Math 485
Fall 2018
Midterm exam 2
10/15/18

Name (Print): ____________________

This exam contains 9 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	15	
3	20	
4	30	
5	20	
Total:	100	

1. (15 points) Consider the multiperiod binomial model with $r = 0.01, u = 1.03, d = .95$ and $\Delta T = 1$. Suppose $S_0 = 100$. (Note $r \neq 0$ here). Find the price of an American put option on S with $K = 110$ and $N = 2$.

   Ans: $\tilde{p} = \frac{e^{r\Delta T} - d}{u-d} \approx 0.75$.

Event	$V_{exercise}$	V_{cont}	V	Decision
uu	3.91	NA	3.91	NA
$ud = du$	12.15	NA	12.15	NA
dd	19.75	NA	19.75	NA
u	7	5.91	7	Exercise
d	15	13.91	15	Exercise

   Finally, $V_0^E = 10, V_0^C = 8.91$ and thus $V_0 = 10$.

2. (15 points) Consider the multiperiod binomial model with $r = 0, u = 1.5, d = .5$ and $\Delta T = 1$. Suppose $S_0 = 100$. Consider a look back option with expiry $N = 2$. Find V_0 using the replicating portfolio approach. Note: you will NOT receive credit for this problem if you use the risk neutral pricing approach.

   We have

$$\begin{aligned}
\Delta_1(u)S_2(uu) + y_1(u)e^{r\Delta T} &= V_2(uu) \\
\Delta_1(u)S_2(ud) + y_1(u)e^{r\Delta T} &= V_2(ud).
\end{aligned}$$

   Thus $\Delta_1(u) = 1/2, y_1(u) = 225/2$.

$$V_1(u) = \Delta_1(u)S_1(u) + y_1(u) = 375/2.$$

$$\begin{aligned}
\Delta_1(d)S_2(du) + y_1(d)e^{r\Delta T} &= V_2(du) \\
\Delta_1(d)S_2(dd) + y_1(d)e^{r\Delta T} &= V_2(dd).
\end{aligned}$$

   Thus $\Delta_1(d) = 0, y_1(d) = 100$.

$$V_1(d) = \Delta_1(d)S_1(d) + y_1(d) = 100.$$

$$\begin{aligned}
\Delta_0 S_1(u) + y_0 e^{r\Delta T} &= V_1(u) \\
\Delta_0 S_1(d) + y_0 e^{r\Delta T} &= V_1(d).
\end{aligned}$$

   Thus $\Delta_0 = 7/8, y_0 = 225/4$

$$V_0 = \Delta_0 S_0 + y_0 = 575/4.$$

3. Let S_t follow the Black-Scholes model

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t.$$

where r is the risk free rate and σ is a constant. Let S_0 be the initial stock price at time 0 that is given. Find V_0 where

(a) (10 points) $V_T = \log \frac{S_T}{S_0}$.

Ans: $S_T = S_0 e^{(r-\frac{\sigma^2}{2})T+\sigma\tilde{W}_T}$. Thus $V_T = \frac{r-\sigma^2}{2}T + \sigma\tilde{W}_T$ and

$$V_0 = \tilde{E}(e^{-rT}V_T) = e^{-rT}(r - \frac{1}{2}\sigma^2)T.$$

(b) (10 points) $V_T = \frac{\int_0^T S_u du}{T} - K$ where K is a constant.

Ans:

$$\begin{aligned} V_0 = \tilde{E}(e^{-rT}V_T) &= e^{-rT}(\frac{\int_0^T \tilde{E}(S_u)du}{T} - K) \\ &= e^{-rT}(\frac{\int_0^T e^{ru}du}{T} - K) \\ &= e^{-rT}(\frac{e^{rT}-1}{rT} - K). \end{aligned}$$

4. Let W_t be a Brownian motion. Compute

   (a) (10 points) $E(e^{2W_t} \Big| W_s), 0 < s < t$

   Ans:

$$\begin{aligned} E(e^{2W_t}|W_s) &= E(e^{2(W_t - W_s + W_s)}|W_s) = e^{2W_s} E(e^{2(W_t - Ws)}) \\ &= e^{2W_s} e^{2(t-s)}. \end{aligned}$$

   (b) (10 points) $E(|W_t|)$.

$$\begin{aligned} E|W_t| &= \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \, dx \\ &= 2\int_0^{\infty} x \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \, dx \\ &= \sqrt{\frac{2t}{\pi}}. \end{aligned}$$

(c) (10 points) $E((K - e^{W_t})^+), K$ a constant. Hint: We knew what $E((e^{W_t} - K)^+)$ is. On the other hand, what is $E((K - e^{W_t})^+) - E((e^{W_t} - K)^+)$? (What is $x^+ - (-x)^+$?) You can do this problem without writing $E((e^{W_t} - K)^+)$ out explicitly if you want.

Ans: $E((K - e^{W_t})^+) - E((e^{W_t} - K)^+) = E(K - e^{W_t}) = K - e^{t/2}$. Thus $E((K - e^{W_t})^+) = K - e^{t/2} + E((e^{W_t} - K)^+)$.

5. Let W_t be a Brownian motion. Compute

   (a) (10 points) $d\ (e^{W_t} W_t)$.
   Ans: $d(e^{W_t} W_t) = e^{W_t}(W_t + 1)dW_t + \frac{1}{2}e^{W_t}(W_t + 2)dt$.

   (b) (10 points) $d\ (e^t \int_0^t W_s dW_s)$.
   Ans: $d\ (e^t \int_0^t W_s dW_s) = e^t(\int_0^t W_s dW_s)dt + e^t W_t dW_t$.

Scratch (Won't be graded)

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