MATH 351 SECTION 2: IDEALS WHICH ARE NOT PRINCIPAL

TAMAR BLANKS

This note is meant to fill in the details of something we started discussing during the last workshop: examples of ideals which are not principal.

Example from class. The ring $\mathbb{Z}[x]$ has the non-principal ideal

$$I = \langle x, 2 \rangle = \{ f(x) \cdot x + g(x) \cdot 2 \ : \ f, g \in \mathbb{Z}[x] \}.$$

To check that this is really not principal, suppose that $I = \langle p(x) \rangle$ for some polynomial $p(x)$. Then $f(x)p(x) = 2$ for some $f(x) \in \mathbb{Z}[x]$, so $p(x)$ must be 1, -1, 2, or -2. But $1 \notin I$, so $p(x) = 2$ or -2. We also know that $x \in I$, so $g(x) \cdot 2 = x$ for some $g(x) \in \mathbb{Z}[x]$. But there is no $g(x) \in \mathbb{Z}[x]$ which satisfies this equation. So I cannot be principal.

A new example. Let F be a field. The polynomial ring in two variables is defined to be

$$F[x, y] = \left\{ \sum_{0 \le i, j \le n} a_{ij} x^i y^j \ : \ a_{ij} \in F, \ n \in \mathbb{Z}_{\ge 0} \right\}.$$

You can check that this is the same as the polynomial ring $F[x][y]$ of polynomials in y with coefficients in the ring $F[x]$, so this isn't too different from rings that were covered in class.

For any field F, the ideal $I = \langle x, y \rangle$ of $F[x, y]$ is a non-principal ideal. The proof that I is not principal is a good exercise. Hint: suppose $I = \langle p(x, y) \rangle$ for some polynomial $p(x, y)$. Then $f(x, y)p(x, y) = x$ for some f. What are the possibilities for $p(x, y)$ that could satisfy this equation? Could any of those generate I?

The ring $F[x, y]$ has a lot of non-principal ideals. As another exercise, you can try to come up with some more examples.

Some non-examples. Why was it so hard to generate examples for rings other than $\mathbb{Z}[x]$? One reason is that many of the rings discussed in this class so far are examples of a kind of ring known as “principal ideal domains” (PIDs). These are integral domains in which every ideal is principal. For example, $\mathbb{Z}$ is a PID, and so is $F[x]$ for any field F. It turns out that every ideal in $\mathbb{Z}_n$ is principal also. So there are no examples to find in these rings.

Date: March 7, 2022.