Math 485
Fall 2018
Midterm exam 1
10/11/18

Name (Print): ______________________

This exam contains 6 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (one sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	25	
Total:	100	

1. (15 points) Consider a one period trinomial model with two assets S^1, S^2. There are three future states of the universe : $\omega_1, \omega_2, \omega_3$. Suppose that

$$S^1_T(\omega_1) = S^1_0 u_1, S^2_T(\omega_1) = S^2_0 d_2,$$
$$S^1_T(\omega_2) = S^1_0 m_1, S^2_T(\omega_2) = S^2_0 m_2,$$
$$S^1_T(\omega_3) = S^1_0 d_1, S^2_T(\omega_3) = S^2_0 u_2.$$

for $i = 1, 2$. Let $u_1 = 1.05, m_1 = 1, d_1 = 0.95, u_2 = 1.15, m_2 = 0.95, d_2 = .90, r = 0, T = 1, S^1_0 = S^2_0 = 100$ (note that as S^1 goes up S^2 goes down and vice versa here) The risk neutral probability condition is

$$\tilde{E}(e^{-rT} S^i_T) = S^i_0, i = 1, 2.$$

where we designate $p_1 = \tilde{P}(\omega_1), p_2 = \tilde{P}(\omega_2), 1 - p_1 - p_2 = \tilde{P}(\omega_3)$.

a. Is there a unique solution $0 < p_1, p_2 < 1$ (that form a probability distribution) in this model?

Ans: We have

$$\begin{aligned} 105p_1 + 100p_2 + 95(1 - p_1 - p_2) &= 100 \\ 90p_1 + 100p_2 + 115(1 - p_1 - p_2) &= 100. \end{aligned}$$

There exists a unique solution : $p_1 = p_2 = 1/3$.

b. Is every derivative on S^1, S^2 replicable in this model? Why or why not?

Ans: Yes, because the risk neutral probability is unique.

c. Is this model arbitrage free ? Why or why not? If you answer is no, construct an arbitrage opportunity for this model.

Ans: Yes, because the risk neutral probability exists in this model.

2. (15 points) Consider a one period trinomial model with two assets S^1, S^2. There are three future states of the universe : $\omega_1, \omega_2, \omega_3$. Suppose that

$$S^1_T(\omega_1) = S^1_0 u_1, S^2_T(\omega_1) = S^2_0 u_2,$$
$$S^1_T(\omega_2) = S^1_0 m_1, S^2_T(\omega_2) = S^2_0 m_2,$$
$$S^1_T(\omega_3) = S^1_0 d_1, S^2_T(\omega_3) = S^2_0 d_2.$$

for $i = 1, 2$. Let $u_1 = 1.05, m_1 = 1, d_1 = 0.95, u_2 = 1.15, m_2 = 1.02, d_2 = .95, r = 0, T = 1, S^1_0 = 100, S^2_0 = 50$. Is there an arbitrage opportunity in this model? If yes, construct one. If no explain why not.

Ans: No. Because S^2 percentage is always at least as high (and sometimes higher) than S^1. An arbitrage opportunity would be to short 1 share of S^1 and long 2 shares of S^2. The portfolio initial value is 0. We have

$$\begin{aligned} \pi_T(\omega_1) &= 115 - 105 = 10 \\ \pi_T(\omega_2) &= 102 - 10 = 2 \\ \pi_T(\omega_1) &= 95 - 95 = 0. \end{aligned}$$

So it is an arbitrage opportunity.

3. (15 points) Consider the multiperiod binomial model with $r = 0, u = 1.05, d = .98$ and $\Delta T = 1$. Suppose $S_0 = 100$. Consider a look back option with expiry $N = 4$. Find $V_1(u), V_2(du)$.

   Ans: We use $\tilde{p} = \frac{e^{r\Delta T} - d}{u - d} = \frac{2}{7}$.

Future path for $V_1(u)$	Probability	Option value
uuu	$\tilde{p}^3$	121.55
uud	$\tilde{p}^2(1-\tilde{p})$	115.76
udu	$\tilde{p}^2(1-\tilde{p})$	113.45
udd	$\tilde{p}(1-\tilde{p})^2$	110.25
duu	$\tilde{p}^2(1-\tilde{p})$	113.45
dud	$\tilde{p}(1-\tilde{p})^2$	108.05
ddu	$\tilde{p}(1-\tilde{p})^2$	105.88
ddd	$(1-\tilde{p})^3$	105

   From the table, $V_1(u) \approx 108.3367$.

Future path for $V_2(du)$	Probability	Option value
uu	$\tilde{p}^2$	113.45
ud	$\tilde{p}(1-\tilde{p})$	108.05
du	$\tilde{p}(1-\tilde{p})$	105.88
dd	$(1-\tilde{p})^2$	102.90

   From the table, $V_2(du) \approx 105.42$

4. (15 points) Consider the multiperiod binomial model with $r = 0, u = 1.05, d = .98$ and $\Delta T = 1$. Suppose $S_0 = 100$. Consider a up and in put option with barrier $L = 106$, strike $K = 120$ and expiry $N = 4$. Find $V_2(ud), V_2(du)$.

   Ans: We use $\tilde{p} = \frac{e^{r\Delta T} - d}{u - d} = \frac{2}{7}$.

Future path for $V_2(ud)$	Probability	Option value
uu	$\tilde{p}^2$	6.55
ud	$\tilde{p}(1-\tilde{p})$	14.21
du	$\tilde{p}(1-\tilde{p})$	0
dd	$(1-\tilde{p})^2$	0

   From the table, $V_2(ud) \approx 3.43$

Future path for $V_2(du)$	Probability	Option value
uu	$\tilde{p}^2$	6.55
ud	$\tilde{p}(1-\tilde{p})$	14.21
du	$\tilde{p}(1-\tilde{p})$	0
dd	$(1-\tilde{p})^2$	0

   From the table, $V_2(du) \approx 3.43$

5. (15 points) Consider the multiperiod binomial model with $r = 0, u = 1.05, d = .98$ and $\Delta T = 1$. Suppose $S_0 = 100$. Consider a up and out call option with barrier $L = 106$, strike $K = 95$ and expiry $N = 4$. Find $V_2(ud), V_2(du)$.

   Ans: We use $\tilde{p} = \frac{e^{r\Delta T} - d}{u-d} = \frac{2}{7}$.

Future path for $V_2(ud)$	Probability	Option value
uu	$\tilde{p}^2$	0
ud	$\tilde{p}(1-\tilde{p})$	0
du	$\tilde{p}(1-\tilde{p})$	10.88
dd	$(1-\tilde{p})^2$	3.83

   From the table, $V_2(ud) \approx 4.17$

Future path for $V_2(du)$	Probability	Option value
uu	$\tilde{p}^2$	0
ud	$\tilde{p}(1-\tilde{p})$	0
du	$\tilde{p}(1-\tilde{p})$	10.88
dd	$(1-\tilde{p})^2$	3.83

   From the table, $V_2(du) \approx 4.17$

6. Consider the multiperiod binomial model with S_0, r, u, d, N and ΔT to be determined.

   (a) (10 points) What is (are) the condition(s) on S_0, r, u, d, N and ΔT so that there is no arbitrage in this model?

   Ans: The risk neutral probability is $\tilde{p} = \frac{e^{r\Delta T} - d}{u-d} = \frac{2}{7}$. So it is the usual condition that $d < e^{r\Delta T} < u$ so that $0 < \tilde{p} < 1$.

   (b) (15 points) Does the put call parity at any time point : $V_k^{\text{call}} - V_k^{\text{put}} = V_k^{\text{forward}}$, for any $0 \leq k \leq N$ hold in this model?

   Ans: Yes. If we consider portfolio π^1 of longing 1 share of call and shorting one share of put and portfolio π^2 of longing 1 share of forward. Then $\pi^1_T(\omega) = \pi^2_T(\omega)$ for any ω. Therefore $\pi^1_t(\omega) = \pi^2_t(\omega)$ for any $0 \leq t \leq T$ by the no arbitrage principle.

Scratch (Won't be graded)

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