Math 485
Fall 2018
Final exam
12/18/18

Name (Print): ______________________

This exam contains 9 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 pages of note (two sided) on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
7	20	
8	20	
9	20	
10	20	
Total:	200	

1. (20 points) Consider the Hull-White model for interest rate:

$$dr_t = k(\mu - r_t)dt + \sigma d\tilde{W}_t.$$

Find $\lim_{t\to\infty} \widetilde{E}(r_t)$.

Ans: Recall from homework 11 solution:

$$\begin{aligned} de^{kt}r_t &= e^{kt}(kr_t dt + k(\mu - r_t)dt + \sigma d\tilde{W}_t) \\ &= e^{kt}(k\mu dt + \sigma d\tilde{W}_t). \end{aligned}$$

Thus

$$\begin{aligned} e^{kt}r_t - r_0 &= \int_0^t ke^{ks}\mu ds + \int_0^t e^{ks}\sigma d\tilde{W}_s \\ &= \mu(e^{kt} - 1) + \int_0^t e^{ks}\sigma d\tilde{W}_s. \end{aligned}$$

Thus

$$r_t = e^{-kt}r_0 + \mu(1 - e^{-kt}) + \int_0^t e^{-k(t-s)}\sigma d\tilde{W}_s.$$

Therefore $\tilde{E}(r_t) = e^{-kt}r_0 + \mu(1 - e^{-kt})$ and $\lim_{t\to\infty} \widetilde{E}(r_t) = \mu$ (assuming $k > 0$).

2. (20 points) Consider a one period trinomial model with two assets S^1, S^2. There are three future states of the universe : $\omega_1, \omega_2, \omega_3$. Suppose that

$$\begin{gathered} S^1_T(\omega_1) = S^1_0 u_1, S^2_T(\omega_1) = S^2_0 d_2, \\ S^1_T(\omega_2) = S^1_0 m_1, S^2_T(\omega_2) = S^2_0 u_2, \\ S^1_T(\omega_3) = S^1_0 d_1, S^2_T(\omega_3) = S^2_0 m_2. \end{gathered}$$

for $i = 1, 2$. Let $u_1 = 1.05, m_1 = 1, d_1 = 0.95, u_2 = 1.15, m_2 = 0.95, d_2 = .90, r = 0, T = 1, S^1_0 = S^2_0 = 100$ (note that the states of S^1, S^2 are not in sync in this model) The risk neutral probability condition is

$$\tilde{E}(e^{-rT}S^i_T) = S^i_0, i = 1, 2.$$

where we designate $p_1 = \tilde{P}(\omega_1), p_2 = \tilde{P}(\omega_2), 1 - p_1 - p_2 = \tilde{P}(\omega_3)$.

a. Is there a unique solution $0 < p_1, p_2 < 1$ (that form a probability distribution) in this model?

$$\begin{aligned} 105p_1 + 100p_2 + 95(1 - p_1 - p_2) &= 100 \\ 90p_1 + 115p_2 + 95(1 - p_1 - p_2) &= 100. \end{aligned}$$

There exists a unique solution : $p_1 = p_2 = 1/3$.

b. Is every derivative on S^1, S^2 replicable in this model? Why or why not?

Yes because the risk neutral probability is unique.

c. Is this model arbitrage free ? Why or why not? If you answer is no, construct an arbitrage opportunity for this model.

Yes because a risk neutral probability exists.

3. (20 points) Consider the Black-Scholes model in which the interest rate is stochastic:

$$dS_t = r_t S_t dt + \sigma_S S_t d\tilde{W}_t.$$

Suppose $S_0 = 100, K = 110, \sigma_S = 0.1, T = 1, B(0,T) = 0.98$ where $B(0,T)$ is the price of the zero coupon bond with maturity T. Find the price of the call option on S with maturity T and strike K, suppose that $\sigma_F = 0.12$ wher $F(t,T) = \frac{S_t}{B(t,T)}$. You can leave the answer in the form of $N(x)$ where x is a concrete number.

Ans:

Using Black's formula:

$$V_0 = B(0,T)[F(0,T)N(d+) - KN(d-)]$$

where

$$d\pm = \frac{\pm\frac{1}{2}\sigma_F^2 T - \log\frac{K}{F(0,T)}}{\sigma_F\sqrt{T}}$$

we have

$$\begin{aligned} d_+ &= -0.5659 \\ d_- &= -0.6859 \end{aligned}$$

and $V_0 \approx 2.01$.

4. Consider the multiperiod binomial model with $r = 0, u = 1.5, d = .5$ and $\Delta T = 1$. Suppose $S_0 = 100$. Consider an Asian option with expiry $N = 2$.

   (a) (10 points) Find V_0 using the replicating portfolio approach. Note: you will NOT receive credit for this problem if you use the risk neutral pricing approach.

   We have

$$\begin{aligned} \Delta_1(u)S_2(uu) + y_1(u)e^{r\Delta T} &= V_2(uu) \\ \Delta_1(u)S_2(ud) + y_1(u)e^{r\Delta T} &= V_2(ud). \end{aligned}$$

   Thus $\Delta_1(u) = 1/3, y_1(u) = 83.33$

$$V_1(u) = \Delta_1(u)S_1(u) + y_1(u) = 133.33.$$

$$\begin{aligned} \Delta_1(d)S_2(du) + y_1(d)e^{r\Delta T} &= V_2(du) \\ \Delta_1(d)S_2(dd) + y_1(d)e^{r\Delta T} &= V_2(dd). \end{aligned}$$

Thus $\Delta_1(d) = 1/3, y_1(d) = 50$.

$$V_1(d) = \Delta_1(d)S_1(d) + y_1(d) = 200/3.$$

$$\begin{aligned} \Delta_0 S_1(u) + y_0 e^{r\Delta T} &= V_1(u) \\ \Delta_0 S_1(d) + y_0 e^{r\Delta T} &= V_1(d). \end{aligned}$$

Thus $\Delta_0 = 2/3, y_0 = 100/3$

$$V_0 = \Delta_0 S_0 + y_0 = 100.$$

(b) (10 points) Find V_0 using the risk neutral probability approach. Verify that the answers are the same in both parts.
$p = \frac{e^{r\Delta T}-d}{u-d} = 1/2$. Thus

$$V_0 = \frac{1}{4}(158.33 + 108.33 + 58.33 + 75) = 100.$$

5. (20 points) Consider the multiperiod binomial model with $r = 0, u = 1.05, d = .98$ and $\Delta T = 1$. Suppose $S_0 = 100$. Consider a down and out put option with barrier $L = 97$, strike $K = 120$ and expiry $N = 4$. Find $V_2(ud), V_2(du)$.

Ans: We use $\tilde{p} = \frac{e^{r\Delta T}-d}{u-d} = \frac{2}{7}$.

Future path for $V_2(ud)$	Probability	Option value
uu	$\tilde{p}^2$	6.55
ud	$\tilde{p}(1-\tilde{p})$	14.12
du	$\tilde{p}(1-\tilde{p})$	14.12
dd	$(1-\tilde{p})^2$	21.17

From the table, $V_2(ud) \approx 17.1$

Future path for $V_2(du)$	Probability	Option value
uu	$\tilde{p}^2$	6.55
ud	$\tilde{p}(1-\tilde{p})$	14.12
du	$\tilde{p}(1-\tilde{p})$	14.12
dd	$(1-\tilde{p})^2$	21.17

From the table, $V_2(du) \approx 17.1$

6. Consider the forward contract with expiry T and strike $K : V_T = S_T - K$.

(a) (10 points) Find $V_t, 0 \le t \le T$.
Ans: $V_t = S_t - Ke^{-r(T-t)}$

(b) (10 points) The V_t in part a can be expressed as $f(t, S_t)$. Show that $f(t, x)$ satisfies the

Black-Scholes PDE:

$$\begin{aligned} -rf + f_t + f_x rx + \frac{1}{2} f_{xx}\sigma^2 x^2 &= 0 \\ f(T,x) &= x - K. \end{aligned}$$

Ans: $f(t,x) = x - Ke^{-r(T-t)}, f_t = -rKe^{-r(T-t)}, f_x = 1, f_{xx} = 0.$. Plugging in :

$$-r(x - Ke^{-r(T-t)}) - rKe^{-r(T-t)} + rx = 0.$$

7. (20 points) Consider the multiperiod binomial model with $r = 0.02, u = 1.03, d = .98$ and $\Delta T = 1$. Suppose $S_0 = 100$. (Note $r \neq 0$ here). Find the price of an American put option on S with $K = 115$ and $N = 2$.

   Ans: $\tilde{p} = \frac{e^{r\Delta T} - d}{u-d} \approx 0.8040$.

Event	$V_{exercise}$	V_{cont}	V	Decision
uu	8.91	NA	8.91	NA
$ud = du$	14.06	NA	14.06	NA
dd	18.96	NA	18.96	NA
u	12	9.7230	14.7230	Exercise
d	17	14.7230	15	Exercise

   Finally, $V_0^E = 15, V_0^C = 12.7230$ and thus $V_0 = 15$.

8. Let S_t follow the Black-Scholes model

$$dS_t = rS_t dt + \sigma S_t d\widetilde{W}_t.$$

   where r is the risk free rate and σ is a constant. Let S_0 be the initial stock price at time 0 that is given. Find V_0 where

   (a) (10 points) $V_T = \log \frac{S_T}{K}, K$ a constant .

   Ans: $S_T = S_0 e^{(r - \frac{\sigma^2}{2})T + \sigma \tilde{W}_T}$. Thus $V_T = \log \frac{S_0}{K} + \frac{r - \sigma^2}{2} T + \sigma \tilde{W}_T$ and

$$V_0 = e^{-rT} \log \frac{S_0}{K} + \tilde{E}(e^{-rT} V_T) = e^{-rT}(\log \frac{S_0}{K} + (r - \frac{1}{2}\sigma^2)T).$$

   (b) (10 points) $V_T = \int_0^T (S_u - K) du$ where K is a constant.

   Ans:

$$\begin{aligned} V_0 = \tilde{E}(e^{-rT} V_T) &= e^{-rT}(\int_0^T \tilde{E} S_u du - KT) \\ &= e^{-rT}(\int_0^T S_0 e^{ru} du - KT) \\ &= e^{-rT}(S_0 \frac{e^{rT} - 1}{r} - KT). \end{aligned}$$

9. Let W_t be a Brownian motion. Compute

   (a) (10 points) $E(e^{|W_t|})$.

$$
\begin{aligned}
Ee^{|W_t|} &= \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{\infty} e^{|x|}e^{-\frac{x^2}{2t}}dx \\
&= \frac{2}{\sqrt{2\pi t}}\int_{0}^{\infty} e^{x}e^{-\frac{x^2}{2t}}dx \\
&= \frac{2e^{t/2}}{\sqrt{2\pi t}}\int_{0}^{\infty} e^{-\frac{x^2-2tx+t^2}{2t}}dx \\
&= \frac{2e^{t/2}}{\sqrt{2\pi t}}\int_{0}^{\infty} e^{-\frac{(x-t)^2}{2t}}dx \\
&= 2e^{t/2}P(N(t,t)>0) = 2e^{t/2}P(Z > \frac{-t}{\sqrt{t}}) \\
&= 2e^{t/2}N(\frac{t}{\sqrt{t}}).
\end{aligned}
$$

   (b) (10 points) $E(\int_0^t W_s^2 dW_s)$.

$$E(\int_0^t W_s^2 dW_s) = 0.$$

10. Let W_t be a Brownian motion. Compute

   (a) (10 points) $d\ e^{W_t}(t+W_t)$.

   Ans:

$$
\begin{aligned}
d\ e^{W_t}(t+W_t) &= d(te^{W_t}) + d(e^{W_t}W_t) \\
&= e^{W_t}dt + t(e^{W_t}dW_t + \frac{1}{2}e^{W_t}dt) + (e^{W_t} + W_te^{W_t})dW_t + \frac{1}{2}(2e^{W_t} + W_te^{W_t})dt \\
&= e^{W_t}(2 + \frac{1}{2}W_t + \frac{1}{2}t)dt + e^{W_t}(1 + W_t + t)dW_t.
\end{aligned}
$$

   (b) (10 points) $d\ (e^{W_t}\int_0^t s dW_s)$.

   Ans: We have $\int_0^t s dW_s = tW_t - \int_0^t W_s ds$. Now

$$d(e^{W_t}\int_0^t W_s)ds = W_te^{W_t}dt + (\int_0^t W_s ds)(e^{W_t}dW_t + \frac{1}{2}e^{W_t}dt).$$

   And

$$d(tW_te^{W_t}) = W_te^{W_t}dt + t[(e^{W_t} + W_te^{W_t})dW_t + \frac{1}{2}(2e^{W_t} + W_te^{W_t})dt].$$

   Putting these together we get :

$$
\begin{aligned}
d\ \left(e^{W_t}\int_0^t s dW_s\right) &= e^{W_t}(tW_t - \int_0^t W_s ds + t)dW_t + \frac{1}{2}e^{W_t}(tW_t - \int_0^t W_s ds + 2t)dt \\
&= e^{W_t}(\int_0^t s dW_s + t)dW_t + \frac{1}{2}e^{W_t}(\int_0^t s dW_s + 2t)dt
\end{aligned}
$$

Scratch (Won't be graded)

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