strategy ! not the usual buy low sell high?). When we have a portfolio of options with the same underlying, transaction cost can be reduced by doing a single transaction in the underlying to make the delta of the whole portfolio neutral.

Theta: Sensitivity of the portfolio to the time remaining. Not a hedge parameter as it does not make sense to hedge against the passage of time. Is usually used as a proxy for gamma : in a delta neutral portfolio, if Gamma is high and positive, Theta is high and negative and vice versa. The reason is the Black-Scholes PDE : $\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$.

Gamma: Using Taylor expansion, assuming Delta is 0: $\Delta\Pi = \Theta\Delta t + \sigma^2 \Delta S^2 \Gamma$. Thus combined with the remark in Theta, if Gamma is positive and there is no change is S, portfolio tends to decline in value, but increases in value if there is a large positive or negative change in S. If Gamma is negative, the situation is reverse. Gamma of both put and call are positive (think about why, using the payoff profile of put and call.)

Dynamic hedging in practice : Each trader (or team) assigned to all options on an underlying. Limits are defined for each Greek letter, special permission required to exceed limits. Delta is quoted in the equivalent max dollar amount in the underlying position. Ex : Delta limit is 1 million, stock price is 50 so delta cannot exceed 20k in absolute value (related, but not the same as max tolerance for the change in portfolio). Vega quoted as max dollar exposure per 1 % change in vol. Traders always make themselves delta neutral at the end of each day. Gamma and vega are monitored, but not managed on day to day basis. Financial institutions usually have negative gamma and vega since they write options. They manage gamma and vega risks by buying options at competitive prices. Usually options are at the money at the start so gamma and vega are high. As time progresses, options might be deep in or out of the money so gamma and vega are low and of little consequence. Nightmare scenario : close to expiry and options are also close to the money.

Delta of forward contract: Delta is 1 if no dividend and e^{-qT} if pays dividend.

Delta of futures contract: Delta is e^{rT} if no dividend and e^{r-qT} if pays dividend. Thus Δ of futures and forward contract is different ! This is because futures contract gets settled daily on the futures price stream. Forward contract changes according to the forward contract price (not the forward price) of $S_0 - Ke^{-rT}$.

Using futures for Delta hedging : Since futures is more actively traded than underlying, for delta hedging we can use futures instead of underlying. The number of shares need to be modified by either $e{-}rT$ if underlying does not pay dividend and $e^{-(r-q)T}$ if the underlying pays dividend from the original delta. If it is a currency then the modification is $e^{-(r-r_f)T}$.

Synthetic option using dynamic hedging: Portfolio managers tend to buy put options for portfolio insurance. If the put options are not available, one can be synthetically created by holding Delta of put in the underlying and the rest in the money market account. The action required to dynamically hedge everyday may have an impact on the overall market supply / demand and thus create unintended market size effect (see business snapshot 19.2). Index futures can be used to synthesize the put option since index futures tends to have lower transaction cost than regular spot futures. In this case two expiries need to be distinguished: the option expiry T_1 and the futures expiry T_2 where $T_1 < T_2$.

16 Chapter 20: Volatility smiles

Deriving implied risk neutral distribution from call option price: We have

$$c(K) = \int_K^\infty e^{-rT}(x-K)f(x)dx.$$

Thus

$$c'(K) = -\int_K^\infty e^{-rT}f(x)dx$$

and

$$c''(K) = e^{-rT}f(K).$$

One can approximate $f(K)$ with the central difference quotient :

$$f(K) \approx e^{rT}\frac{c(K+\delta) - 2c(K) + c(K-\delta)}{\delta^2}.$$

This is also the same as the price of $\frac{e^{rT}}{\delta^2}$ share of the butterfly spread. Thus observing the butterfly spread at different strikes gives us the implied risk neutral distribution.

Volatility smile is the same for call and put : By put call parity :$c - p = S_0 - Ke^{-rT}$ which applies for both theoretical and market price, we have $c_T - c_M = p_T - p_M$.

Suppose $\sigma^C_{implied}$ is such that the LHS is 0 (by its definition). Then it must also make the RHS 0 and thus $\sigma^C_{implied} = \sigma^P_{implied}$.

Volatility smile for foreign currency options: Both deep in and out of the money has higher implied volatility than close to the money options. This leads to the implied distribution having fatter tails than the theoretical log normal distribution. (The probability of being in the money for a call option (which is originally out of the money) by finishing over K_2 is higher than the theory which implies higher price than the theory. Similarly the probability of being in the money for a put option (which also is originally out of the money) by finishing below K_1 is higher than the theory which leads to higher implied price).

Emprical evidence : One can test the evidence of volatility smile by testing the implied distribution versus the log normal distribution using empirical data. We record how often the percentage change in exchange rate exceeds 1, 2, $\cdots$, 6 standard deviations of the daily percentage change (this percentage change is NOT the same for physical versus risk neutral probability. However, since $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$, the probability of exceeding the SD depends only on σ). The result shows that the tails of the distribution are fatter than we would expect from a log normal distribution (see Table 20.1).

Reason for volatility smile in foreign currency options: Exchange rate has non-constant volatility (in time? need to see how that relates to strikes) and exchange rate has jumps (in response to central banks actions, for example). Both of these effects are less pronouce as the maturity increases. Thus the vol smile becomes less pronouced as maturity increases (also referred to as vol term structure). When there is a jump, the implied distribution can be viewed as a mixture of two log normal distributions (see figure 20.5).

We can use the one-step binomial model to simulate the effect of a jump in volatility. (See e.g. Table 20.3). The implied vol is actually a frown, not a smile.

Vol smile of equity options: Actually a decreasing concave up function, thus the lower the strike the higher the implied volatility. This results in the implied distribution with fatter tail on the left and skinnier tail on the right. The reason for this kind of smile may be because of leverage : the higher the equity, the lower the vol (vol is a decreasing function of S). This leads to the skinner tail on the right and fatter tail on the left of the implied distribution picture, as well as the fact that option with lower strike has higher price due to higher volatility when underlying is close to the money. Another is fear of crash : traders price option according to the fear of crash, thus lower strike tends to be priced more expensively. Empirical evidence supporting this: Declien in S&P 500 tends to be accompanied by a steepening of the vol skew, while increasing of the 500 tends to have a less steep skew.

Alternative ways of characterizing the vol smile: If graph implied vol vs K, the lowest point tends to be S_0, which leads to movement of the smile when S changes. If we graph against $\frac{K}{S_0}$ this becomes more stable. An alternative is against $\frac{K}{F_0}$ (some traders define at the money option as when $K = F_0$). Yet another is against option Δ and at the money is when $Delta = 0.5$ for call and equals -0.5 for put.

Vol surface and term structure: We can graph implied vol against $\frac{K}{S_0}$ and T. In the direction of T, the vol smile becomes less pronouced, thus the time to maturity needs to be taken into account for option pricing. Thus one can also graph implied vol against $\frac{1}{\sqrt{T}} \log \frac{K}{S_0}$. This way the smile is less dependent on time to maturity.

Greek letters: The Greek in the presence of implied vol needs to be modified to (i.e. Delta)

$$\frac{\partial C_{BS}}{\partial S} + \frac{\partial C_{BS}}{\partial \sigma_{imp}} \frac{\partial \sigma_{imp}}{\partial S}.$$

This reflects the fact that as the price of the underlying changes, the implied vol changes to reflect the option's moneyness (or just simply σ depending on S, via $\frac{K}{S_0}$ or $\frac{K}{F_0}$?). Thus if we believe that implied vol is a decreasing function of $\frac{K}{S}$ (in contradiction to the leverage effect mentioned above !, which may be strange since the graph should be obtained by changing K, even if it's graphed against $\frac{K}{S}$) then this implies that the Delta here is higher than the Delta given by BS. Hull: in practice, trader ensures their exposure to the most commonly observed change in vol surface is small.

Role of model: Sophisticated interpolation tools ? Arguably if BS is not used and other models adopted, vol surface would change, but the market price would not change appreciably. Even Delta, if calculated using the above formula, would not change as much. Model has most effect when derivatives are not actively traded.

17 Chapter 21 : VaR

Statement: I am X percent certain that there won't be a loss of more than V dollars in the next N days. V is the VaR of the portfolio. Banks are required to calculate VaR for market risk with $N = 10$ and $X = 99$ and hold a capital k times of the VaR ($k \geq 3$ and is bank specific). It is the 1st percentile of the distribution of the gain (left tail) and equivalently 99th percentile of the distribution of the loss (right tail) of the portfolio.

Time horizon: Market risk VaR is calculated using $N = 1$ because there is not enough data available to estimate directly the behavior of the market variables over periods longer than 1 day. The $N-$ day VaR then is calculated as $\sqrt{N}$ 1-day VaR. This is based on the assumption that daily changes of portfolio value are iid Normal. Since then the 99th percentile of the total change is $\sqrt{N}\times$ the 99th percentile of the one day change.

General theory: We look at the general market variables affecting the portfolio : interest rates, equity prices, commodity prices. VaR then is calculated based on the historical changes of these variables or on the models of these variables.

Historical simulation: We look at the 501 days of data and consider each daily change as a possible scenarios (for a total of 500 scenarios). Suppose today is day n and the historical day is day i. Then

$$\text{Portfolio value under the ith scenario } = v_n \frac{v_i}{v_{i-1}},$$

where v denotes the value of the market variable at the specific date. For example, the market variables in section 22.2 are the 4 stock indices: DJIA, FTSE 100, CAC 40, Nikkei 225. The change of the portfolio value in the ith scenario is

$$\sum_{k=1}^{4} V^k I_n^k \frac{I_i^k}{I_{i-1}^k} - \Pi_n,$$

where I^k is the k th index value and V^k is the portfolio dollar value invested in the ith index. From the 500 such changes calculated from the 500 scenarios, the 5th worst loss would correspond to the 99 % daily VaR. The 99 % 10 day VaR is $\sqrt{10}$ times this number.

Model building approach :

General theory for portfolio of assets : Daily percentage changes of each asset are iid Normal. We assume that σ_{daily} of each asset is known. Then letting α_i to be the number of shares in asset i :

$$\begin{aligned}
\Delta\Pi &= \sum_i \alpha_i \Delta S_i \\
&= \sum_i \alpha_i S_i \frac{\Delta S_i}{S_i} \\
&= \sum_i V_i \frac{\Delta S_i}{S_i},
\end{aligned}$$

where again V_i is the dollar amount in asset i. Thus

$$SD(\Delta\Pi) = \sqrt{\sum_{ij} \rho_{ij}(V_i\sigma_i)(V_j\sigma_j)},$$

where rho_{ij} is the correlation of assets i,j. This is the estimate of one standard deviation change in the portfolio value. The 99 % one day VaR is then 2.236 $SD(\Delta\Pi)$ where 2.236 is the 99th percentile of the standard Normal. The 99 % 10 day VaR is $\sqrt{10}$ times this.

Trivia: $\sigma_{year} = \sigma_{day}\sqrt{252}$ as there are 252 trading days in a year. For ex if daily vol is 2 % then annual vol is about 32 %.

General theory for portfolio of options: Consider a portfolio of options on different underlyings. Denoting δ_i as the (total) delta of the options with the ith underlying then

$$\begin{aligned} \Delta\Pi &= \sum_i \frac{\Delta\Pi}{\Delta S_i}\Delta S_i \\ &= \sum_i \delta_i S_i \frac{\Delta S_i}{S_i}. \end{aligned}$$

Thus

$$SD(\Delta\Pi) = \sqrt{\sum_{ij} \rho_{ij}(\delta_i S_i\sigma_i)(\delta_j S_j\sigma_j)},$$

The 99 % one day VaR is then 2.236 $SD(\Delta\Pi)$ where 2.236 is the 99th percentile of the standard Normal. The 99 % 10 day VaR is $\sqrt{10}$ times this.

General theory for portfolio with bonds: Assuming parallel shift in the yield curve is unrealistic. Instead, the market variables are the prices of zero coupon bonds with standard maturitires: 1 month, 3 months, 6 months, 1 year, 2 years, 5 years, 7 years, 10 years and 30 years. For the purposes of calculating VaR, the cash flows from instruments in the portfolio are mapped into cashflows occuring on the standard maturity dates. Ex: suppose a portfolio with 1 million dollars position in a bond with coupons in 0.2, 0.7, 1.2 years. The cashflow in 0.2 year is apporximated by position in 1 month and 3 month zero-coupon bonds, 0.7 year is aproximated by 6 month and 1 year bonds, 1.2 years is approximated by 1 year and 2 years bond. This is known as cash-flow mapping. Cash-flow mapping is not necessary if historical simulation is used.

Dimension reduction - PCA approach : Using historical data on movements in market variables such as the swap rates with various maturities (or equivalently the bond yields in the previous example) we can reduce the number of variables by the PCA approach. Table 22.7 gives the factor loadings. It can be understood as followed: $(\Delta S)_t = PC\bar{\alpha}_t$, where ΔS is the vector of daily interest rate changes, PC is the principle component matrix (matrix of factors) and $\bar{\alpha}_t$ is the solution so that the equation holds, also known as factor scores of the day.

To illustrate, suppose S_i is the swap rate and F_i are the factors. Then

$$\begin{aligned} \Delta\Pi &= \sum_i \frac{\Delta\Pi}{\Delta S_i}\Delta S_i \\ &= \frac{\Delta\Pi}{\Delta S}^T PC\bar{\alpha}_t, \end{aligned}$$

where $\frac{\Delta\Pi}{\Delta S}$ is the vector of the change of portfolio value under one basis movement of a particular rate (can be computed simply, thus deterministic). The co-variance matrix of $\bar{\alpha}_t$ is a diagonal matrix as a consequence of principle component decomposition, denoted by Σ. We then have

$$SD(\Delta\Pi) = \sqrt{\frac{\Delta\Pi}{\Delta S}^T PC\Sigma PC^T \frac{\Delta\Pi}{\Delta S}}.$$

This then gets translated to 99 % daily VaR and 99 % 10 day VaR exactly as above. In Hull's example, PC only consists of the first 2 factors of the component matrix. $SD(\Delta\Pi)$ on the other hand can be calculated from the covariance matrix of ΔS_i. What is the advantage of using PCA?

Alternative to VaR: Expected shortfall : the expected loss during the N-day period conditional on the loss being worse than VaR loss. The expected short fall may be helpful to detect large loss that exceeds VaR level (outliers). See also figure 22.1 in Hull.

Stress test: Estimating how the portfolio would perform under some of the most extreme market movements (5 standard deviation moves or more of market variables).

Back testing: Testing how well VaR estimates would have performed in the past. For example : Looking at how often the loss in 1 day exceeded 1 day 99 % VaR that would have been calculated for that day. If this happens significantly more than 1 % then the VaR methodology needs revision.

18 Chapter 23: Estimating Volatilites and Correlations

Daily percentage change distribution: $u_i = \frac{S_i - S_{i-1}}{S_{i-1}}$. u_i is assumed to be Normal with mean 0 and Variance σ_{i-1}^2. (Mean zero actually means the mean is much smaller than the standard deviation). u_i will be the basic building blocks for $v_i = \sigma_i^2$.

Constant variance model: $v_N = \frac{\sum_{i=1}^N u_i^2}{N}$. This can also be viewed as giving equal weights to u_i^2.

General weighting scheme model:

No long run average: $v_N = \sum_{i=1}^N \alpha_i u_{N-i}^2$ where $alpha_1 > \alpha_2 > \cdots$ (note the subscripts of α and u) and the sum of the alphas equalling 1. The idea is giving more weight to the most recent percentage change and less weight to further data points.

Long run average : $v_N = \gamma v_L + \sum_{i=1}^N \alpha_i u_{N-i}^2$ where $alpha_1 > \alpha_2 > \cdots$ and $\gamma + \sum_i \alpha_i = 1$. This is also known as the ARCH(N) model (autoregressive models that posit structure on the variance of the errors).

EWMA (exponential weighted moving average) model : Example of the no long run average scheme with $\alpha_{i+1} = (1-\lambda)\alpha_i$ for $0 < \lambda < 1$. By the requirement that $\sum_i \alpha_i = 1$ it actually follows that $\alpha_i = (1-\lambda)\lambda^{i-1}$. The recurrence relation for updating v_i is

$$v_i = \lambda v_{i-1} + (1-\lambda)u_{i-1}^2$$

λ governs how responsive the estimate of daily vol to most recent daily perecentage change. A low λ gives more weight to u_{i-1}^2. In this case the estimates themselves are highly volatile. A large λ produces estimates that respond relatively slowly to new information provided by daily percentage change.

GARCH(1,1) model : Example of the long run average scheme. The updating recurrence relation is

$$v_i = \gamma v_L + \beta v_{i-1} + \alpha u_{i-1}^2,$$

where $\gamma + \alpha + \beta = 1$. In general, the weight applied to u_{N-i}^2 is $\alpha\beta^{i-1}$ (so that they sum up to $\frac{\alpha}{1-\beta}$. The weight of the v_L is $\frac{\gamma}{1-\beta}$. For a stable Garch(1,1) process we require $\alpha + \beta < 1$ otherwise $\gamma < 0$. EWMA is a special case of Garch(1,1) with $\beta = \lambda$ and $\alpha = 1 - \lambda, \gamma = 0$.

In general, GARCH(p,q) model would use p observations on u^2 and the most recent q estimates on v.

GARCH(1,1) is a discretization of a mean reversion model:

$$dv = \gamma(v_L - v)dt + \sqrt{2}\,\alpha\, v\, dz.$$

Reason : from the above we get

$$v_i - v_{i-1} = \gamma(v_L - v_{i-1}) + \alpha(u_{i-1}^2 - v_{i-1}).$$

Now u_{i-1} is Normal with mean 0 and variance v_{i-1}. Therefore u_{i-1}^2 has mean v_{i-1} and variance $2v_{i-1}^2$. Or $u_{i-1}^2 - v_{i-1}$ is approximately $\sqrt{2}v_{i-1}dz$ where dz represents some noise distribution.

Fitting the model : Whether it's EWMA or Garch(1,1), we basically need to choose λ or α, β, γ (or $\omega = \gamma v_L$) to maximize

$$\prod_{i=1}^{N} \frac{1}{\sqrt{2\pi v_i}} e^{-\frac{u_i^2}{2v_i}}.$$

We start by setting $v_2 = u_2^2$ and fill in the table of values for v_i, u_i. An iterative method can be used to then figure out the optimal parameters of α, β, ω. If the optimal ω is negative, we should choose EWMA for stability reason.

Estimating covariance matrix : Denoting x_i, y_i as the percentage change of assets X, Y. We can similarly update v_i^X, v_i^Y as

$$\begin{aligned} v_i^X &= \gamma v_L^X + \beta v_{i-1}^X + \alpha\, x_{i-1}^2 \\ v_i^Y &= \gamma v_L^Y + \beta v_{i-1}^Y + \alpha\, y_{i-1}^2 \\ cov_i &= \gamma cov_L + \beta cov_{i-1} + \alpha\, x_{i-1}y_{i-1} \end{aligned}$$

In doing this, we should keep the the covariance matrix to be positive semi-definite.

Autocorrelation of daily percentage change : It is observed that when u_i^2 is high, $u_{i+1}^2, u_{i+2}^2 \cdots$ also tend to be high and vice versa. That is u_i^2 do exhibit autocorrelation. The Garch(1,1) model works well if it removes the autocorrelation in $\frac{u_i^2}{v_i}$. Table 23.2 shows that this is indeed the case. u_i is the daily percentage change, σ_i is its standard deviation. Thus $\frac{u_i^2}{v_i}$ is a "normalization" of u_i^2 which may remove the autocorrelation structure (the GARCH structure) in its variance.

Using Garch(1,1) to forecast volatility: We have

$$v_n - v_L = \alpha(u_{n-1}^2 - v_L) + \beta(v_{n-1} - v_L),$$

true for all n so that

$$v_{n+t} - v_L = \alpha(u_{n+t-1}^2 - v_L) + \beta(v_{n+t-1} - v_L).$$

Since $E(u_{n+t-1}^2) = v_{n+t-1}$,

$$\begin{aligned} E(v_{n+t} - v_L) &= (\alpha + \beta)(v_{n+t-1} - v_L) \\ &= (\alpha + \beta)^t (v_n - v_L) \end{aligned}$$

or

$$E(v_{n+t}) = v_L + \;=\; (\alpha + \beta)^t (v_n - v_L).$$

This also shows that $E(v_n)$ tends to V_L as n gets large if $\alpha + \beta < 1$.

Volatility term structures: Denoting $V(t) = E(v_t)$ and setting $n = 0$ in the above equation gives

$$\begin{aligned} V(t) &= v_L + e^{-at}(V(0) - v_L), \\ a &= -\log(\alpha + \beta). \end{aligned}$$

V_t is an estimate of the instantaneous variance rate in t days. The average variance rate perday between $[0, T]$ is

$$\frac{1}{T} \int_0^T V(t) dt = v_L + \frac{1 - e^{-aT}}{aT} [V(0) - v_L].$$

Denoting $\sigma(T)^2$ as the volatiltiy per annum that should be used to price a T day option under the Garch(1,1). Then

$$\sigma(T)^2 = 252\Big(v_L + \frac{1 - e^{-aT}}{aT} [V(0) - v_L]\Big).$$

When the current volatility is above the long-term vol, the Garch(1,1) model estimates a downward sloping vol term structure and vice versa. This equation can also be used to analyze the impact of volatility changes. If $\sigma(0)$ moves to $\Delta\sigma(0), \sigma(T)$ changes by approximately

$$\frac{1 - e^{-aT}}{aT} \frac{\sigma(0)}{\sigma(T)} \Delta\sigma(0).$$

Financial institutions then relate the size of the volatility increase that is considered to the maturity of the option, rather than consider an across the board increae of 1 % implied vol in calculating vega.

Case study of EWMA model when estimating VaR: (Section 23.8) Because the date is right after a period of high volatility, the vol estimate using EWMA gives much higher vol than the equal weight approach. Correlation is also higher. This gives a twice as high VaR than the previous estimate using equal weights.

19 Credit risk

Credit ratings: (Moody's) Investment grade: Aaa, Aa, A. Baa. Others: Ba, B, Caa-C. For investment grade bonds, probability of default in a year tends to be an increasing function of time (the longer the time, the greater the possibility that financial health will decline). For the poor credit rating, it tends to be a decreasing function of time (the next couple years are critical, if survived, financial health will improve).

Altman's Z-Score: For publicly traded manufacturing companies, based on discriminant analysis, using five accounting ratios:

$$\begin{aligned} X_1 &: \text{Working capital / Total assets} \\ X_2 &: \text{Retained earnings / Total assets} \\ X_3 &: \text{Earnings before interest and taxes / Total assets} \\ X_4 &: \text{Market value of equity / Book value of total liabilities} \\ X_5 &: \text{Sales / Total assets} \\ Z &= 1.2X_1 + 1.4X_2 + 3.3X_3 + 0.6X_4 + 0.999X_5. \end{aligned}$$

If $Z \geq 3.0$, the company is unlikely to default. If $2.7 \leq Z < 3.0$ it should be "on alert." If $1.8 \leq Z < 2.7$ there is a good chance of default. If $Z < 1.8$ the chance of default is very high.

Conditional and unconditional probability of default: Moody publishes tables for average cumulative default rates Q for different credit ratings (see e.g. Hull Table 19.1). The unconditional probability of default in year n if $Q(n)-Q(n-1)$ (where data for both years $n-1$ and n are available). The conditional probability of defaulting in year n, conditioned on the fact that the company has survived until year $n-1$ is $\frac{Q(n)-Q(n-1)}{1-Q(n-1)}$. The conditional probability of default is the basis for defining the hazard rates.

Hazard rates: (a.k.a default intensity) $\lambda(t)$ so that

$$P(\text{ default in } (t, t+\Delta t) | \text{no default before t}) = \lambda(t)\Delta t.$$

If τ is the time of default then

$$\mathbb{P}(t < \tau < t + \Delta t \mid \tau \geq t) = \lambda(t)\Delta t.$$

Let $Q(t) = \mathbb{P}(\tau \leq t)$ and $V(t) = 1 - Q(t)$. Then

$$V(t + \Delta t) - V(t) = -V(t)\lambda(t)\Delta t.$$

Pushing Δt to 0 gives the solution $V(t) = e^{-\int_0^t \lambda(u)du}$. Thus

$$Q(t) = 1 - e^{-\int_0^t \lambda(u)du} = 1 - e^{-t\bar{\lambda}(t)},$$

where $\bar{\lambda}(t)$ is the average hazard rate for the duration $[0, t]$. Note : In this case the density of τ is $\lambda(t)e^{-\lambda\bar{(t)}t}$.

Recovery rate: denoted by R is the percentage of the asset value recovered in case of default. For a bond, typically defined as the market value a couple days after a default, as a percentage of its face value. Hull's example in table 24.2 has R ranges from 24.7 % to 51.6 %. Recovery rate typically is negatively related to default rate (the more defaults, the less the recovery). In fact, implied probability of defaults are approximately proportional to $\frac{1}{1-R}$. See below for more discussions.

Estimating $\bar{\lambda}(t)$ from bond yield spread: Approximately we have

$$\bar{\lambda}(T)(1 - R) = s(T),$$

where $s(T)$ is the excess of the bond yield over the risk free rate (typically Treasury rate) per annum. This equation means the average loss rates equals the average default rate times the estimated loss percentage. A variation will appear in Merton's model below. Mathematical reason : the value of the bond is

$$\begin{aligned} R\,\tilde{\mathbb{E}}(e^{-r\tau}\mathbf{1}_{\{\tau \le T\}}) + e^{-rT}P(\tau > T) &= R\int_0^T \lambda(t)e^{-(r+\bar{\lambda}(t))t}dt + e^{-(r+\bar{\lambda}(T))T} \\ &= e^{-(r+s(T))T}. \end{aligned}$$

So

$$R\int_0^T \lambda(t)e^{-(r+\bar{\lambda}(t))t}dt + e^{-(r+\bar{\lambda}(T))T} \approx e^{-\left(r+(1-R)\bar{\lambda}(T)\right)T}?$$

If we take the case where λ is constant and look at the first order expansion this is approximately true.

Example : matching bond price assuming piecewise constant hazard rate. Consider a 1 year bond with semiannual coupon rate of 8% per annum and yield 6.5 %. Its present value is 101.33. The risk free rate is 5 % per annum. Thus the bond's risk free value is 102.83. The expected default loss is 1.5. Suppose the recovery rate is 40 % (in reality this rate R may need to be estimated as well. See the CDS section for more discussion). That is the bond is worth 40 dollars in the event of default. Assume $\lambda(t)$ is piecewise constant during any 1 year period. We want to estimate

λ_1 based on this. We assume that bond can only default at the mid point of each 6 month interval (of coupon paying date). The 3 month value of the bond is

$$e^{-0.05\times0.25}4 + e^{-0.05\times0.75}104 = 104.12$$

If default happens at 3 month point the present value of the loss is

$$e^{-0.05\times0.25}(104.12 - 40) = 63.33$$

Similarly the 9 month value of the bond is

$$e^{-0.05\times0.25}104 = 102.71$$

If default happens at 3 month point the present value of the loss is

$$e^{-0.05\times0.75}(102.71 - 40) = 60.40$$

The probability of default happens at 3 month point is $1 - e^{-0.5\lambda1}$ and at 9 month point is $e^{-0.5\lambda1} - e^{-\lambda1}$. Thus λ_1 satisfies

$$(1 - e^{-0.5\lambda1})63.33 + (e^{-0.5\lambda1} - e^{-\lambda1})60.40 = 1.5.$$

If we have a 2 year bond with a semiannual 8% per annum coupon and yield 6.8 % we can calculate λ_2 in a similar way, having known λ_1 to calculate the probability of the 2 year bond defaulting in year 1.

Risk neutral versus physical probability: The default probabilities or hazard rates implied from credit spreads are risk-neutral estimates (Table 24.3, column 3). Default probabilities or hazard rates calculated from historical data are physical default probabilities (Table 24.3, column 2). The risk neutral probability estimate tends to give higher estimates in hazard rates and default probabilities than the real world estimates. This also corresponds to excess return over risk free rate (even after accounting for the spread of historical default). This excess return can be explained by the risks that the holder has to bear, which include systemic risks and idiosyncratic risks that cannot be diversified away. (If there was no expected excess return, the real-world and risk-neutral probabilities would be the same, and vice verssa) For pricing purpose, risk neutral probabilities should be used. When carrying out scenario analyses to calculate potential future loss from defaults, real world probabilities should be used. Thus the abstract reason for the difference in real world versus risk neutral world default probabilities is the same reason as why corporate bond traders earn more than the risk free rate on average. Viewing it from this angle, we then see that one reason is that corporate bonds are relatively illiquid and the higher returns are to compensate for this. By far the most important reason is that bonds do not default independently of each other.Bond traders earn excess expected return for bearing this risk. These reasonings again can be captured in the observation that the expected return (the drift term in the dynamics of the asset) in the risk neutral world is the risk free rate, which is (generally) lower than the expected return of the same asset in the physical world. Viewing it this way, it is not surprising that the default probability in the risk neutral world is higher than the physical world.

Estimating default probabilities using equity price: Let V denotes the value of the company's asset and σ_V its volatility (assumed constant), E the value of its equity and σ_E its instantaneous volatility, D the debt repayment at time T. Then

$$E_T = \max(V_T - D, 0)$$

and

$$\begin{aligned} E_0 &= V_0 N(d+) - De^{-rT} N(d-), \\ d\pm &= \frac{r \pm \frac{1}{2}\sigma_V^2 T - \log \frac{D}{V_0}}{\sigma_V \sqrt{T}}. \end{aligned}$$

The probability of default is $1 - N(d-)$, which requires (non directly observable) σ_V and V_0 to calculate. We also have the concept of distance to default, which is represented by $d-$:

$$d- \quad = \quad \frac{r - \frac{1}{2}\sigma_V^2 T - \log \frac{D}{V_0}}{\sigma_V \sqrt{T}}.$$

As the distance to default declines, the company becomes more likely to default (as $1 - N(d-)$ increases). On the other hand, E_0 and σ_E can be directly observed. We then have

$$E_{\Delta t} - E_0 \approx \frac{\partial E}{\partial t}\Delta t + \frac{\partial E}{\partial V}\Delta V + \frac{1}{2}\frac{\partial^2 E}{\partial V^2}\sigma_V^2 V_0^2 \Delta t.$$

Thus

$$\begin{aligned} \sigma_E = SD(\frac{E_{\Delta t} - E_0}{E_0}) &= \frac{1}{E_0}\frac{\partial E}{\partial V} SD(V_h - V_0) \\ &= \frac{1}{E_0}\frac{\partial E}{\partial V}\sigma_V V_0 \sqrt{\Delta t}. \end{aligned}$$

Taking $\Delta t = 1$ we arrive at

$$\sigma_E E_0 = \frac{\partial E}{\partial V} \sigma_V V_0 = N(d+)\sigma_V V_0.$$

This provides another equation to solve for σ_V and V_0. For example, $E_0 = 3, \sigma_E = 0.8, D = 10, T = 1, r = 0.05$. Then $V_0 = 12.4$ and $\sigma_V = 0.2123$. Probability of default is $N(d-) = 12.7\%$. Market value of the debt is $V_0 - E_0 = 9.4$. Present value of the debt assuming no default is $10e^{-0.05} = 9.51$. Expected loss percentage is $\frac{9.51-9.4}{9.51} = 1.2\%$ of the no default value. Since expected loss (EL) equals prob of default (PD) times (1 - recovery rate),

$$R = 1 - \frac{EL}{PD} = 1 - \frac{1.2}{12.7} = 91\%$$

of the debt's no default value. (Can the 91 % be calculated in another way?)

Performance of the Merton's model: Merton's model provides a good ranking of default probabilities either in risk neutral or the real world. The default probability $1 - N(d-)$ is in theory a risk neutral probability. Thus the ranking of default probabilities in the risk neutral world obtained via Meron's model can be directly translated to the ranking (but not the probabilities itself) in the real world. Moody's KMV and Kamakura provide a service that transforms a default probability produced by Merton's model into a real world probability. CreditGrades uses Merton's model to estimate credit spreads, which are closely linked to risk-neutral dfefault probabilities.

Credit value adjustment (CVA) calculation: If f_{nd} is the no default value to the bank of its outstanding derivative transactions with the counterparty, the value of the outstanding transactions after taking into account possible defaults is $f_{nd} - CVA + DVA$. CVA is the expected loss of the bank from default by the counterparty. DVA is the counterparty's CVA: it is the expected cost to the counterparty because of the default by the bank. CVA reduces the value of the derivative, while DVA increases its value to the bank (if DVA is a cost to the counterparty it must be a benefit ot the bank). We have

$$CVA = \sum_i q_i v_i,$$

where v_i is the present value of the expected loss to the bank if the counterparty defaults at the midpoint of the ith interval and q_i is the risk neutral probability of the counterparty defaulting in the ith interval. Alternatively, we can write

$$CVA = (1 - R) \sum_i q_i v_i,$$

where R is the recovery rate and v_i is the present value of the expected net exposure of the dealer to the counterparty (after collateral) at the mid point of the ith interval, conditional on a default. If $s(t_i)$ is the counterparty credit spread then

$$q(t_i) = \exp(-\frac{s(t_{i+1})t_{i+1}}{1-R}) - \exp(-\frac{s(t_i)t_i}{1-R})$$

v_i is usually calculated via Monte Carlo by simulating different market scenarios and calculating the exposure of the bank to the counterparty at the midpoint of each interval. The exposure E is equal to $\max(V - C, 0)$ where V is the total value of the outstanding transaction to the bank and C is the collateral posted by the counterparty at the time of a default. If C is negative, $-C$ is he collateral posted by the bank with the counterparty at the time of the default. Consider first the three scenarios where $C = 0$ for simplicity:

1. The bank has a short position in an option with the counterparty. In this case the derivative is a liability to the bank and the bank (always) has no credit exposure to the counterparty.

2. The bank has a long position in an option with the counterparty. In this case the derivative is an asset to the bank and the bank (always) has positive credit exposure to the counterparty.

3. The bank has a long (or short) position in a forward contract with the counterparty. In this case, the bank may have a positive or no credit exposure to the counterparty in the future, depending on the price of the underlying asset.

Next consider 4 scenarios where collateral is posted. In this case, it is usually assumed that a period of time elapses between the time when a counterparty stops posting collateral and the close out of transactions. This period is referred to as the cure period (or margin period of risk), typically of 10 or 20 days. The effect of the cure period is that the collateral at the time of default does not reflect the value of the portfolio at the time of default. It reflects the value 10 or 20 days earlier. Thus the Monte Carlo simulation to calculate v_i mentioned above should be structured so that the value of the derivatives portfolio with the counterparty is calculated at time $t_i^* - c$ as well as time t_i^* where t_i^* is the midpoint between t_{i-1} and t_i and c is the length of the cure period. The following 4 scenarios illustrate the possibilities. It assumes a two way zero threshold collateral agreement between the bank and its counterparty. This means that the collateral posted by one side at the time of a default equals $\max(V, 0)$ where V is the value to the other side 20 days earlier (the cure period).

1. On a particular simulation trial, the value of outstanding transaction to the

bank at time τ is 50 and their value 20 days earlier is 45. The calculation assumes that the bank has collateral worth 45 in the event of a default at time τ. The bank's exposure is 5.

2. On a particular simulation trial, the value of outstanding transaction to the bank at time τ is 45 and their value 20 days earlier is 50. The calculation assumes that the bank has collateral worth 50 in the event of a default at time τ. The bank's exposure is 0.

3. On a particular simulation trial, the value of outstanding transaction to the bank at time τ is -50 and their value 20 days earlier is -45. The calculation assumes that the bank has posted collateral worth 45 which is less than 50 in the event of a default at time τ. The bank's exposure is 0.

4. On a particular simulation trial, the value of outstanding transaction to the bank at time τ is - 45 and their value 20 days earlier is -50. The calculation assumes that the bank has posted collateral worth 50 which is less than 45 in the event of a default at time τ. The bank's exposure is 5.

Two special cases for CVA:

a) The portfolio consists of a single uncollateralized derivative that provides payoff to the bank at time T (ex: the bank bought a European option with maturity at time T from the counterparty). Then $v_i = V_{nd}(1-R)$ where V_{nd} is the no-default value of the derivative today. Thus

$$CVA = (1-R)f_{nd}\sum_i q_i.$$

Note also that in this case DVA = 0 and thus the value f of the derivative today after adjusting for credit risk is

$$f = f_{nd} - (1-R)f_{nd}\sum_i q_i.$$

On the other hand, this equation also applies to a T year zero coupon bond issued by the counterparty (assuming the same recovery rate):

$$B = B_{nd} - (1-R)B_{nd}\sum_i q_i.$$

This implies that $\frac{f}{f_{nd}} = \frac{B}{B_{nd}}$. But there are simpler expression for B and B_{nd} : $B = e^{-yT}$ and $B_{nd} = e^{-y_{nd}T}$ where y_{nd} is the yield of a riskless similar bond. Thus we have a simpler expression for f in terms of f_{nd} :

$$f = f_{nd}e^{-(y-y_{nd})T}.$$

That is the risk adjusted value of the derivative can be evaluated by its no default value and the credit spread of the counterparty.

b) The portfolio consists of a long position in a forward contract with strike K for an asset with futures price F_t. The value of the transaction at time t is

$$(F_t - K)e^{-r(T-t)}.$$

The bank's exposure at time t is

$$\max((F_t - K)e^{-r(T-t)}, 0) = e^{-r(T-t)} \max((F_t - K), 0).$$

The expected present value of this exposure (from Black's formula) is

$$\begin{aligned} w(t) &= e^{-rt}e^{-r(T-t)}(F_0 N(d+) - K(d-)) \\ &= e^{-rT}(F_0 N(d+) - K(d-)) \end{aligned}$$

where

$$d\pm = \frac{\pm\frac{1}{2}\sigma^2 t - \log\frac{K}{F_0}}{\sigma\sqrt{t}}.$$

Finally $v(t_i) = (1 - R)w(t_i)$.

Peak exposure: In addition to CVA, the bank may also be interested in the peak exposure at a time point t_i^*, which is a high percentile (97.5 %) of the exposure given by the Monte Carlo simulation. The maximum peak exposure is the maximum of the peak exposures across all time t_i^*. On the other hand, there is a theoretical issue with the approach, since peak exposure is about scenario analysis (the bank is asking "how bad can the exposure get in the future") which should be carried out under the real world probability. On the other hand, the CVA calculation is done under the risk neutral probability since it is essentially about "pricing" (evaluating) the exposure at time t_i^* at the present time.

Wrong way risk: The calculation of CVA assumes the indpendence between default and exposure level. This needs not be the case. In fact, there may be a positive correlation between the two, which is referred to as the wrong way risk. There may also be a negative correlation between the two, which is referred to as the right way risk. An example of wrong way risk is when a counterparty is using a credit default swap to sell protection (to a different reference entity) to the bank. Suppose the credit spread of the entity increases, the swap's value to the bank becomes positive (the bank's exposure to the counterparty increases). On the other hand, since credit spreads of different companies tend to be correlated, the default probability of the counterparty also increases. An example o wright way risk is when a counterparty buys credit protection from the bank, with similar reasoning. A simple way of dealing with wrong way risk is to increase the value of CVA by an alpha factor, which is typically about 1.07 to 1.10 as reported by the banks.

Expected exposure of interest rate swaps vs currency swaps: The expected exposure of interest rate swaps starts at zero, increases then decreases. By contrast, the expected exposure on the currency swaps increases steadily with the passage of time. The main reason is because principals are exchanged at the end of the life of a currency swap and there is uncertainty about the exchange rate at that time. By contrast, toward the end of the life of the interest rate swap, there is very little remaining to be exchanged. The impact of default risk for a currency swap dealer is therefore much higher than for an interest rate swap dealer. The probability of defaults q_i are the same (if with the same counterparty) but the exposure v_i are on average greater for currency swaps.

Modeling correlated default using Gaussian copula : Let $Q_i(t)$ be the cumulative distribution of $\tau_i, i = 1, \cdots, n$. Then $N^{-1}(Q_i(\tau_i))$ has standard Normal distribution. To simulate n default times with the same marginal distribution but with a correlation structure, we first simulate a n multivariate Gaussian distribution with a correlation structure. To decide whether the ith default has happened by time T, we compare X_i with $N^{-1}(Q_i(T))$. Since

$$P(X_i < N^{-1}(Q_i(T))) = P(N(X_i) < Q_i(T)) = Q_i(T).$$

(If X is a RV then $F_X(X)$ has Uniform distribution :

$$P(F_X(X) < u) = P(X < F_X^{-1}(u)) = F_X(F_X^{-1}(u)) = u.)$$

Modeling correlated default using factor based structure : Instead of generating a multivariate Normal distribution, we can also defined

$$X_i = a_i F + \sqrt{1 - a_i^2} Z_i,$$

where F, Z_i are iid standard Normals. Then $Cor(X_i, X_j) = a_i a_j$. The probability of the ith default having happened by time T becomes

$$P(X_i < N^{-1}(Q_i(T))) \quad = \quad P\Big(Z_i \leq \frac{N^{-1}(Q_i(T)) - a_i F}{\sqrt{1 - a_i^2}}\Big) = Q_i(T),$$

as before. We express the default probability of the ith company as:

$$Q_i(T|F) \quad = \quad N\Big(\frac{N^{-1}(Q_i(T)) - a_i F}{\sqrt{1-a_i^2}}\Big).$$

Note that the expression $N\Big(\frac{N^{-1}(Q_i(T))-a_iF}{\sqrt{1-a_i^2}}\Big)$ by itself is random while the expression $P\Big(Z_i \leq \frac{N^{-1}(Q_i(T))-a_iF}{\sqrt{1-a_i^2}}\Big)$ is a constant. Thus $Q_i(T|F)$ expresses the procedure to generate a set of correlated distributions whose marginals will be the same as $Q_i(T)$. That is $E(Q_i(T|F)) = Q_i(T)$. It comes from the following observation :

$$\int_y P(X < g(Y)|Y) f_Y(y) dy = P(X < g(Y)),$$

where the RHS is over a pair of (X, Y) whose marginals equal to the LHS X, Y and jointly independent. The quantity $Q_i(T|F)$ is usually used in between the modelling for correlation effect. The final result (e.g. price of a derivative) will be integrated over F to obtain the unconditional value (again see the CDS chapter for example).

If $a_i = \rho$ and $Q_i(T) = Q(T), \forall i$ then the above equation reduces to

$$Q(T|F) \quad = \quad N\Big(\frac{N^{-1}(Q(T)) - \rho F}{\sqrt{1-\rho^2}}\Big).$$

Credit risk VaR: Credit loss over a certain time period that will not be exceeded within a certain confidence level. Credit risk VaR models may either consider losses only from defaults or losses from downgrades / credit spread changes as well as defaults.

Vasicek's model: For a portfolio of loans with individual marginal default distribution $T(t) := P(\tau \leq t)$ and default correlation ρ, the propotion of loans defaulting by time t that will not be exceeded within X confidence level is

$$WCD(t, X) = N(\frac{N^{-1}(T(t)) + \sqrt{\rho} N^{-1}(X)}{\sqrt{1-\rho}}).$$

Therefore, the t-days $X-$ level credit VaR given by Vasicek model for a portfolio of loans is

$$WCD(t, X) \times (1 - R) \times EAD,$$

where EAD is the exposure (of the portfolio) at default and R is the recovery rate. The portfolio of loans can also be viewed as a portfolio on assets of companies whose default distribution and correlation are given by $T(t)$ and ρ. If we have n such (sub) portfolios to form a large portfolio (each sub-portfolio may represent an area of the industry), it is approximately true that the Xth percentile of the loss distribution is

$$\sum_{i=1}^{n} WCD_i(t, X) \times (1 - R_i) \times EAD_i.$$

Using Merton's model of company's default, the correlation ρ between two companies' default times can be showed to be roughly equal to the correlation between the returns on their assets. This correlation again can be approximated by the correlation between the returns of their equities.

Credit risk plus: A methodology to calculate credit VaR from Credit Suisse Financial Products. It relies on the Gamma-Poisson mixture result in probability (whhich is also used in the insurance industry) which says that if X has a Possion(λ) distribution where λ is also a Gamma($r, \frac{p}{1-p}$) distribution (where r and $\frac{p}{1-p}$ are the shape and scale parameters) then X equivalently has a Negative Binomial (r,p) distribution. See e.g. Negative Binomial. In the context of defaults, we assume that the average default rate over a period of time T is $\lambda = np$ where p is small and n is large (n is the total number of companies and p is the probability of individual company defaulting). The number of defaults in this period of time can be approximated as a Poission (λ) distribution. The expected number of defaults is not a deterministic constant. We thus assume it has a Gamma distribution with mean μ and variance σ^2. (A Gamma (k, θ) distribution has mean $k\theta$ and variance $k\theta^2$ - when k is an integer it is equivalent to the sum of k iid Exp($\frac{1}{\theta}$) distribution). The reason for Gamma distribution assumption may come from the Gamma process model for the average defaults at different time intervals. That is we assume the average number of defaults follow a Gamma process. See e.g. Gamma process. The number of defaults M in the time interval T then follows a negative binomial distribution:

$$\begin{aligned} P(M = m) &= p^m (1-p)^\alpha \frac{\Gamma(m+\alpha)}{\Gamma(m+1)\Gamma(\alpha)} \\ \alpha &= \frac{\mu^2}{\sigma^2}, p = \frac{\sigma^2}{\mu + \sigma^2}. \end{aligned}$$

Note that $\Gamma(n) = (n-1)!$ for n integer. But α may not be an integer so the above formulation is more general. The $X-$ percentile of the number of losses M can be computed using the negative binomial model, which depends on μ, σ. When $\sigma \to 0$ the distribution converges to a Poisson distribution. The X percentile of the loss distribution is then calculated based on the X percentile of M.

Credit plus model can be characterized as a default rate uncertainty model (the uncertainty is the distribution of the average default rate λ). Without this uncertainty (namely when $\sigma = 0$), there is very little chance of a large number of defaults. As the uncertainty increases, a large number of defaults become more likely. Thus Credit plus captures default correlation with the uncertainty of the default rate. The loss distribution under credit plus model also has positive skewness when the uncertainty is non-zero (which may agree with empirical data). Without default correlation, the loss probability distribution is fairly symmetric.

Monte Carlo simulation: Vasicek and Credit risk plus provides models for computing percentiles of the loss distribution based on either the proportion of defaults (Vasicek) or the number of defaults (Credit risk plus). In practice, a bank may have a number of different categories of exposures with different default rate for each category. In this case, Monte Carlo simulation provides a flexible alternative to the modelling approach. A simulation may proceed as follows:

1. Develop a model relating the default rate in each category to the overall default rate. This could be via regressing the specific default rate against the overall default rate (over a period of time) or from some structural model.

2. Sample an overall default rate. This could be from a table for the annual percentage default frate for all rated companies in a period of time (table 11.4 in Hull) or from a default rate model. In particular, annual default rates are not independent. Thus randomly sampling a default rate from a table to determine next year's default rate may not be the best approach. It may be preferrable to develop a model that relates the rate of one year to the rate or other economic variables of the previous year .

3. Sample a number of defaults for each category using the rates in step 1.

4. Sample a loss given default for each default in each category. This could be from a model of loss distribution, exposure level and recovery. It could also be from the nature of the financial products / contracts the bank has in each category.

5. Calculate the total loss from defaults.

6. Repeat steps 1-5 to construct a total loss probability distribution.

7. Calculate the VaR from the total lss distribution.

Credit metrics: Vasicek and Credit risk plus gives the loss percentile based on default events, but not on credit downgrading. In practice, a change in credit rating would affect the default probability (or equivalently the credit sread) which would in turn affect the credit risk VaR. CreditMetrics, proposed my JPMorgan in 1997 is a credit VaR model that takes into account both defaults and downgradings. It is based on a (annual) rating transition matrix such as the one given in Hull table 21.1. Monte Carlo simulation is used to calculate a one year credit VaR for a portfolio of transactions with many counterparties. A simulation may proceed as follows:

1. Use the rating transition matrix to simulate the credit ratings of all counterparties at the end of the year. A correlation structure of credit ratings change may need to be incorporated using the Gaussian copula model. Specifically if there are n counterparties with a correlation structure Σ, we can generate a multivariate Normal $(X_1, \cdots, X_n)$ with covariance matrix Σ and the rating change of the ith counterparty is decided by how $N(X_i)$ is compared with its corresponding transition matrix entries.

2. Calculate the credit loss for each counter party. If the end-of-year credit rating is default, the credit loss is the exposure at default times $(1 - R)$. If the rating is not default, the credit loss is

$$\sum_{i=j}^{n}(1 - R)(q^*i - q_i)v_i.$$

Here we follow the set up of CVA calculation, where the time horizon is divided into n transactions. The jth interval represents the 1 year point. v_i is the present value of the exposure if default at the midpoint of the ith interval. q_i is the original probability of default (based on the present rating) and q_i^* is the new probability of default based on the end of year rating. Determination of q_i^* requires the term structure of credit spreads at the one year point and determination of q_i requires the term structure of the credit spreads at the present time. (Recall that $\bar{\lambda}_i(1 - R) = s_i$ and $q_i = e^{-\bar{\lambda}_{i-1}t_{i-1}} - e^{-\bar{\lambda}_i t_i}$.)

Example: Suppose a company owns a two year zero coupon bond with principal of 1,000. Suppose the risk free rate is 3 % and the current credit spread is 200 basis points (2 %). Thus the bond yield is 5 % and its current price (with annual compounding) is $1,000/1.05^2 = 907.03$. Uppose that the bond's current rating is BB and during the next year there is 0.3 % that it wil increase to BBB, 99.2 % that it will stay the same and 0.4 % chnace that it will decrease to B and 0.1 % chance that it will default. If defaults, the bond is worth 400 (so $R = 40\%$). For each possible rating category there are 2 equally likely credit spreads. In basis points there are 100 and 120 for BBB, 200 and 240 for BB and 450 and 500 for B.

If the bond defaults, it is worth 400 and the present value is 400/ 1.03 = 388.34. The credit loss is 907.03 - 388.34 = 518.69. This event has probability 0.1 %.

If the bond's rating is BBB, there are two possibilities each with probability 0.15 %. The first is when the credit spread is 100 with expected present value

$$\frac{1000}{1.04 \times 1.03} = 933.53.$$

The credit loss is 907.03 - 933.53 = -26.5. Note that here the credit loss is negative since actually it is a "gain." The second is when the credit spread is 120 with expected present value

$$\frac{1000}{1.042 \times 1.03} = 931.74.$$

The credit loss is 907.03 - 931.74 = -24.71. With other similar calculations the distribution of the credit loss is calculated and the credit VaR is the appropriate percentile of the credit loss.

20 Chapter 25: Credit derivatives

Credit default swap (CDS): A contract that provides insurance against the risk of default of a particular company (a reference entity). The default is known as a credit event. The buyer of the CDS makes periodic **payments** (determined by the notional principal L, the CDS spread s and the frequency of payment (typically quarterly)) to the seller until the end of the life of the CDS or until the credit event occurs. The buyer receives the **payoff** of $L(1-R)$ upon default.

Use of CDS in hedging: Suppose an investor buys a 5 year corporate bond with 7 % yield per annum and at the same time enters into a 5 year CDS against the bond default. The effect of the CDS is to convert the corporate bond into a risk free bond. Therefore in principle, the CDS spread should be equal to the bond yield spread. In reality this is rarely the case. CDS-bond basis = CDS spread - Bond yield spread was positive pre-2007 and negative 2007-2009. This sign can also depend on a number of factors:

1. The bond may sell for a price that is significantly different from par (above par leads to negative basis while below par leads to positive basis)

2. There is counterparty default risk in a CDS (negative basis direction)

3. Cheapest to deliver bond in a CDS (positive basis direction)

4. Payoff in CDS does not include accrued interest on the bond that is delivered (negative basis direction)

5. Restrucuring clause in a CDS contract may lead to a payoff when there is no default (positive basis direction)

6. LIBOR is greater than the risk free rate assumed by the market (positive direction).

Evaluation of CDS: To find the CDS spread of a particular reference entity. Standard assumption: Default only happens at midpoint of the periodic payments. Recall that

$$Q(t) = \mathbb{P}(\tau \leq t)$$

and

$$V(t) = 1 - Q(t) = \mathbb{P}(\tau > t)$$

are the probabilities of default and survival of the entity by time t. We suppose that the payment for the CDS is made at times t_i and $Q(t_i), V(t_i), i = 1, \cdots, n$ are given (or calculated from the hazard rate). Suppose that the risk free rate is r and the CDS spread is s. Then s satisfies

$$A(s) + B(s) = C,$$

where $A(s)$ is the present value of the expected payments, $B(s)$ is the present value of the accrual payment (from the time of the previous payment to default) and C is the present value of the expected payoff. $A(s), B(s)$ are from the buyer of the CDS and C is from the seller of the CDS. The notional amount L does not appear in the equation since it gets cancelled out on both sides. We have

$$\begin{aligned} A(s) &= \sum_{i=1}^{n} e^{-rt_i} V(t_i) s \, \Delta t \\ B(s) &= \sum_{i=0}^{n} e^{-r\frac{t_i+t_{i+1}}{2}} (Q(t_{i+1}) - Q(t_i)) s \, \frac{\Delta t}{2} \\ C &= (1-R) \sum_{i=0}^{n} e^{-r\frac{t_i+t_{i+1}}{2}} (Q(t_{i+1}) - Q(t_i)). \end{aligned}$$

Here we assume $t_{i+1} - t_i = \Delta t$ is a constant.

CDS marked to market: The CDS spread s is a function of time. Thus a CDS has zero value at the beginning but may have positive or negative value after. In fact, a similar caculation as above (provided the present time is t_0) can be used to show that the current value of the CDS to the buyer is

$$L\big(A(s_2) + B(s_2) - [A(s_1) + B(s_1)]\big)$$

where s_1 is the orginal CDS spread and s_2 is the current CDS spread. The value of the CDS to the seller is the opposite. Thus if the spread goes down, the CDS becomes negative in value to the buyer and positive to the seller.

Dependence of implied probability of defaults on recovery rate R : We need to estimate the recovery rate to evaluate the CDS spread. However, as long as we use the same recovery rate in estimating the default probabilities and valuing the CDS, the CDS spread is not sensitive to this rate. We have discussed that implied probability of defaults are approximately proportional to $\frac{1}{1-R}$. On the other hand, the payoffs from the CDS are proportional to $1-R$. This argument does not apply to the binary CDS below.

Binary CDS: A CDS where the payoff is 1 instead of 1 - R (or equivalently $R = 0$).

CDS forward and options: Similar to forward and option on other assets, where the strike is quoted in basis points for the CDS spread to be entered with.

Total return swaps: A credit derivative to exchange the total return on a bond (coupons, interests, gain or loss on the bond value) for LIBOR plus a spread. **The payer** pays coupons earned on the bond. **The receiver** pays LIBOR plus a spread on the same principal. At the end of life of the swap T, the payer pays $V_T - V_0$ to the receiver where V is the value of the bond (the payer receives the amount $V_T - V_0$ if it is negative). Thus if there is default on the bond, the payer would receive $(1-R)L$ from the receiver and the swap is terminated. If the payer actually owns the bond, the total return swap allows it to pass the credit risk of the bond to the receiver. The total return swap spread reflects the credit risk of the receiver (the receiver itself can default on the swap.) The total return swap is also a financial tool for the receiver to invest in the corporate bond without having to make the actual investment (the payer would do the actual investment in the bond). The receiver in effect is taking out a loan on the bond principal with rate LIBOR plus spread, which reflects its own credit risk.

Synthetic CDOs: Recall : (cash) CDO is an ABS on bonds. Synthetic CDOs create the CDOs wihtout actually acquiring the bonds, based on the observation that a long position on a corporate bond is the same as a short position on the CDS on the same corporation. The synthetic CDO originator chooses a portfolio of companies and a maturity T (for example, an idex of companies such as the iTRaxx Europe or the CDX NA IG in America). It sells CDS protection on each company with the same maturity T (thus in effect as buying bonds from the corporations). The synthetic CDO principal is the total of the notional principals underlying the CDSs. The originator has cash inflows equal to the CDS spreads and cash outflows when the companies in the portfolio default. Tranches are formed and the cash inflows and outflows are distributed to tranches. The tranche cashflow rules are as followed: Each tranche is responsible for the payoffs on the CDSs above its attachment point α_L and below its detachment point α_H of the synthetic CDO principal. It earns its own spread of s basis points per year on the outstanding principal of its tranche. For example, suppose there are 3 tranches: senior, mezzanie and equity with proportions (80 %, 15 %, 5%) respectively on a principal of 100 millions. There is a loss of 2 millions in the first year by the CDSs payout. The equity tranche principal is reduced to 3 millions and it earns its spread on 3 millions after that. If the loss is above 5 millions then the equity tranche is wiped out and the mezzanine tranche principal is reduced etc. In this way, the CDO originator sells protection on the companies to some market participants and the tranche holders sell protection to the originator.

Single tranche trading: trading of a specific tranche of the synthetic CDO without the portfolio of CDSs being created. The CDS portfolio is used as reference for cashflows. The buyer pays the tranche spread (on the tranche principal) to the seller and the seller pays the CDS payouts that the tranche is responsible for to the buyer. Thus the buyer can be viewed as buying protection for the tranche (by paying the tranche spread) and the seller can be viewed as selling protection for the tranche (by receiving the tranche spread).

Valuing of a particular synthetic CDO tranche: Here we assume E_i as the expected tranche principal at time $t_i, i = 0, \cdots, n$. The calculation is very similar to CDS calculation with

$$A(s) + B(s) = C,$$

where $A(s)$ is the present value of the expected tranche payment, $B(s)$ is the present value of the expected accrual payment and C is the present value of the expected payoffs. We also assume defaults of the reference entities only happen at the midpoint of the payments. We have

$$\begin{aligned}
A(s) &= \sum_{i=1}^{n} e^{-rt_i} E_i s(t_i - t_{i-1}) \\
B(s) &= \sum_{i=1}^{n} e^{-r\frac{t_i+t_{i-1}}{2}} (E_{i-1} - E_i) s \frac{t_i - t_{i-1}}{2} \\
C &= \sum_{i=1}^{n} e^{-r\frac{t_i+t_{i-1}}{2}} (E_{i-1} - E_i).
\end{aligned}$$

Here $E_{i-1} - E_i$ is the expected tranche loss in between the (i-1)th and ith payments.

Calculation of the expected tranche principal: We use the Gaussian copula model in the previous chapter to model N reference entities in the portfolio. Suppose all companies have the same probability $Q(t|F)$ of defaulting by time t where

$$Q(t|F) = N\Big(\frac{N^{-1}(Q(t)) + \rho N^{-1}(X)}{\sqrt{1-\rho^2}}\Big),$$

where $Q(t) = 1 - e^{-\lambda t}$, for example. The probability of exactly k defaults by time t is then

$$P(k, t|F) = \binom{n}{k} Q(t|F)^k (1 - Q(t|F))^{N-k}.$$

Let α_L, α_H be the attachment and detachment points of the tranche. Also assume the notional principal L on each company and their recovery rates are the same. The tranche is responsible for the defaults amount k such that

$$nL\alpha_L \leq (1-R)Lk \leq nL\alpha_H.$$

In other words

$$\lceil n_L \rceil := \lceil \frac{n\alpha_L}{1-R} \rceil \leq k < \lceil \frac{n\alpha_H}{1-R} \rceil = \lceil n_H \rceil.$$

Thus the tranche principal is 1 if $k < \lceil n_L \rceil$ and 0 if $k \geq \lceil n_L \rceil$. It is

$$\frac{\alpha_H - \frac{k(1-R)}{n}}{\alpha_H - \alpha_L}$$

otherwise. This is the tranche percentage, but in calculation of CDS spread the notional principal does not enter. Then the expected tranche principle E_i (conditional on the factor F) at time t_i is

$$E_i(F) = \sum_{k=0}^{\lceil n_L \rceil - 1} P(k, t_i|F) + \sum_{\lceil n_L \rceil}^{\lceil n_H \rceil - 1} P(k, t_i|F) \frac{\alpha_H - \frac{k(1-R)}{n}}{\alpha_H - \alpha_L}.$$

Thus the quantity $A(s), B(s), C$ we calculated above will be dependent on F. To obtain the tranche spread, we integrate these quantities over F and use the relation $A(s) + B(s) = C$ again.

kth to default CDS: a CDS that provides payoff when the kth default occurs in a number of reference entities. It is part of a family called basket CDS. The evaluation of a kth to default CDS is similar to the vanilla CDS :

$$\begin{aligned} A(s) &= \sum_{i=1}^{n} e^{-rt_i} V(t_i) s \, \Delta t \\ B(s) &= \sum_{i=0}^{n} e^{-r\frac{t_i+t_{i+1}}{2}} (Q(t_{i+1}) - Q(t_i)) s \, \frac{\Delta t}{2} \\ C &= (1-R) \sum_{i=0}^{n} e^{-r\frac{t_i+t_{i+1}}{2}} (Q(t_{i+1}) - Q(t_i)), \end{aligned}$$

where we replace

$$\begin{aligned} Q(t_i) &= \sum_{m=k}^{N} P(m, t_i | F) \\ V(t_i) &= 1 - Q(t_i). \end{aligned}$$

21 Exotic options

Variance swap : Consider an asset with dynamics

$$dS_t = (r-q) S_t dt + \sigma(t) S_t d\tilde{W}_t.$$

A variance swap with expiration T pays the holder

$$V_T = L\left(\frac{\int_0^T \sigma(t)^2 dt}{T} - V_K\right)$$

at time T where L is some notional amount (taken to be 1 here for convenience) and V_K is some fixed variance rate.

Remark : If we discretize $0 = t_0 < t_1 < \cdots < t_n = T$ days then

$$\bar{v}(T) := \frac{\int_0^T \sigma(t)^2 dt}{T}$$

can be approximated as

$$\frac{252}{n-2} \sum_{i=0}^{n-1} \left[\log \frac{S_{t_{i+1}}}{S_{t_i}} \right]^2.$$

Reason:

$$
\begin{aligned}
\log \frac{S_{t_{i+1}}}{S_{t_i}} &= \int_{t_i}^{t_{i+1}} (r - q - \frac{1}{2}\sigma^2(t)dt + \int_{t_i}^{t_{i+1}} \sigma(t)d\tilde{W}_t \\
\sum_{i=0} \big[\int_{t_i}^{t_{i+1}} \sigma(t)d\tilde{W}_t \big]^2 &\approx \int_0^T \sigma^2(t)dt \\
T &= \frac{n-2}{252},
\end{aligned}
$$

where the last equation converts the number of trading days to years.

Pricing variance swap : We have

$$
d\log S_t = (r - q - \frac{1}{2}\sigma^2(t))dt + \sigma(t)d\tilde{W}_T.
$$

Thus

$$
\int_0^T \frac{1}{2}\sigma^2(t)dt = (r-q)T + \int_0^T \sigma(t)d\tilde{W}_T - \log \frac{S_T}{S_0}.
$$

That is

$$
\begin{aligned}
\tilde{\mathbb{E}}\bar{v}(T) &= \frac{2}{T}(r-q)T - \frac{2}{T}\tilde{\mathbb{E}} \log \frac{S_T}{S_0} \\
&= \frac{2}{T} \log \frac{F_0}{S_0} + \frac{2}{T}\tilde{\mathbb{E}} \log \frac{S_0}{S_T}
\end{aligned}
$$

where we recall that $F_0 = e^{(r-q)T} S_0$.

We claim that for any value S^* :

$$
\int_0^{S^*} \frac{1}{K^2}(K - S_T)^+ dK + \int_{S^*}^{\infty} \frac{1}{K^2}(S_T - K)^+ dK = \log \frac{S^*}{S_T} + \frac{S_T}{S_*} - 1.
$$

Reason: If $S^* < S_T$ then

$$
\begin{aligned}
\int_0^{S^*} \frac{1}{K^2}(K - S_T)^+ dK &= 0 \\
\int_{S^*}^{\infty} \frac{1}{K^2}(S_T - K)^+ dK &= \int_{S^*}^{S_T} \frac{1}{K^2}(S_T - K)dK \\
&= \log \frac{S^*}{S_T} + \frac{S_T}{S_*} - 1.
\end{aligned}
$$

If $S^* \geq S_T$ then

$$
\begin{aligned}
\int_{S^*}^{\infty} \frac{1}{K^2}(S_T - K)^+ dK &= 0 \\
\int_0^{S^*} \frac{1}{K^2}(K - S_T)^+ dK &= \int_{S_T}^{S^*} \frac{1}{K^2}(K - S_T)dK \\
&= \log \frac{S^*}{S_T} + \frac{S_T}{S_*} - 1.
\end{aligned}
$$

Thus choosing $S^* = F_0$ (which may not possible in practical application) we have

$$\frac{2}{T}\Big[\int_0^{F_0} \frac{1}{K^2}e^{rT}p(K)dK + \int_{F_0}^{\infty} \frac{1}{K^2}e^{rT}c(K)dK\Big] = \tilde{E}\log\frac{F_0}{S_T} = \tilde{\mathbb{E}}\bar{v}(T).$$

More generally we have

$$\tilde{\mathbb{E}}\bar{v}(T) = \frac{2}{T}\Big[1 + \log\frac{F_0}{S^*} - \frac{F_0}{S^*} + \int_0^{S^*} \frac{1}{K^2}e^{rT}p(K)dK + \int_{S^*}^{\infty} \frac{1}{K^2}e^{rT}c(K)dK\Big].$$

In practice, we generally choose $S*$ to equal to the first strike price below F_0 and approximate

$$\int_0^{S^*} \frac{1}{K^2}(K - S_T)^+ dK + \int_{S^*}^{\infty} \frac{1}{K^2}(S_T - K)^+ dK$$

with

$$\sum_{i=1}^{n} \frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i),$$

where $Q(K_i)$ is the price of Euro put when $K_i < S^*$ and the price of Euro call when $K_i = S^*$. If $K_i = S^*$ we set $Q(K_i)$ to be the average of the price of Euro call and put. This also suggests a way to replicate a variance swap with a portfolio of Euro calls an puts.

Volatility swap: A volatility swap with expiration T pays the holder

$$\begin{aligned} V_T &= L\Big(\sqrt{\frac{\int_0^T \sigma(t)^2 dt}{T}} - \sigma_K\Big) \\ &= L(\sqrt{\bar{v}(T)} - \sigma_K) = L(\bar{\sigma}(T) - \sigma_K) \end{aligned}$$

at time T where L is some notional amount (taken to be 1 here for convenience) and V_K is some fixed variance rate.

Pricing of volatility swap: We need to evaluate

$$\tilde{E}\sqrt{\bar{v}(T)}.$$

We have

$$\sqrt{\bar{v}(T)} = \sqrt{\tilde{E}\bar{v}(T)}\sqrt{1 + \frac{\bar{v}(T) - \tilde{E}\bar{v}(T)}{\tilde{E}\bar{v}(T)}}.$$

The second order xxpansion of $\sqrt{1+x}$ around 0 is

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2.$$

Thus

$$\sqrt{1 + \frac{\bar{v}(T) - \tilde{E}\bar{v}(T)}{\tilde{E}\bar{v}(T)}} \approx 1 + \frac{1}{2}\frac{\bar{v}(T) - \tilde{E}\bar{v}(T)}{\tilde{E}\bar{v}(T)} - \frac{1}{8}\left(\frac{\bar{v}(T) - \tilde{E}\bar{v}(T)}{\tilde{E}\bar{v}(T)}\right)^2.$$

Thus

$$\tilde{E}\bar{\sigma}(T) \approx \sqrt{\tilde{E}\bar{v}(T)}\Big(1 - \frac{1}{8}\frac{v\tilde{a}r[\bar{v}(T)]}{[\tilde{E}\bar{v}(T)]^2}\Big).$$

Thus in pricing the volatility swap, the variance of the average variance rate during the life of the contract is required.

VIX index: The CBOE publishes indices of implied volatility. The most popular index, SPX VIX is the index of the implied vol of 30 day options on the S&P 500 calculated from a wide range of calls and puts. Trading on futures and options on the VIX is available in the mid 2000s. A trade involving futures or options on the S&P 500 is a bet on both the future level of the S&P 500 and its volatility. On the other hand, a futures or options contract on the VIX is a bet only on volatility. In the equation

$$\tilde{\mathbb{E}}\bar{v}(T) = \frac{2}{T}\Big[1 + \log\frac{F_0}{S^*} - \frac{F_0}{S^*} + \int_0^{S^*}\frac{1}{K^2}e^{rT}p(K)dK + \int_{S^*}^{\infty}\frac{1}{K^2}e^{rT}c(K)dK\Big]$$

we can approximate $\log(x)$ around $x = 1$

$$\log x \approx (x-1) - \frac{1}{2}(x-1)^2$$

to have

$$\tilde{\mathbb{E}}\bar{v}(T)T \approx -(\frac{F_0}{S^*} - 1)^2 + 2\sum_{i=1}^{n}\frac{\Delta K_i}{K_i^2}e^{rT}Q(K_i).$$

This is the equation that is based on to calculate the VIX. On any given day, $\tilde{\mathbb{E}}\bar{v}(T)T$ is calculated for options traded in the market with maturities immediately above and below 30 days. The 30 day risk neutral expected cumulative variance is calculated from these two numbers using interpolation. The result is multiplied with 365/30 and the index is set equal to the square root of the result.

22 Chapter 21, 27: Models and Numerical Procedures

Determination of p, u, d in tree model: p, u, d must match the first two moments of the asset return during a time interval of length δt. An additional condition is Cox, Ross, Rubinstein :

$$d = \frac{1}{u}.$$

In a binomial tree for an asset with dividend rate q, the model is

$$\begin{aligned} S_{i+1} &= S_i u \text{ with probability } p \\ &= S_i d \text{ with probability } 1-p. \end{aligned}$$

On the other hand, the log return is

$$\log \frac{S_{t+\Delta t}}{S_t} = (r-q)\Delta t + \sigma(\tilde{W}_{t+\Delta t} - \tilde{W}_t).$$

Hence

$$\begin{aligned} \widetilde{\mathbb{E}} \log(\frac{S_{i+1}}{S_i}\Big|\mathcal{F}_i) &= (r-q)\Delta t \\ \widetilde{Var} \log(\frac{S_{i+1}}{S_i}\Big|\mathcal{F}_i) &= \sigma^2 \Delta t. \end{aligned}$$

That is

$$\begin{aligned} p \log u + (1-p) \log d &= (r-q)\Delta t \\ p(\log u)^2 + (1-p)(\log d)^2 &= \sigma^2 \Delta t + [(r-q)\Delta t]^2. \end{aligned}$$

A solution to these equations, when terms of higher order than Δt are ignored is

$$\begin{aligned} p &= \frac{e^{(r-q)\Delta t} - d}{u-d} \\ u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}}. \end{aligned}$$

On the other hand, if the asset has a probability of $1 - e^{-\lambda \Delta t}$ of defaulting in the interval of length Δt :

$$P(t < \tau < t + \Delta t | \tau \geq t) = \frac{\int_t^{t+\Delta t} e^{-\lambda s} ds}{e^{-\lambda t}} = 1 - e^{-\lambda \Delta t}$$

the above equations are modified to

$$
\begin{aligned}
S_{i+1} &= S_i u \text{ with probability } p \\
&= S_i d \text{ with probability } 1-\lambda\Delta t - p \\
&= 0 \text{ with probability } \lambda\Delta t.
\end{aligned}
$$

That is

$$
\begin{aligned}
\frac{S_{i+1}-S_i}{S_i} &= u-1 \text{ with probability } p \\
&= d-1 \text{ with probability } 1-\lambda\Delta t - p \\
&= -1 \text{ with probability } \lambda\Delta t.
\end{aligned}
$$

The continuous model is

$$
S_{t+\Delta t} - S_t = S_t\Big[(r-q+\lambda)\Delta t + \sigma_W(\tilde{W}_{t+\Delta t} - \tilde{W}_t)\Big] - S_t \mathbf{1}_{\{t\le\tau<t+\Delta t\}}.
$$

Hence

$$
\begin{aligned}
\widetilde{\mathbb{E}}\Big(\frac{S_{i+1}-S_i}{S_i}\Big|\mathcal{F}_i\Big) &= (r-q)\Delta t \\
\widetilde{Var}\Big(\frac{S_{i+1}-S_i}{S_i}\Big|\mathcal{F}_i\Big) &= \sigma_W^2\Delta t + \lambda\Delta t(1-\lambda\Delta t) = \big(\sigma_W^2+\lambda\big)\Delta t := \sigma^2\Delta t.
\end{aligned}
$$

Note : The vol of the stock by definition is just the square root of $\widetilde{\mathbb{E}}(\frac{S_{i+1}-S_i}{S_i}\Big|\mathcal{F}_i)$ and may NOT be σ_W if there are other noise factors. This way of viewing it may cause problem in estimating historical volatility, since all we observe is the movement in BM and not the default. Also in this sense the vol given in example 27.1 in the book should be viewed as the historical vol and thus only belonging to BM. Unless the vol is the implied vol in which case the hazard rate has been factored into the option price??? We have

$$
\begin{aligned}
p(u-1) + (1-\lambda\Delta t - p)(d-1) - \lambda\Delta t &= (r-q)\Delta t \\
p(u-1)^2 + (1-\lambda\Delta t - p)(d-1)^2 + \lambda\Delta t &= \sigma^2\Delta t,
\end{aligned}
$$

(ignoring higher order terms in Δt.) Rewriting these equations :

$$
\begin{aligned}
p_u(u-1) + p_d(d-1) = (r-q+\lambda)\Delta t \\
p_u(u-1)^2 + p_d(d-1)^2 = (\sigma^2-\lambda)\Delta t.
\end{aligned}
$$

A solution to these equations, when terms of higher order than Δt are ignored is

$$\begin{aligned}
p_u &= e^{-\lambda\Delta t}\frac{e^{(r-q+\lambda)\Delta t}-d}{u-d}\\
p_d &= e^{-\lambda\Delta t}\frac{u-e^{(r-q+\lambda)\Delta t}}{u-d}\\
u &= e^{\sqrt{(\sigma^2-\lambda)\Delta t}}\\
d &= \frac{1}{u}.
\end{aligned}$$

$e^{-\lambda\Delta t}$ is the probability of not defaulting in the time interval $[t, t+\Delta t]$. The term $e^{(r-q+\lambda)\Delta t}$ represents the drift of the continous time dynamics when the asset does not default given above. The u, d term reflecting that when there's no default in $[t, t+\Delta t]$ then the vol contributed by the BM is $\sqrt{\sigma^2-\lambda}$.

The implied volatility function (IVF) model: a stochastic vol model such that the option price matches all Euro option prices on any given day, regardless of the shape of the vol surface. The model is

$$dS_t = (r_t - q_t)Sdt + \sigma(S,t)Sd\tilde{W}_t,$$

where

$$[\sigma(K,T)]^2 = 2\frac{\frac{\partial c_{mkt}}{\partial T} + q(T)c_{mkt} + K[r(T)-q(T)]\frac{\partial c_{mkt}}{\partial K}}{K^2\frac{\partial^2 c_{mkt}}{\partial K^2}}.$$

There is also the implied tree methodology (by Derman) involves constructing a tree for the asset price that is consistent with option prices in the market.

Convertible bond: Corporate bonds where the holder has the option to exchanged the bonds for the company's stock at certain times in the future. The conversion ratio is the number of shares of stock obtained for one bond (can be a function of time). The bonds are almost always callable (the issuer has the right to buy them back at a certain times at a predetermined prices). The holder always has the right to convert the bond once it has been called. The call feature therefore is a way of forcing conversion earlier than the holder would otherwise choose. Credit risk plays an important role in the valuation of convertibles (thus modelling the equity with a hazard rate as above is relevant in pricing a concertible bond). See example 27.1 in Hull. We need to model the stock price with default using the binomial tree model. If default happens the bond is worth 40 % of its face value. At each point n where conversion is possible, the value of the bond is valued as

$$\max(\min(V_{roll(n)}, K), \alpha(n)S_n),$$

where $V_{roll(n)}$ is obtained by rolling back from step $n+1$, K is the callable strike, $\alpha(n)$ is the conversion ratio at time n. K is at least the face value of the bond. Therefore at the terminal node N, if conversion is allowed then $\min(V_{roll(N)}, K) = F$, the face value and the only decision to make is whether conversion happens at that node.

Trinomial trees: Trinomial trees can be used as an alternative for binomial trees. A calibration for p_u, p_m, p_d that matches the two first moments of $d \log S_i$ are

$$\begin{aligned}
p_u &= \sqrt{\frac{\Delta t}{12\sigma^2}}(r - q - \frac{\sigma^2}{2}) + \frac{1}{6} \\
p_u &= -\sqrt{\frac{\Delta t}{12\sigma^2}}(r - q - \frac{\sigma^2}{2}) + \frac{1}{6} \\
p_m &= \frac{2}{3} \\
u &= e^{\sigma\sqrt{3\Delta t}} \\
d &= \frac{1}{u}.
\end{aligned}$$

The trinomial tree approach proves to be equivalent to the explicit finite difference method.

23 Chapter 28: Martingales and numéraires

Market price of risk : Suppose there is an asset with dynamics

$$dS_t = \mu^S S_t dt + \sigma^S S_t dW_t$$

(under some UNSPECIFIED measure, the measure actually is equivalent to the market price of risk λ see below). Suppose V^1, V^2 are two derivatives based on S and

$$\begin{aligned}
dV_t^1 &= \mu_1 V_t^1 dt + \sigma^1 V_t^1 dW_t \\
dV_t^2 &= \mu_2 V_t^2 dt + \sigma^2 V_t^2 dW_t.
\end{aligned}$$

(This is true, if e.g. $V^i = f^i(t, S_t)$.) Then

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} = \lambda.$$

This result , of course, also applies to S_t itself. λ is referred to as the market price of risk of S_t (under the choice of a particular measure). Thus more precisely it should be λ is the market price of risk associated with a particular risk measure (or a particular investors' preference). See below.

Reason: If we form a self financing portfolio π with V^1, V^2 then

$$\begin{aligned} d\pi_t &= \alpha_t^1 dV_t^1 + \alpha_t^2 dV_t^2 \\ &= (\cdots)dt + (\alpha_t^1 \sigma^1 V_t^1 + \alpha^2 \sigma^2 V_t^2) dW_t. \end{aligned}$$

By choosing $\alpha^1 = \sigma^2 V^2, \alpha^2 = -\sigma^1 V^1$ we have

$$d\pi_t = (\sigma^2 \mu_1 - \sigma^1 \mu^2) V_t^1 V_t^2 dt = r\pi_t dt = r(\sigma^2 - \sigma^1) V_t^1 V_t^2 dt$$

by no arbitrage argument. Thus

$$\sigma^2(\mu_1 - r) = \sigma^1(\mu_2 - r).$$

The conclusion follows. In multivariate asset $S^i, i = 1, \cdots N$ case, where

$$dS_t^i = \mu^{S,i} S_t^i dt + \sigma^i S_t^i dW_t^i,$$

the result is

$$\begin{aligned} \mu^1 - r &= \sum_{i=1}^{N} \lambda_i \sigma_i^1 \\ \mu^2 - r &= \sum_{i=1}^{N} \lambda_i \sigma_i^2, \end{aligned}$$

where λ_i is the market price of risk of asset S^i.

Remark: A more precise statement is, when we have m sources of risk and n assets then we have m market prices of risk $\lambda_1, \lambda_2, \cdots, \lambda_m$ that have to satisfy the system of equations

$$\sum_{k=1}^{m} \sigma_k^i \lambda^k = \mu^i - r, i = 1, \cdots, n.$$

Here σ_k^i is the risk factor of the ith asset in the kth risk component. Thus one should refer to λ^k as the market price of risk of the kth risk factor rather than the kth asset in general.

Risk measures and market price of risk: The dynamics of the price of a derivative V is

$$dV_t = \mu V_t dt + \sigma V_t dW_t.$$

The value of μ depends on the risk preferences of investors. By the above result it can be written as

$$dV_t = (r + \lambda\sigma)V_t dt + \sigma V_t dW_t.$$

When $\lambda = 0$ this is the dynamics of V_t under the risk neutral measure and thus the probability under consideration is risk neutral. For other choices of λ we have different views of the market, including the physical and other choices of numéraire.

Change of numéraire: There are two fundamental principles in change of numéraire.

The pricing principle: Let V_t, N_t be the price processes (in the physical world) of two non-dividend paying assets V, N. The t-value of an asset V denoted in the unit of any numéraire N at some future point T, that is the t- value of $\frac{V_T}{N_T}$, must be exactly the t-price of the asset denominated in N, that is $\frac{V_t}{N_t}$. This principle can be proven by no arbitrage argument: Conisder a self-financing portfolio investing α_t^1 shares in V_t and α_T^2 shares in N_t. The value of this portfolio in the denomination of N is

$$\pi_t^{(N)} = \alpha_t^1 V_t^{(N)} + \alpha_t^2,$$

where $\pi_t^{(N)} = \frac{\pi_t}{N_t} \cdots$. Viewed in this way, this is exactly the same as the traditional value of a portfolio investing in a stock and the money market account. The traditional no arbitrage argument applies in this case.

The mathematical expression of the pricing principle is that the price process of a non dividend paying asset denominated under a numéraire is a martingale under the risk measure associated with that numéraire. Note that the risk neutral pricing formula is a special case of this when the numéraire is the money market account (usually denoted as $N_t = e^{rt}$ when r is constant.)

Remark: In the case that the numéraire is some nondividend paying foreign asset (such as the foreign money market or foreign bond), the same martingale pricing principle applies. That is the discounted price of the foreign bond or foreign money market denominated in the domestic currency unit is a martingale : the present value of the future price of the foreign bond is its current price, expressed in domestic unit. In symbols: $B^f(t,T)Q_t$ and $e^{\int_0^t r^f du}Q_t$ are $\tilde{\mathbb{P}}$ martingales.

The risk measure principle: The risk measure associated with a numéraire N is such that the market price of risk is the volatility $\rho_{NS}\sigma^N$, where ρ_{NS} is the correlation of the percentage return of S and N. (Thus when N is the money market and has no correlation with the asset $\lambda = 0$). More precisely, ρ_{NS} should be the correlation between N and the (only) risk factor of the market. Thus more preciesly, if the money market risk factor is uncorrelated with the security market risk factor, we have $\lambda = 0$. Is this true? This should be ! So that $e^{-\int_0^t r_s ds} S_t$ is still a martingale under the "risk neutral" measure.

Reason: We only need to show that given a price process V_t :

$$dV_t = (r + \sigma_N \sigma_V) V_t dt + \sigma_V V_t dW_t$$

and the price process of N_t itself :

$$dN_t = (r + (\sigma_N)^2) N_t dt + \rho_{NS} \sigma_N N_t dW_t,$$

then $V_t^{(N)} = \frac{V_t}{N_t}$ is a martingale. In fact,

$$dV_t^{(N)} = (\sigma_V - \sigma_N \rho_{NS}) V_t^{(N)} dW_t.$$

This can be verified straightforwardly using Ito's formula.

In the multivariate case as set up above, $\lambda_i = \rho_{NS_i} \sigma_i^N$ where ρ_{NS_i} is the correlation between N and the ith risk factor (which coincides with the ith asset in the above model, but not in general, see the example below).

Effect of change of numéraire (multivariate case): Suppose

$$\begin{aligned}
dV_t &= (r + \sum_{i=1}^n \lambda_i \sigma_i^V) V_t dt + V_t \sum_{i=1}^n \sigma_i^V dW_t^i \\
dN_t &= (r + \sum_{i=1}^n \lambda_i \sigma_i^N) N_t dt + N_t \sum_{i=1}^n \sigma_i^N dW_t^i. \\
dM_t &= (r + \sum_{i=1}^n \lambda_i \sigma_i^M) M_t dt + N_t \sum_{i=1}^n \sigma_i^M dW_t^i.
\end{aligned}$$

Using N_t as numéraire results in a dynamics where $\lambda_i = \sigma_i^N, i = 1, \cdots, n$:

$$dV_t = (r + \sum_{i=1}^n \sigma_i^N \sigma_i^V) V_t dt + V_t \sum_{i=1}^n \sigma_i^1 dW_t^i$$

Similarly using M_t as numéraire results in a dynamics where $\lambda_i = \sigma_i^M, i = 1, \cdots, n$:

$$dV_t = (r + \sum_{i=1}^n \sigma_i^M \sigma_i^V) V_t dt + V_t \sum_{i=1}^n \sigma_i^1 dW_t^i.$$

If we change from numéraire M to numéraire N, the change in the expected growth rate of V is

$$\sum_{i=1}^n (\sigma_i^N - \sigma_i^M) \sigma_i^V.$$

On the other hand, the dynamics of the numéraire ratio $\frac{N}{M}$ is

$$d\frac{N_t}{M_t} = (\cdots)dt + \frac{N_t}{M_t}\sum_{i=1}^{n}(\sigma_i^N - \sigma_t^M)dW_t^i.$$

Thus the result can be summarized as: the adjustment α_V to the expected growth rate of an asset V when we change from one numéraire M to another numéraire N is the covariance between the percentage change of V with the percentage change of the numéraire ratio $W = \frac{N}{M}$:

$$\alpha_V = \rho_{VW}\sigma_V\sigma_W.$$

Note that in the multi-dimensinal case, this really means:

$$\rho_{VW}\sigma_V\sigma_W = \sum_i \sigma_V^i\sigma_W^i.$$

σ_V really is $\|\underline{\sigma}_V\|$ and similarly for σ_W. For example :

$$\begin{aligned} dS_t^1 &= \mu^1 S_t^1 dt + \sigma^1 S_t^1 dW_t^1 \\ dS_t^2 &= \mu^2 S_t^2 dt + \sigma^2 S_t^2(\rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2). \end{aligned}$$

Here

$$\begin{aligned} vol(S_t^1) &= SD(\frac{dS_t^1}{S_t^1}) = \sigma^1 \\ vol(S_t^2) &= SD(\frac{dS_t^2}{S_t^2}) = \sigma^2 \\ Cov(\frac{dS_t^1}{S_t^1}, \frac{dS_t^2}{S_t^2}) &= \rho\sigma^1\sigma^2 \\ corr(\frac{dS_t^1}{S_t^1}, \frac{dS_t^2}{S_t^2}) &= \rho. \end{aligned}$$

In this example, we can define $dW_t^3 = \rho dW_t^1 + \sqrt{1-\rho^2}dW_t^2$. Thus ρ_{VW} can also be looked at as the correlation between the risk factors of V and W (this may be the easiest definition to follow).

Note that ρ_{VW} can be negative, as in the example below.

Application: Siegel's paradox

Suppose Q_t is the exchange rate at time t of Y for X. That is Q_t is the number of units of Y per one unit of X. Under the risk measure of the Y-money market, we have

$$dQ_t = (r_Y - r_X)Q_t dt + \sigma Q_t dW_t.$$

By Ito's formula, we can calculate that

$$d\frac{1}{Q_t} = (r_Y - r_X + \sigma^2)\frac{1}{Q_t}dt - \sigma\frac{1}{Q_t}dW_t.$$

$\frac{1}{Q_t}$ is the exchange rate of X for Y. We would expect that its growth rate is $r_X - r_y$ by a similar argument to the growth rate of Q_t. The reason this did not show is because the above dynamics is given under the $Y-$ money market numéraire. If we use the above result, we have M is the Y money market, N is the X money market and $W_t = e^{(r_X - r_Y)t}Q_t$ and $V = \frac{1}{Q_t}$. Thus $\rho_{VW} = -1, \sigma_V = \sigma_W = \sigma$. Therefore, the dynamics of $\frac{1}{Q_t}$ under the $X-$money market numéraire must be

$$\begin{aligned} d\frac{1}{Q_t} &= (r_Y - r_X + \sigma^2 - \sigma^2)\frac{1}{Q_t}dt - \sigma\frac{1}{Q_t}dW_t^X \\ &= (r_Y - r_X)\frac{1}{Q_t}dt - \sigma\frac{1}{Q_t}dW_t^X \end{aligned}$$

as expected.

Forward price versus futures price: Consider a forward contract on an asset S_t with strike K with maturity T. We want to find the forward price $F(t,T)$ of S_t. Since $V_t = 0$ it is the same under the T forward measure :

$$0 = E^{(T)}(S_T - F(t,T)|\mathcal{F}_t).$$

Thus

$$\begin{aligned} F(t,T) &:= E^{(T)}(S_T|\mathcal{F}_t) \\ &= \tilde{E}(e^{-\int_t^T r_u du}S_T|\mathcal{F}_t). \end{aligned}$$

These relations can be viewed as the definitino of forward price. On the other hand, the futures price $Fu(t,T)$ of S_t is such that

$$Fu(t,T) = \tilde{E}(S_T|\mathcal{F}_t).$$

(Note the non discounting part). Thus future price process itself is martingale under the risk neutral world. The reason is by construction, the futures prices process is an asset paying dividend with rate r_t which converges to S_T.

Black's model with random interest rate : Consider a call option on an asset S_t where the risk free rate is random. The price of this option under the foward measure (the measure with the zero coupon bond $B(t,T)$ as numéraire) is

$$\begin{aligned} V_0^{(T)} &= E^{(T)}(S_T - K)^+ \\ &= E^{(T)}(S_T)N(d+) - KN(d-), \end{aligned}$$

since $B(T,T)=1$ and

$$N(d\pm) = \frac{\pm\frac{\sigma^2 T}{2} - \log\frac{K}{E^{(T)}(S_T)}}{\sigma\sqrt{T}}.$$

Now $E^(T)(S_T) = F_0$, the forward price of S_T. Thus we recover Black's result :

$$\begin{aligned} V_0 &= B(0,T)F_0 N(d+) - KN(d-), \\ N(d\pm) &= \frac{\pm\frac{\sigma^2}{2} - \log\frac{K}{F_0}}{\sigma\sqrt{T}}. \end{aligned}$$

Note that $\sigma\sqrt{T} = SD^{(T)}(\log(S_T))$ but it is equal to the physical volatility by the change of measure result.

Option to exchange one asset for another: Suppose S_t^1, S_t^2 each has volatility σ^1, σ^2 and correlation ρ for their percentage change.

$$\begin{aligned} dS_t^1 &= \mu^1 S_t^1 dt + \sigma^1 S_t^1 dW_t^1 \\ dS_t^2 &= \mu^2 S_t^2 dt + \sigma^2 S_t^2 (\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2) \\ V_T &= (S_T^1 - S_T^2)^+. \end{aligned}$$

Then

$$V_0^{(2)} = E^{(2)}(S^{1,(2)} - 1)^+,$$

where under the risk measure of numéraire S^2

$$\begin{aligned} dS_t^{1,(2)} &= d\frac{S_t^1}{S_t^2} = \frac{S_t^1}{S_t^2}((\sigma_1 - \rho\sigma_2)dW_t^1 - \sqrt{1-\rho^2}\sigma_2 dW_t^2) \\ &= \frac{S_t^1}{S_t^2}\sigma_3 dW_t^3, \end{aligned}$$

where

$$\begin{aligned} \sigma_3^2 &= (\sigma_1 - \rho\sigma_2)^2 + (1-\rho^2)(\sigma_2)^2 \\ &= (\sigma_1)^2 - 2\rho\sigma_1\sigma_2 + (\sigma_2)^2. \end{aligned}$$

Thus

$$\begin{aligned} V_0 &= S_0^2(\frac{S_0^1}{S_0^2}N(d+) - N(d-)), \\ &= S_0^1 N(d+) - S_0^2 N(d-) \\ N(d\pm) &= \frac{\pm\frac{(\sigma_3)^2}{2} - \log\frac{S_0^2}{S_0^1}}{\sigma_3\sqrt{T}} \end{aligned}$$

24 Chapter 29: Interest rates derivatives

Embedded bond options: Callable bond: the issuer has the right to repurchase the bond at a predetermined price (usually as a decreasing function of time, also not exercisable in the first few years in the bond's life : lock out period). Effectively the purchaser has sold the issuer a call option on the bond. The yield of a callable bond therefore is higher than bond with no call features. Puttable bond (retractable bond) : the purchaser has the right to redeem the bond before the maturity at a predetermined price. The issuer effectively has sold the purchaser a put option on the bond. The yield of a puttable bond therefore is lower than bond with no put features.

Examples of products with call / put features: Prepayment previleges on loans and mortgages are call options on bonds. Loan commitment (a rate quote that is good until a certain time T in the future) made by a financial institution is a put option on a bond.

Pricing of Euro bond option: By Black's formula:

$$\begin{aligned}
c &= \tilde{E}^T((B(T,T*) - K)^+) \\
c &= B(0,T)(F_B N(d+) - KN(d-)) \\
p &= B(0,T)(KN(-d-) - F_B N(-d+)) \\
d\pm &= \frac{\pm\frac{1}{2}\sigma_B^2 T - \log\frac{K}{F_B}}{\sigma_B\sqrt{T}}
\end{aligned}$$

where F_B is the forward bond price :

$$F_B = \frac{S_0 - I}{B(0,T)},$$

S_0 is the bond price at time 0 and I is the present value of the income provided by the bond.

Note : This assumes that the forward bond price $F_B(0,T)$ has a log normal distribution with volatility σ_B (and mean 0 under the forward risk measure T). These and other similar assumptions in this chapter are sufficient for valuing Euro style interest rate derivatives; but not sufficient for American style interest rate derivatives or structured notes. For these derivatives, short rate models such as CIR, Vasicek, Hull-White, Ho-Lee etc are relevant. See chapter 31.

Determination of σ_B: σ_B is the vol of the forward bond price. In practice it is calculated from the forward bond yield via the duration equation :

$$\frac{\Delta F_B}{F_B} \approx -Dy_F \frac{\Delta y_F}{y_F},$$

where D is the modified duration and y_F is the forward yield. Thus we have

$$\sigma_B = Dy_0\sigma_y,$$

where y_0 is the current value of the forward yield.

Interest rate caps and floors: Consider a cap with a total life of T, principal of P and cap rate of R. Suppose the reset dates are $t_1, t_2, \cdots, t_n$. An interest rate cap pays the holder at time $t_{k+1}, k = 1, \cdots, n$ an amount of

$$P(L(t_k, t_{k+1}) - R)^+(t_{k+1} - t_k).$$

A cap can be viewed as n call options on the LIBOR rate observed at time t_k with payoff at time t_{k+1}. The n call options underlying the cap are known as caplets.

The payoff above at time t_{k+1} is equivalent to the following payment at time t_k :

$$\frac{P}{1 + L(t_k, t_{k+1})\Delta t}(L(t_k, t_{k+1}) - R)^+\Delta t = \big[P - \frac{P(1 + R\Delta t)}{1 + L(t_k, t_{k+1})\Delta t}\big]^+.$$

The quantity $\frac{P(1+R\Delta t)}{1+L(t_k,t_{k+1})\Delta t}$ is the value at time t_k of a zero coupon bond with face value $P(1 + R\Delta t)$. Thus the cap can also be viewed as a portfolio of put options with expiry t_k on zero coupon bond with maturity at t_{k+1} and face value $P(1 + R\Delta t)$.

A floor is the reverse of a cap. Thus we have the put call parity for Caps and Floors:

$$\text{Value of cap - Value of floor} = \text{Value of swap},$$

where the all contracts have the fixed rate R and floating rate being the prevailing LIBOR.

Valuation of caplet: By Black's formula the value of the caplet is

$$\begin{aligned} V_0^k &= L\Delta_t B(0, t_{k+1})[F_k N(d_1) - RN(d-)], \\ d\pm &= \frac{\pm\frac{\sigma_k^2 t_k}{2} - \log\frac{R}{F_k}}{\sigma_k\sqrt{t_k}}, \end{aligned}$$

where F_k is the LIBOR forward rate of at time 0 for the period $[t_k, t_{k+1}]$ and σ_k is the vol of this forward rate.

Remark: This is why forward price is relevant since Black's formula allows to price option with random interest rate once the forward price is available. Again, the assumption here is the LIBOR forward rate has a log-normal distribution with volatility σ_k.

Flat volatility vs spot volatility: Each caplet (floorlet) must be valued separtely using the equation above. One approach is to use a different vol for each caplet. This is referred to as spot volatilities. An alternative is to use the same vol for all caplets comprising of a cap, but vary this according to the life of the cap. This is referred to as flat volatility. The vols quoted in the market are usually flat volatilities.

European swaptions: Options on interest rate swaps, which give holder the right to enter into a certain interest rate swap at a certain time in the future (with a predetermined fixed rate s_K). Suppose that there are m payments per year under the swap and the notional amount is L. Thus each fixed payment is $\frac{L}{m}$ times the fixed rate. Suppose that the realized swap rate at time T for this swap is s_T. The payoff from the swaption is a series of cash flows equal to

$$\frac{L}{m}(s_T - s_K, 0),$$

received at times $t_1, t_2, \cdots, t_{nm}$ for a total of mn payments. The value of the swaption is

$$\begin{aligned} V_0 &= \sum_{i=1}^{mn} \frac{L}{m} B(0, t_i)(s_0 N(d+) - s_k N(d-)) \\ d\pm &= \frac{\pm\frac{\sigma^2 T}{2} - \log\frac{s_k}{s_0}}{\sigma\sqrt{T}}, \end{aligned}$$

where s_0 is the forward rate of the swap rate s_T.

Remark : The forward swap rate s_0 may need to be calculated using the whole swap structure in the future, i.e. using the forward LIBOR rates etc. so that the value of the future swap at time T is 0. (See next chapter for some finer details on convexity and time adjustments). Indeed, let $T_1 < T_2 < \cdots < T_N$ be the payment dates , $t = T_0$ is the start date of the swap and $\tau_i = T_{i+1} - T_i$ then the swap rate $s(t)$ satisfies

$$\sum_{i=0}^{n-1} B(t, T_{i+1}) F(t, T_i, T_{i+1}) \tau_i = \sum_{i=0}^{n-1} B(t, T_{i+1}) s(t) \tau_i.$$

On the other hand, $F(t,T_i,T_{i+1})\tau_i = \frac{B(t,T_i)-B(t,T_{i+1})}{B(t,T_{i+1})}$. Thus

$$\begin{aligned} s(t) &= \frac{\sum_{i=0}^{n-1} B(t,T_i) - B(t,T_{i+1})}{\sum_{i=0}^{n-1} B(t,T_{i+1})\tau_i} \\ &= \frac{B(t,T_0) - B(t,T_n)}{\sum_{i=0}^{n-1} B(t,T_{i+1})\tau_i}. \end{aligned}$$

Note that $s(0)$ is NOT $\tilde{E}^T(s(T))$. Recall that a swap rate is the par yield when LIBOR discounting is used. Thus we can consider a bond with coupon rate $s(0)$ with life time equals the life time of the swap whose yield at that time is $s(T)$. We can write down the price of that bond $B(T)$ as a function of its yield $G(y_T)$ (since we use s_0 as the coupon rate and NOT LIBOR the price of the bond $B(T)$ is not 1). We can then use convexity adjustment which says

$$\tilde{E}^T(s_T) = s_0 - \frac{1}{2}s_0^2\sigma_S^2 T\frac{G''(s_0)}{G(s_0)}.$$

σ_S can be implied from the prices of swaption with the same maturity.

25 Chapter 30: Convexity and timing adjustments

Main question: For $T_1 < T_2$ how to relate $E^{T_2}(S_{T_1})$ and $E^{T_1}(S_{T_1})$. By definition, we already know the forward price of S at time T_1, which could be $E^{T_1}(S_{T_1})$ OR $E^{T_2}(S_{T_1})$, depending on the situation (see below).

Case 1: Interest rates. Note: interest rate is NOT a traded asset. Thus if R_{T_1} is the prevaling rate (LIBOR, swap etc) for some time interval $[T_1,T_2]$ $E^{T_1}(R_{T_1})$ is NOT the forward rate of R_{T_1}. Indeed, suppose $R_{T_1} = L(T_1,T_2)$ is the prevailing LIBOR rate at T_1. the usual approach is to relate the amount received at T_2, which can be viewed as some traded asset (a zero coupon bond) with the bond prices at time T_1, T_2. Indeed, we have showed that the forward rate F_0 of R_{T_1} at time 0 (compound once) is such that

$$(1 + F_0(T_2 - T_1))B(0,T_2) = B(0,T_1).$$

And thus

$$F_0 = \frac{B(0,T_1)}{B(0,T_2)} - 1.$$

In the T_2 forward risk measure notation, this calculation is

$$E^{T_2}\Big[[L(T_1,T_2)-F_0](T_2-T_1)\Big]=0,$$

and thus by definition $E^{T_2}[R_{T_1}]=E^{T_2}[L(T_1,T_2)]=F_0$. And thus one easily see that $E^{T_2}[R_{T_1}]$ is NOT F_0, the forward rate. Now since R_{T_1} is available at time T_1, one can have a contract that pays

$$V_{T_1}=R_{T_1}(T_2-T_1)$$

at time T_1. The value of this contrat is

$$V_0=B(0,T_1)E^{T_1}(V_{T_1})=B(0,T_1)(T_2-T_1)E^{T_1}(R_{T_1}).$$

Thus we need to evaluate (or at least approximate) $E^{T_1}(R_{T_1})$.

The most consistent approach is time adjustment (that would be used also to evaluate quanto option below) : In changing from $E^{T_1}(R_{T_1})$ to $E^{T_2}(R_{T_1})$ we change from the numéraire $B(t,T_1)$ to $B(t,T_2)$. Another observation is that the forward rate $F(t,T_1)$ is such that $F(T_1,T_1)=R_{T_1}$. Thus we are changing from $E^{T_1}(F(T_1,T_1))$ to $E^{T_2}(F(T_1,T_1))$. This is helpful because $F(t,T_1)$ is a process with growth rate and volatility that we can discuss. The numéraire ratio is $W=\frac{B(t,T_2)}{B(t,T_1)}$. and the "asset" is $F(t,T_1)$. According to the change of numéraire result, the growth rate of the asset is changed by

$$\alpha_F=\rho_{FW}\sigma_F\sigma_W.$$

We need to be able to express σ_W in terms of σ_F. Suppose that the interst is structured in a frequency of m times (per annum). Then similar to the above we have

$$(1+\frac{F(t,T_1)}{m})^{m(T_2-T_1)}=\frac{B(t,T_1)}{B(t,T_2)}=\frac{1}{W}.$$

A side remark about geometric BM processes: not all Ito processes are of the geoemetric form:

$$dX_t=\mu_X(t)X_tdt+\sigma_X(t)X_tdW_t.$$

But if we are reasonably certain that X_t is not zero in some interval of time, then it can be turned into a geometric form by redefining $\mu_X(t),\sigma_X(t)$. Thus the vol and growth rate, defined as the mean and standard deviation of the percentage return, can always be found for any process. The growth rate and the vol may be dependent on the process itself.

A side remark about Ito's formula:

If $Y = f(X)$ are two Ito processes then we always have

$$\sigma_Y Y = \sigma_X X f'(X).$$

Note that σ_y, si_X here are the "generalized" vol as in the above sense. They might even contain $\pm$ sign and so they may be one sign away from what we think of as vol. The reason is by the generalized geometric structure:

$$\begin{aligned} dY_t &= (\cdots)dt + \sigma_Y Y_t dW_t \\ &= (\cdots)dt + f'(X)\sigma_X X dW_t. \end{aligned}$$

Equating the dW_t terms we have the result. (Again in reality W_t may even be a multi dimensional BM).

Applying these remarks, we have

$$\sigma_W(t) = \frac{-\sigma_F F(t, T_1)(T_2 - T_1)}{1 + \frac{F(t,T_1)}{m}}.$$

Here we assume that σ_F is a constant. The negative sign in front shows that the correlation ρ_{FW} between **the risk factors** of F, W is -1 (in this case we are really taking

$$\sigma_W(t) = \frac{\sigma_F F(t, T_1)(T_2 - T_1)}{1 + \frac{F(t,T_1)}{m}}).$$

And thus

$$\alpha_F \approx -\frac{(\sigma_F)^2 F(0, T_1)(T_2 - T_1)}{1 + \frac{F(0,T_1)}{m}},$$

where we have approximated $F(t, T_1)$ with $F(0, T_1)$.

Lastly, since changing from $E^{T_1}(F(T_1, T_1))$ to $E^{T_2}(F(T_1, T_1))$ only affects the growth rate by α_F, we can approximate :

$$\begin{aligned} E^{T_2}(F(T_1, T_1)) &\approx E^{T_1}(F(T_1, T_1))e^{\alpha_F(T_2 - T_1)} \\ &\approx E^{T_1}(F(T_1, T_1)) \exp\big(-\frac{(\sigma_F)^2 F(0, T_1)(T_2 - T_1)}{1 + \frac{F(0,T_1)}{m}} T_1\big). \end{aligned}$$

In particular, if $T_2 - T_1 = \Delta t$ is one period of compounding then $\frac{1}{m} = \Delta t$ and thus

$$E^{T_1}(F(T_1,T_1)) \quad \approx \quad F(0,T_1)\exp\big(\frac{(\sigma_F)^2 F(0,T_1)(T_2-T_1)}{1+F(0,T_1)(T_2-T_1)}T_1\big).$$

Case 2: Equity: Consider a derivative that provides a pay off in T_2 equalling to the value of an index observed at $T_1 < T_2$. Here we need to evaluate $E^{T_2}(S_{T_1})$ and we already know $E^{T_1}(S_{T_1})$ which is the forward price of S at time T_1. Using the same argument as above, but now

$$\alpha_S = \rho_{SW}\sigma_S\sigma_W = -\frac{\rho_{SF}\sigma_S\sigma_F F(t,T_1)(T_2-T_1)}{1+\frac{F(t,T_1)}{m}}.$$

Remark: The correlation between the risk factors of F, W is -1 as in the above remark. This literally means if the BM factor of F is W_t then the BM factor of W is $-W_t$. Therefore, $\rho_{SW} = -\rho_{SF}$ naturally.

The result then is

$$E^{T_2}(S_{T_1}) \quad \approx \quad E^{T_1}(S_{T_1})\exp\big(-\frac{\rho_{SR}\sigma_S\sigma_F F(0,T_1)(T_2-T_1)}{1+\frac{F(t,T_1)}{m}}T_1\big).$$

Finally,

$$V_0 = B(0,T_2)F^S(0,T_1)\exp\big(-\frac{\rho_{SR}\sigma_S\sigma_F F(0,T_1)(T_2-T_1)}{1+\frac{F(t,T_1)}{m}}T_1\big).$$

Quantos : Quantos are derivatives where the payoff is defined in terms of a variable that is measured in one of the currencies and the payoff is made in another currency. Specifically, consider an index S_t that is defined in terms of a currency X (say Nikkei 250 in yen) and a derivative that pays S_T at time T in the denomination of Y-currency (e.g. US dollars). The value of the derivative is

$$B^(0,T)E^{Y,T}(S_T)$$

while what we know is $E^{X,T}(S_T)$ which is the forward price of S in the currency of X. $E^{Y,T}$ is the risk measure associated with the zero coupon bond in $Y-$ currency and similarly for X. Thus we want to relate $E^{X,T}(S_T)$ with $E^{Y,T}(S_T)$. Changing from $E^{X,T}$ to $E^{Y,T}(S,T)$ the ratio W of the numéraires is $\frac{B^Y(t,T)Q_t}{B^X(t,T)}$ where Q_t is the exchange of X for Y (that is the price of 1 unit of Y currency denominated in X). In fact, W_t here is the forward exchange rate of X for Y with expiry T quoted at time t : The forward exchange rate $F(t,T)$ of X for Y must satisfy $B^Y(t,T)Q_t = F(t,T)B^X(t,T)$.

The change in growth rate of S_T is

$$\alpha_S = \rho_{SW}\sigma_S\sigma_W.$$

And we have

$$E^{Y,T}(S_T) \approx E^{X,T}(S_T)exp(\rho_{SW}\sigma_S\sigma_W T).$$

26 Chapter 31: Interest rate derivatives: Model for short rates

For Euro style interest rate derivatives evaluation, log normal distribution assumption on the forward rate (or forward price, in case of a bond) is sufficient. On the other hand, these structures does not directly provide a description of how interest rates evolve through time (knowing the distribution of $F(t, T_1)$ and $F(t, T_2)$ where they are the forward rates prevaling at time T_1 and T_2 does not provide a description on how short rate evolves in $[T_1, T_2]$. In other words, we don't have $R(T_1, T_2)$ in the notation below (be careful to distinguish this with $F(t, T_1, T_2)$, the forward rate for the borrowing period $[T_1, T_2]$ available at time t). This is unless we model the forward rate directly for any T as in the HJM approach.) The convexity and time adjustment in the previous chapter is one way to address this problem. Alternatively, we can model the short rate directly. This provides a description of the evolution of all zero-coupon interest rates $r(t, T)$ (for all T), also known as a term structure model. Using term structure models, convexity and timing adjustments are not required. They are also useful for evaluating American style interest rate derivatives or structure notes.

Basics of instantaneous short rate: Derivative pricing depends only on the dynamics of $r(t)$ in a risk-neutral world. The process of $r(t)$ in the real world is not used (this is not quite precise, see below for the fitting the equilibrium models section). The pricing formula for a zero coupon bond with maturity T is

$$\begin{aligned} B(t,T) &= \tilde{E}(e^{-\int_t^T r_u du}|\mathcal{F}_t) \\ &= e^{-R(t,T)(T-t)}. \end{aligned}$$

Or

$$\begin{aligned} R(t,T) &= -\frac{1}{T-t}\log \tilde{E}(e^{-\int_t^T r_u du}|\mathcal{F}_t) \\ &= -\frac{1}{T-t}\log B(t,T). \end{aligned}$$

This is the term structure equation. If we assume the Markov property of r_t, the term structure only depends on the value of r at t and its dynamics on $[t, T]$.

Black-Scholes equation for any interest rate derivatives: suppose r follows the dynamics

$$dr = m(r,t)dt + \sigma(r,t)dW_t.$$

Then any derivative dependent on r, if providing no income must satisfy

$$-rV + V_t + mV_r + \frac{1}{2}\sigma^2 V_{rr} = 0.$$

One particular solution to the equation is the zero-coupon bond $B(t,T)$ (under the Markov property of $r(t)$.)

Equilibrium models: Equilibrium models start with assumptions about economic variables (factors ?) and derive a process for r. This process in turn implies about bond prices and option prices. The general dynamics of r_t under the risk neutral measure is of the form

$$dr = m(r)dt + \sigma(r)dWt.$$

The distinction is m, σ do not depend on t (especially m). (Is this what is meant by equilibrium? When $t \to \infty$?). Examples in clude Vasicek :

$$dr = a(b-r) + \sigma dW_t$$

and CIR:

$$dr = a(b-r) + \sigma\sqrt{r}dW_t.$$

Both Vasicek and CIR gave bond price of the form

$$B(t,T) = A(t,T)e^{-C(t,T)r(t)}$$

with explicit solutions for $A(t,T), C(t,T)$.

Equilibrium models (Vasicek, CIR) and term structure : In both Vasicek and CIR model

$$\begin{aligned} R(t,T) &= -\frac{1}{T-t}\log B(t,T) \\ &= \frac{-\log A(t,T)}{T-t} + \frac{C(t,T)}{T-t}r(t). \end{aligned}$$

Since $A(t,T), C(t,T)$ can be explicitly determined by a, b, σ in both models, the entire term structure is determined by these coefficients as well. Moreover, $R(t,T)$ is linearly dependent on $r(t)$. Thus the **level** of the term structure at time t depends only the value of $r(t)$. On the other hand, the **shape** of the term structure at time t (as a function of T) is independent of $r(t)$ but does depend on t. Moreover, as $A(t,T), C(t,T)$ are really functions of $T-t$, we get a homogenous term structure. So the the **shape** of the term structure at time t (as a function of T) is really just a shift of the original term structure (thus for the same time to maturity $\tau = T-t$ we get the same $R(t,T)$).

Question: Can a, b, σ be chosen so that the equilibrium models fit the initial term structure as the no arbitrage models below? That is can they be chosen so that

$$R(0,T) = -\frac{1}{T}\log B_m(0,T),$$

where $B_m(0,T)$ is the market bond prices at time 0? This does not seem possible as we have only 3 parameters a, b, σ and the infinitely many equations for T. This can also be explained from the fact that a, b are independent of t. The no arbitrage models allow for this possibility and hence can fit the initial term structure exactly.

Fitting equilibrium models: We can fit the models with historical short rate data or with bond price data. The first approach results in the dynamics of the short rate in the real world, which is appropriate for scenario analysis (insurance company interested in the value of its portfolio in 20 years). The second approach results in the dynamics in the risk neutral world.

Example: Vasicek's model:

$$\Delta r = a(b-r)\Delta t + \sigma\epsilon\sqrt{\Delta t}.$$

We can fit this model on weekly data of short temr interest rate over a period of 10 years by regressing Δr (the change in the short rate in 1 week) against r. The slope is $a\Delta t$, the intercept is $ab\Delta t$ and the standard error of the estimate is $\sigma\sqrt{\Delta t}$. Interesting enough, this approach can also give the risk neutral dynamics of r_t. The approach is to note that changing to risk neutral measure is the same as reducing the growth rate (the proportional drift) of r_t by $\lambda\sigma_R$ where $\sigma_R = \frac{\sigma}{r_t}$ in Vasicek's model. Thus the dynamics of r in the risk neutral world is

$$\begin{aligned} dr &= (a(b-r) - \lambda\sigma)dt + \sigma d\tilde{W}_t \\ &= a(b^* - r)dt + \sigma d\tilde{W}_t. \end{aligned}$$

where $b* = b - \frac{\lambda\sigma}{a}$ and λ is the market price of risk to be determined by some other methods.

Example: CIR model :

$$dr = a(b-r)dt + \sigma\sqrt{r}d\tilde{W}_t$$

can be used to value bonds of any maturity using the model's analytic results. One can choose a, b, σ that minimize the sume of squared differences between the market prices of a set of bonds and the prices given by the model. On the other hand, using a similar argument with the above, we get the real world dynamics of r_t as

$$dr = [a(b-r) + \lambda\sigma\sqrt{r_t}]dt + \sigma\sqrt{r}d\tilde{W}_t.$$

This also can be derived from the fact that

$$dW_t - \lambda dt = d\tilde{W}_t.$$

No arbitrage model: Models designed to be exactly consistent with today's term structure of interest rates. In an equilibrium model, today's term structure is an output. Furthermore, the drift is usuallly not a function of time. In a no arbitrage model, today's term structure of interest rates in an input. Furthermore, the drift is usually a function of time. This is because the shape of the initial zero curve governs the average path taken by the short rate in the future in a no-aribtrage model. If the zero curve is steeply upward sloping for maturities betwen t_1 and t_2 (that is $\frac{\partial}{\partial T}R(t,T) < 0$ for $t_1 \leq T \leq t_2$)then r has a positive drift between these times and the reverse also holds. See Ho-Lee model for example.

Ho-Lee model:

$$\begin{aligned} dr &= \theta(t)dt + \sigma d\tilde{W}_t \\ \theta(t) &= F_t(0,t) + \sigma^2 t. \end{aligned}$$

The choice of $\theta(t)$ is so that the model fits the initial term structure and is independent of r. Note that as an approximation, $\theta(t) \approx F_t(0,t)$, where $F(0,t)$ is the **instantaneous** forward rate for a maturity t **as seen** at time 0 :

$$F(0,T) = -\frac{\partial}{\partial T} log B(0,T).$$

Thus the average direction of the short rate in the future is approximately the slope of the instantaneous forward rate.

Reason: Note that $F(0,s,t)$ satisfies

$$B(0,t)e^{F(0,s,t)(t-s)} = B(0,s).$$

Thus

$$\begin{aligned} F(0,s,t) &= \frac{1}{t-s}\log\frac{B(0,s)}{B(0,t)} \\ &= -\frac{B(0,t)-B(0,s)}{t-s}. \end{aligned}$$

Thus the instantaneous forward rate $F(0,t)$ satisfies

$$F(0,T) = -\frac{\partial}{\partial T}\log B(0,T).$$

Note: $F(0,0) := r_0$ by definition. We can NOT plug in $B(0,0) = 1$ to deduce that $F(0,0) = 0$ since that's NOT the rate of change of $\log B(0,T)$ around 0.

On the other hand, the initial term structure equation is

$$R(0,T) = -\frac{1}{T}\log\tilde{E}(e^{-\int_0^T r_u du})$$

Plugging in

$$\begin{aligned} r_u &= r_0 + \int_0^u F_s(0,s) + \sigma^2 s ds + \sigma W_u \\ &= r_0 + F(0,u) - F(0,0) + \frac{1}{2}\sigma^2 u^2 + \sigma W_u \\ &= F(0,u) + \frac{1}{2}\sigma^2 u^2 + \sigma W_u \end{aligned}$$

where we have used $r_0 := F(0,0)$ gives

$$-\int_0^T r_u du = -\int_0^T F(0,u)du - \int_0^T \frac{1}{2}\sigma^2 u^2 + \sigma W_u du.$$

It can be showed that

$$\tilde{\mathbb{E}}\, e^{-\int_0^T \frac{1}{2}\sigma^2 u^2 + \sigma W_u du} = 1$$

and

$$\int_0^T F(0,u)du = \log B(0,T).$$

Thus

$$R(0,T) = -\frac{\log B(0,T)}{T},$$

which is the initial term structure.

The bond pricing formula for Ho-Lee model is

$$\begin{aligned} B(t,T) &= A(t,T)e^{-r(t)(T-t)} \\ \log A(t,T) &= \log\frac{B(0,T)}{B(0,t)} + (T-t)F(0,t). \end{aligned}$$

Here $A(t,T)$ does not depend only on $T-t$ (through the term $\frac{B(0,T)}{B(0,t)}$ and thus Ho-Lee term structure is not time homogenous.

Hull-White one factor model:

$$dr_t = (\theta(t) - ar)dt + \sigma d\tilde{W}_t.$$

Hull-White is Ho-Lee model with mean reversion at rate a (thus Ho-Lee is Hull-White with $a = 0$). Alternatively, Hull-White is Vasicek's model with time dependent mean reversion level. The choice of θ_t that matches the initial term structure is

$$\theta(t) = F_t(0,t) + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

The bond pricing formula for Hull-White one factor model is

$$\begin{aligned} B(t,T) &= A(t,T)e^{-C(t,T)r(t)} \\ C(t,T) &= \frac{1 - e^{-a(T-t)}}{a} \\ \log A(t,T) &= \log\frac{B(0,T)}{B(0,t)} + B(t,T)F(0,t) - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{-at})^2(e^{2at} - 1). \end{aligned}$$

Thus $A(t,T)$ is not dependent only on $T-t$ and the term structure curve is not time homogenous.

Options on bonds: For Vasicek, Ho-Lee and Hull-White one factor models, the price V_0 at time 0 for a call option with strike K that matures at time T on a zero-coupon bond with principal P maturing at time T^* is

$$\begin{aligned} V_0 &= PB(0,T^*)N(d+) - KB(0,T)N(d-) \\ d\pm &= \frac{\pm\frac{\sigma_P^2}{2} - \log\frac{B(0,T)K}{B(0,s)P}}{\sigma_P} \\ \sigma_P &= \frac{\sigma}{a}[1-e^{-a(T^*-T)}]\sqrt{\frac{1-2e^{-aT}}{2a}} \text{ (Vasicek, Hull White)} \\ \sigma_P &= \sigma(T^*-T)\sqrt{T} \text{ (Ho Lee) }. \end{aligned}$$

This is essentially Black's formula for a bond option with bond forward price having volatility $\frac{\sigma_P}{\sqrt{T}}$.

Volatility structure: The previous section gives the bond forward price volatility to be $\frac{\sigma_P}{\sqrt{T}}$. Suppose that we take $T^* - T = \Delta$ to be fixed (i.e. 3 months). Then we see that as a function of T, the volatility structure of the models are different. In particular, Ho Lee gives a flat volatility structure, while Vasicek and Hull-White one factor gives a decreasing volatiltiy structure. Hull-White two factor gives a hump volatility structure, which is consistent with observations (see Figure 31.5). These correspond (though not exactly) to the **standard deviation** of the instantaneous forward rate, which is not covered here. We will see that (in the next section, when the standard deviation of the forward rate equals σ we obtain the Ho-Lee model and when it is equal $\sigma e^{-a(T-t)}$ we obtain the Hull-White model). The volatility of forward rate refers to the LIBOR forward rate, see next chapter.

Interest rate trees: Trinomial tree building procedure for Hull-White one factor model:

$$dr_t = (\theta(t) - ar_t)dt + \sigma d\tilde{W}_t.$$

The general procedure is to first build a tree for the process r^* :

$$dr_t^* = -ar_t^* dt + \sigma d\tilde{W}_t$$

(r^* is r when $\theta(t) = 0$.). Then we define

$$\alpha(t) := r(t) - r*(t)$$

and decide $\alpha(t)$ so that the initial term structure is exactly matched.

In particular, for the first stage, we construct a tree that mathces the first two moments of $\Delta r^*(t)$:

$$\begin{aligned} \tilde{\mathbb{E}}(r^*(t+\Delta t) - r^*(t)) &= -ar^*(t)\Delta t \\ \widetilde{Var}(r^*(t+\Delta t) - r^*(t)) &= \sigma^2\Delta t. \end{aligned}$$

Define the (i,j) node of the tree as where $t = i\Delta t$ and $r^* = j\Delta R = j\sigma\sqrt{3\Delta t}$. Here $\Delta R := \sqrt{3\Delta t}$ is the spacing between interest rates on the tree, and proves to be a good choice of ΔR in terms of error minimization.

Eg: At (0,0) $r^* = 0$. The next time step can be $(1,-1),(1,0),(1,1)$ if we use the up, straight, down scheme (scheme 0) . We can also use up two, up one, straight scheme (scheme 1) to get $(1,2),(1,1),(1,0)$ (useful to incorporate mean reversion when interest rate is very low) or straight, down one, down two scheme (scheme -1) to get $(1,0),(1,-1),(1,-2)$ (useful to incorporate mean reversion when interest rate is very high).

The normal scheme is scheme 0.We incorporate mean reversion by defining a j_{max} where we switch from scheme 0 to scheme -1 and a j_{min} where we switch from scheme 0 to scheme 1. Hull and White showed that the probabilities are always positive if

$$j_{max} = \frac{0.184}{a\Delta t}$$
$$j_{min} = -j_{max}.$$

The next step is to decide the probabilities p_u, o_m, o_D **at each node** (i,j) that matches the two moments. In particular, recalling $\Delta R = \sqrt{3\Delta t}$ for scheme 0, the system is

$$\begin{aligned} p_u &= \frac{1}{6} + \frac{1}{2}(a^2j^2\Delta t^2 - aj\Delta t) \\ p_m &= \frac{2}{3} - a^2j^2\Delta t^2 \\ p_d &= \frac{1}{6} + \frac{1}{2}(a^2j^2\Delta t^2 + aj\Delta t). \end{aligned}$$

For scheme 1, it is

$$\begin{aligned} p_u &= \frac{1}{6} + \frac{1}{2}(a^2j^2\Delta t^2 + aj\Delta t) \\ p_m &= -\frac{1}{3} - a^2j^2\Delta t^2 - 2aj\Delta t \\ p_d &= \frac{7}{6} + \frac{1}{2}(a^2j^2\Delta t^2 + 3aj\Delta t). \end{aligned}$$

For scheme -1, it is

$$\begin{aligned} p_u &= \frac{7}{6} + \frac{1}{2}(a^2j^2\Delta t^2 - 3aj\Delta t) \\ p_m &= -\frac{1}{3} - a^2j^2\Delta t^2 + 2aj\Delta t \\ p_d &= \frac{1}{6} + \frac{1}{2}(a^2j^2\Delta t^2 - aj\Delta t). \end{aligned}$$

Note that the probabilities by construction depends on j but not on i.

This finishes the first stage. For the second stage, we use an interative process to find $\alpha_i := \alpha(i\Delta t)$. In particular, let $Q_{i,j}$ be the present value of a security that pays 1 dollar if node (i,j) is reached an 0 otherwise. First note that $Q_{0,0} = 1$ and $\alpha_0 = r(0)$. $r(0)$ is determined from the present price of a zero coupon bond with maturity Δt :

$$e^{-r(0)\Delta t} = B(0, \Delta t).$$

Note that $r(0)$ is the prevailing rate compounding continuously on $[0, \Delta t]$. Next, α_1 is such that it gives a precise price for the bond with maturity $2\Delta t$. We walk back to find the value of this bond at time Δt or $i = 1$. This step is done by pure discounting as $r(1,j) := r*(1,j)+\alpha_1$ is the prevaling interest rate on $[\Delta t, 2\Delta t]$. Thus by definition of $Q_{i,j}$

$$V_{i,j} = Q_{i,j} e^{-(r^*(i,j)+\alpha_i)\Delta t},$$

where $V_{i,j}$ is the **present value** of a bond with maturity at time i + 1 if step (i,j) is reached. On the other hand, if step i has been reached, $Q_{i,j}$ is known exactly by the previous probabilities calculation. We then have

$$\sum_j Q_{i,j} e^{-(r^*(i,j)+\alpha_i)\Delta t} = B(0, i+1).$$

From this equation, we can solve for α_i ($B(0, i+1)$ as the initial term structure is part of the given data). This finishes the second stage.

Calibration of models: We want to determine the parameters in the short rate model. They are determined from market data on actively traded options (i.e. caps and swaptions) (aka calibrating instruments). Suppose there are n calibrating instruments. A goodness of fit measure is the square difference:

$$\sum_i (U_i - V_i)^2,$$

where U_i is the market and V_i is the model price of the ith instrument. The number of parameters should not be greater than the number of calibrating instruments. If we have two parameters a, σ as in the Hull-White model, we can choose a to be constant and make σ dependent on time. We can choose times $t_1, t_2, \cdots, t_n$ and let $si(t) = \sigma_i, t_i < t \leq t_{i+1}, 0 \leq i <\leq n-1$ for a total of $n+1$ parameters. The minimization of the goodness of fit measure can be accomplished using the Levenberg-Masquardt procedure. When σ is a function of time, we can use a different objective function for more desirable properties of σ :

$$\sum_i (U_i - V_i)^2 + \sum_{i=1}^{n} w_{1,i}(\sigma_i - \sigma_{i-1})^2 + \sum_{i=1}^{n-1} w_{2,i}(\sigma_{i-1} - 2\sigma_{i-1} + \sigma_{i+1})^2.$$

The second term provides a penalty for large gradient in σ and the third provides a penaly for large curvature in σ. Finally, the calibrating instruments chosen should be as similar as possible to the instrument being valued.

27 Chapter 32 : HJM, LMM. and Multiple zero curves

From the previous chapter, the instantaneous forward rate $F(t,T)$ satisfies :

$$\begin{aligned} F(t,T) &= -\frac{\partial}{\partial T} \log B(t,T) \\ &= -\frac{\partial}{\partial T} \log \tilde{\mathbb{E}}[e^{-\int_t^T r_u du} | \mathcal{F}_t]. \end{aligned}$$

On the other hand, the short rate r_t depends only on one source of noise. If we make the coefficients of the short rate dependent on t then the volitility term structure given by the model in the future may be quite different from that existing in the market today (see the σ_P in the options on bonds in the previous chapter). Also the short rate implies that there is only one way to do risk free discounting (while there are different possibilities that may be used in the same product, such as OIS discounting in a caplet evaluation). A solution is to model the forward rate $F(t,S,T)$ directly and this forward rate can result from different kinds of discounting. This also resolves the source of noice and gives flexibility in modelling the volatility structure (since essentially we model the forward rate for each borrowing period $[T_1, T_2]$ in the future and set the volatility equalling the caplet spot volatility).

Forward rate dynamics: Suppose under the risk neutral measure

$$dB(t,T) = r(t)B(t,T)dt + v(t,T)d\tilde{W}_t.$$

And since

$$f(t,S,T) = -\frac{\log B(t,T) - \log B(t,S)}{T-S},$$

we have

$$df(t,S,T) = \frac{v(t,T)^2 - v(t,S)^2}{2(T-S)}dt - \frac{v(t,T)-v(t,S)}{T-S}d\tilde{W}_t.$$

Pushing $T \to S$ gives

$$dF(t,T) = v(t,T)v_T(t,T)dt - v_T(t,T)d\tilde{W}_t.$$

The negative sign gives some information about the correlation between bond price and forward rate (though we need to be careful since in the forward rate it is $v_T(t,T)$ and not $v(t,T)$). Thus there is a relation between the drift and **standard deviation** of the instantaneous forward rate in the risk neutral world (note: $v_T(t,T)$ or $s(t,T)$ below is NOT the volatility of $F(t,T)$. $v(t,T)$ is the volatility of the bond $B(t,T)$):

$$\begin{aligned} dF(t,T) &= m(t,T)dt + s(t,T)d\tilde{W}_t \\ m(t,T) &= s(t,T)\int_t^T s(t,u)du. \end{aligned}$$

If we set $s(t,T) = \sigma$ we obtain the forward rate corresponding Ho-Lee model for short rate and when $s(t,T) = e^{-a(T-t)}$ we obtain the corresponding Hull-White model for the short rate.

Note that the short rate resulting from a general HJM model is non Markov (possibly due to the relation between the drift and standard deviation of the forward rate, especially if $s(t,T) = -v_T(t,T)$ depends on r_t or $B(t,T)$ itself).

LIBOR market model: One drawback of the HJM model is that the instantaneous forward rate is not directly observable in the market. Also it is difficult to calibrate the model to prices of actively traded instruments. (Due to the exponential discounting, the instantaneous forward rate is NOT the price of some instrument. Hence its dynamics under some forward risk measure is not conveniently given, unlike the forward LIBOR rate. This in turn leads to consequence that Black's formula is not readily usable, since the forward bond price is related to the forward rate, but the forward bond price is naturally given under the forward risk measure.) We instead model the LIBOR forward rate $F_k(t)$ that is available for the period $[t_k, t_{k+1}]$. Since it is essentially the price of the traded asset, under the forward risk measure with respect to $B(t, t_{k+1}), F_k(t)$ is a martingale:

$$dF_k(t) = v_k(t)F_k(t)d\tilde{W}_{k+1}(t).$$

On the other hand, suppose the forward bond price processes has the dynamics

$$dB^k(t, t_k) = \sigma_k(t) B^k(t, t_k) d\tilde{W}_k(t).$$

Denote $m(t)$ to be the index for the next reset date at time t. That is if $t_k < t \leq t_{k+1}$ then $m(t) = k + 1$. When value interest rate derivatives at a time t, it is most convenient to use the forward risk neutral measure with respect to $B(t, m(t))$. This is called the rolling forward risk neutral measure (since it changes with t !). The dynamics of $F_k(t)$ with respect to $B(t, m(t))$ is

$$dF_k(t) = v_k(t)(\sigma_{m(t)} - \sigma_{k+1}) F_k(t) dt + v_k(t) F_k(t) d\tilde{W}_{m(t)}(t)$$

The relationship between forward rates and bond prices is

$$\frac{B(t, t_i)}{B(t, t_{i+1})} = 1 + \delta_i F_i(t)$$

or

$$\log B(t, t_i) - \log B(t, t_{i+1}) = \log(1 + \delta_i F_i(t)).$$

Apply Ito's formula and equating the $d\tilde{W}_{t_{i+1}}$ terms give

$$\sigma_i(t) - \sigma_{i+1}(t) = \frac{\delta_i F_i(t) v_i(t)}{1 + \delta_i F_i(t)}.$$

Plugging this in we have the dynamics of $F_k(t)$ with respect to the rolling forward risk neutral measure as

$$dF_k(t) = F_k(t) v_k(t) \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t) v_i(t)}{1 + \delta_i F_i(t)} + F_k(t) v_k(t) d\tilde{W}_{m(t)}(t).$$

The HJM model is the limiting case of this model as $\delta_i \to 0$.

Forward rate volatilities : For convenience we assume that the volatility $v_k(t)$ of $F_k(t)$ is a step function. Let $\Lambda_i, i = 1, \cdots, n$ be given. We set

$$v_k(t) = \Lambda_{k-m(t)}.$$

Thus $v_k(t) = \Lambda_i$ where i is the nunber of accrual periods remaining between the next reset date at time t_k. For example, consider $k = 4$ and $t_1 < t \leq t_2$. Then $v_4(t) = \Lambda_2$. And if $t_3 < t \leq t_4$ then $v_4(t) = \Lambda_0$.

The Λ_i can be related to the volatilities used to value caplets in Black's model. Suppose that σ_k is the Black volatility for the caplet that corresponds to the period $[t_k, t_{k+1}]$. Thus $\sigma_k \sqrt{t_k}$ IS the standard deviation for $\log F_k(t_k)$. On the other hand, since

$$d \log F_k(t) = \cdots dt + v_k(t) d\tilde{W}_k(t),$$

the variance of $\log F_k(t_k)$ is

$$\int_0^{t_k} v_k^2(t) dt = \sum_{i=1}^{k} \Lambda_{k-i}^2 \delta_i.$$

We then have, by equating variance:

$$\sigma_k^2 t_k = \sum_{i=1}^{k} \Lambda_{k-i}^2 \delta_i.$$

Example: Suppose $n = 3$ and $\delta_i = 1$. Also suppose $si_1 = 24\%, \sigma_2 = 22\%, \sigma_3 = 20\%$. Then $\sigma_1 = \Lambda_0 = 24\%$.

$$\Lambda_0^2 + \Lambda_1^2 = 2\sigma_2^2 = 2 \times 0.22^2.$$

We get $\Lambda_1 = 19.80\%$. Finally

$$\Lambda_0^2 + \Lambda_1^2 + \Lambda_2^2 = 3\sigma_3^2 = 3 \times 0.20^2.$$

We get $\Lambda_2 = 15.23\%$.

Implementation of the forward LIBOR model: We had

$$\begin{aligned} dF_k(t) &= F_k(t) v_k(t) \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t) v_i(t)}{1 + \delta_i F_i(t)} + F_k(t) v_k(t) d\tilde{W}_{m(t)}(t) \\ &= F_k(t) \Lambda_{k-m(t)} \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t) \Lambda_{i-m(t)}}{1 + \delta_i F_i(t)} + F_k(t) \Lambda_{k-m(t)} d\tilde{W}_{m(t)}(t) \end{aligned}$$

So that

$$d \log F_k(t) = \Lambda_{k-m(t)} \sum_{i=m(t)}^{k} \frac{\delta_i F_i(t) \Lambda_{i-m(t)}}{1 + \delta_i F_i(t)} + \Lambda_{k-m(t)} d\tilde{W}_{m(t)}(t).$$

We approximate $F_i(t)$ by a stepwise function : $F_i(t) = F_i(t_j)$ for $t_j < t < t_{j+1}$. Then

$$F_k(t_{j+1}) = F_k(t_j) \exp\Big[\Big(\sum_{i=j+1}^{k} \frac{\delta_i F_i(t_j) \Lambda_{i-j-1} \Lambda_{k-j-1}}{1 + \delta_i F_i(t_j)} - \frac{1}{2} \Lambda_{k-j-1}^2 \Big) \delta_j + \Lambda_{k-j-1} (\tilde{W}_{j+1}(t_{j+1}) - \tilde{W}_{j+1}(t_j)). \Big]$$

Remark on Monte Carlo simulation: in terms of simulation, we stat with the initial values for all rates $F_i(0), i = 1, \cdots n$. Next we use the above formula to simulate $F_i(t_1), i = 2, \cdots n$. Then we simulate $F_i(t_2), i = 3, \cdots, n$ etc. Note that as we move through time, the length of the zero curve gets shorter and shorter. At each interval $[t_i, t_{i+1}]$ we simulate an independent Normal distribution with mean 0 and variane δ_i. Even though $\tilde{W}_j$ and $\tilde{W}_{j-1}$ are related, we are simulating on the interval $[t_i, t_{i+1}]$ the distributions that are independent from the distribution of $[t_{i-1}, t_i]$ (in the physical measure sense, for example). Thus there is ONLY one source of uncertainty. Also note that the "relevant" time of $\tilde{W}_j$ is until time t_j only. Thus the above formula can be shortened to

$$F_k(t_{j+1}) = F_k(t_j) \exp\Big[\Big(\sum_{i=j+1}^{k} \frac{\delta_i F_i(t_j)\Lambda_{i-j-1}\Lambda_{k-j-1}}{1+\delta_i F_i(t_j)} - \frac{1}{2}\Lambda^2_{k-j-1}\Big)\delta_j + \Lambda_{k-j-1}\epsilon\sqrt{\delta_j}\Big],$$

where ϵ is **sample from** a standard normal distribution. The multiple source of uncertainty case is similar, and its formula is

$$\begin{aligned} F_k(t_{j+1}) &= F_k(t_j) \exp\Big[\Big(\sum_{i=j+1}^{k} \frac{\delta_i F_i(t_j)\sum_{q=1}^{p}\lambda_{i-j-1,q}\lambda_{k-j-1,q}}{1+\delta_i F_i(t_j)} - \frac{1}{2}\sum_{q=1}^{p}\lambda^2_{k-j-1,q}\Big)\delta_j \\ &+ \sum_{q=1}^{p}\lambda_{k-j-1,q}\epsilon_q\sqrt{\delta_j},\Big] \end{aligned}$$

where $\lambda_{i,q}$ is the qth component of the volatility when there are i accrual periods between the next reset date and the maturity of the forward contract.

Model calibration: There are two steps to calibration of the forward rates. The first is to calibrate the Λ_i to fit the prices of the calibrating instrumetns (typically caps, swaptions). The penalty function is similar to the one mentioned in the previous chapter:

$$\sum_i (U_i - V_i)^2 + P,$$

where P is chosen so that the $\Lambda's$ have some smooth properties. Next we determine the $\lambda's$ from the $\Lambda's$. This is usually done by PCA. Suppose

$$\Delta F_j = \sum_{q=1}^{M} \alpha_{j,q} x_q,$$

where M is the total number of factors (which equal to then number of different forward rates) , ΔF_j is the change in the jth forward rate F_j and $\alpha_{j,q}$ is the factor loading for the jth forward rate and qth factor and x_q is the factor score of the qth factor. Let s_q be the standard deviation of the qth factor score. If the number of factors (the number of uncertainties) used in the LIBOR model p is equal to the total number of factors M we can set

$$\lambda_{j,q} = \alpha_{j,q} s_q.$$

If, as usual, $p < M$ the $\lambda_{j,q}$ must be scaled so that

$$\Lambda_j = \sqrt{\sum_{q=1}^{p} \lambda_{j,q}^2}.$$

We achieve this by setting

$$\lambda_{j,q} = \frac{\Lambda_j s_q \alpha_{j,q}}{\sqrt{\sum_{q=1}^{p} s_q^2 \alpha_{j,q}^2}}.$$

Modelling multiple zero curves: It is now usual to use the OIS zero curve tas the risk-free zero curve for discounting. This means that more than one zero curve must be modeled for derivatvies such as swaps, interest rate caps and swaptions whose payoffs depend on LIBOR. A LIBOR zero curve is necessary to calculate payoffs, the OIS zero curve is necessary for discounting. Even within LIBOR there might be multiple curves for different maturrity. This reflects credit risk : a 12 month LIBOR loan has more risk than 12 continually refreshed 1 month LIBOR loans.

If we model both LIBOR and OIS curves, it is not possible to assume the no arbitrage condition. One alternative is to model credit risk and liquidity risk so that the spread between LIBOR and OIS is explained. This adds a huge layer of complexity to the model. Practitioners just model the two curves separately and usually ignore the arbitrage opportunities created by the use of the two curves.

Lastly, the curves are obtained by modelling the forward rates as we have done so far. However, we should keep in mind that $F_{LD}(t, S, T)$ (for LIBOR discounting) is a martingale under the forward risk measure associated with $P_{LD}(t, T)$ and $F_{OD}(t, S, T)$ (for OIS discounting) is a martingale under the forward risk measure associated with $P_{OD}(t, T)$ but not necesarily vice versa.

28 Chapter 34: Energy and Commodity Derivatives

Agricultural commodities: A watched stat is the stock-to-use ratio, which is the ratio between year-end inventory and the year's usage, typically between 20 % to 40 %. As the ratio becomes lower, the price becomes more sensitive to supply change and thus volatility increases. Agricultural prices exhibit mean reversion property due to supply and demand. The prices also tend to be seasonal, as storage is expensive and there is a limit to storage time. Weather also plays a role in price change. Some of agricultural products are used to feed livestocks (e.g. corn). The price of livestock then is dependent on agricultural prices, which in turn is subject to weather.

Metals: Metals can be consumption assets (copper) or investment assets (silver, gold). Inventory level stats are also monitored to determine shor-term volatility. Exchange rate can play a role since metal is extracted in a different country than the one where the price is quoted. Investment asset metal price may not exhibit mean reversion property. Consumption asset metal may exhibit mean reversion, again due to supply and demand.

Energy: prices do follow mean reverting processes, again due to supply and demand. Crude oil: virtually any derivative that is available on common stokcs or indices is now available with oil as the underlying asset. Natural gas: A typical OTC contract is for delivery of a specified amount of natural gas at a uniform rate over 1 month period. Forward contracts, options and swaps are also available. The seller is usually responsible for moving the gas through pipelines to the specified location. A popular source for heating, hence price is seasonal. Electriciy: cannot be stored. A major use is for AC systems. Demand is much greater in summer than winter. Because of inability for storage, price can be subject to very large movements (1000 % increase has been observed).

Trinomial tree for commodity price: The commodityp price tree is built so that the futures price induced by the tree matches the futures price observed today.

$$d \log S = (\theta(t) - a \log(S_t) dt) + \sigma d\tilde{W}_t.$$

Also sometimes written as

$$\frac{dS_t}{S_t} = (\theta^*(t) - a \log(S_t) dt) + \sigma d\tilde{W}_t.$$

To build the tree for S, we first build the tree for X :

$$dX_t = -aX_t dt + \sigma d\tilde{W}_t.$$

The procedure is exactly as specified in the trinomial tree for short rate model. We then add a process $\alpha(i)$ so that the expected value of the commodity price at time i is the **futures price** at time 0 (this futures price is $F(0,i)$ and hence dependent on i). More specifically, the equation is

$$F(0,i) = \tilde{\mathbb{E}}(e^{X(i)+\alpha(i)}), i = 1, \cdots, n.$$

Interpolation and Seasonality: When a large number of time steps are used, it is necessary to interpolate between futures price (say $F(0,5)$ and $F(0,7)$) to obtain a futures price at the end of each time step (say for the time step at June, or month 6). When there is seasonality, the interpolation procedure should reflect this. One way is to collect monthly historical data on the spot price and calculate the 12 month moving average of the price. A **percentage season factor** can then be estimated as the average ratio of the spot price for the month to the 12 month moving average of the spot prices that is centered (approximately) on the month (say we have $S(9), S(10), S(11), S(12)$). The percentage seasonal factors are then used to deseasonalize the futures prices that are known (say $F(0,9)$ and $F(0,12)$ are known. We use $S(9)$ and $S(12)$ to deseasonalize it). Monthly deseasonalized futures are then calculated using interpolation (say we calculate $\tilde{F}(0,10)$ and $\tilde{F}(0,10)$ based on $\frac{F(0,9)}{S(9)}$ and $\frac{F(0,12)}{S(12)}$). These futures prices are then seasonalized using the percentage seasonal factors and the tree is built. (say we calculate $F(0,10) = S(10)\tilde{F}(0,10)$ and $F(0,11) = S(11)\tilde{F}(0,11)$).

Weather derivative: The underlying variable is cumulative HDD (heating degree days) or CDD (cooling degree days) during a month. HDD = $\max(0, 65 - A)$, CDD = $\max(0, A - 65)$ where A is the average of the highest and lowest temperature during the day at a specified weather station, measured in Fahrenheit. There is **no systemic risk** (risk priced by the market) in the payoffs of weather and insurance CAT (catastrophic) derivatives. There has been study that showed no statistically significant correlation between the returns from CAT bonds and stock market returns. This means that estimates made from historical data (real world estimates) can also be assumed to apply to the risk neutral world. Therefore, weather and insurance derivatives can be priced by : using historical data to estimate the expected payoff (the distribution of the log normal random variable) and discount the estimated payoff at the risk free rate. The uncertainty in the underlying weather variable also does not grow at the rate of square root of time. For example, the uncertainty about the February HDD at a certain location in 4 years is usually only a little greater than the uncertainty about the February HDD at the same location in 1 year. Finally the Black-Scholes formula can be applied to find the price of call and put derivatives.

Ex: Call option on cumulative HDD in Feb 2016 with strike price of 700 and payment rate of 10,000 USD per degree day . Suppose that the HDD estimated from historical data is log normal with mean 710 and SD of the log equalling 0.07. Thus the expected pay off is

$$\begin{aligned} E = 10,000 \quad &\times \quad [710N(d+) - 700N(d-)] \\ d\pm \quad &= \quad \frac{\log(\frac{710}{700}) \pm \frac{1}{2}0.07^2}{0.07}. \end{aligned}$$

Suppose the option is being valued in Feb 2015, the value of the option is $Ee^{-0.03\times 1}$ where 0.03 is the risk free rate.