Root System Basics

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Vertex Operator Algebras

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We say $\mathfrak{g}$ is a *Lie algebra* if it is a vector space with bilinear multiplication $[\cdot, \cdot]$ that satisfies:

- $[x, x] = 0$ for all x (which implies $[x, y] = -[y, x]$)
- $[[x, y], z] = [x, [y, z]] - [y, [x, z]]$.

For $x \in \mathfrak{g}$, define $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ by $g \mapsto [x, g]$.
The second rule should be thought of as:

$$\mathrm{ad}_{[x,y]} = \mathrm{ad}_x \mathrm{ad}_y - \mathrm{ad}_y \mathrm{ad}_x$$

The prototypical Lie algebra is $End(V)$ with $[A, B] = AB - BA$. Lie algebra homomorphisms are defined as you would expect. A representation is a Lie algebra homomorphism:

$$\phi : \mathfrak{g} \to End(V)$$

for some vector space V. V is also said to be a $\mathfrak{g}$ module (corresponding to this representation):

$$x \cdot v = \phi(x)v$$

Putting these together, V is a $\mathfrak{g}$ module if

$$x \cdot (y \cdot v) - y \cdot (x \cdot v) = [x, y] \cdot v$$

Any Lie algebra $\mathfrak{g}$ is a module over itself via:

$$x \cdot g = [x, g]$$

since

$$x \cdot (y \cdot v) - y \cdot (x \cdot v) = [x, [y, v]] - [y, [x, v]] = [[x, y], v]$$

Any Lie algebra has two trivial ideals - 0 and itself.
The uninteresting one-dimensional Lie algebra that maps all brackets to 0 technically only has these two ideals; thus we say Lie algebra is *simple* if it has non-trivial ideals and is dimension > 1.
From here on out, we always assume vector spaces and Lie algebras are over $\mathbb{C}$.

Last week, we identified an important simple Lie algebra $\mathfrak{s}\ell(2)$ - the subspace of Endomorphisms on $\mathbb{C}^2$ with trace 0. This is the smallest simple Lie algebra (over $\mathbb{C}$; on other fields it is tied for the smallest). Under some choice of basis of $\mathbb{C}$, we set

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and have relations

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h$$

(recall $[x, x] = 0$ and $[x, y] = -[y, x]$ so this gives all basis relations). Note ad_h acts diagonally on $\mathfrak{s}\ell(2)$. Thus we say h is a semisimple element.

We pointed out that all finite-dimensional modules of simple Lie algebras can be written as the direct sum of irreducible modules and that semisimple elements in $\mathfrak{g}$ act diagonally on such modules. We also found that there is exactly one irreducible $\mathfrak{sl}(2)$ module V_{k-1} of each dimension $k > 0$, and it has the following properties:

- V_{k-1} has a basis $\{v_{k-1}, v_{k-3}, \dots v_{-(k-1)}$
- $h \cdot v_i = iv_i$.
- $f \cdot v_i = v_{i-2}$ (or 0 if $i = -k-1$)
- $e \cdot v_i = \frac{k+k^2/2+i-i^2/2}{2} v_{i+2}$

We also introduced an important bilinear form known as the Killing form $(x, y) = \mathrm{tr}(ad_x ad_y)$ on $\mathfrak{g}$. With just algebra manipulation, one can show:

- $(x, y) = (y, x)$ (symmetric)
- $(x, [y, z]) = ([x, y], z)$ ($\mathfrak{g}$-invariant)

but what's a bit harder to show (and very important) is that if $\mathfrak{g}$ is simple, the Killing form is non-degenerate

Let $\mathfrak{g}$ be a simple Lie algebra. One can show that $\mathfrak{g}$ must contain some semisimple elements; take let $\mathfrak{h}$ be a maximal subspace of commuting semisimple elements. This is called a Cartan Subalgebra

Commuting diagonal operators have the same eigenspaces (with possibly different eigenvalues):

$$BAv_{\lambda,B} = ABv_{\lambda,B} = A\lambda v_{\lambda,B}$$

So A preserves eigenspaces of B and vice versa. Thus A has eigenspaces in the eigenspaces of B and vice versa, so their eigenspaces are the same.

With that being said, write

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda$$

where each $\mathfrak{g}_\lambda$ is an eigenspace for all $\mathrm{ad}_{\mathfrak{h}}$ and $\mathrm{ad}_h, h \in \mathfrak{h}$ has eigenvalue $\lambda(h)$ on this space.

Note $\mathfrak{h} \subset \mathfrak{g}_0$ since $\mathfrak{h}$ is abelian. One can show we actually have $\mathfrak{h} = \mathfrak{g}_0$.
Let $\Phi \subset \mathfrak{h}^*$ be the set of $\lambda \neq 0$ for which $\mathfrak{g}_\lambda \neq 0$. These are called the roots. So in this notation:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

Lemma

Φ *spans* $\mathfrak{h}^*$

Proof.

Otherwise there is some $h \in \mathfrak{h}$ for which $\Phi(h) = 0$. Then for all $\alpha \in \Phi$, $x_\alpha \in \mathfrak{g}_\alpha$ we have

$$[h, x_\alpha] = \alpha(h)x_\alpha = 0$$

and $[h, \mathfrak{h}] \subset [\mathfrak{h}, \mathfrak{h}] = 0$. So h spans a 1-dimensional ideal of $\mathfrak{g}$, but $\mathfrak{g}$ is simple (also can't happen in semisimple) □

Lemma

$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$

Proof.

Take $x_\alpha \in \mathfrak{g}_\alpha, x_\beta \in \mathfrak{g}_\beta$.

$$[h, [x_\alpha, x_\beta]] = [[h, x_\alpha], x_\beta] + [x_\alpha, [h, x_\beta]]$$

$$= \alpha(h)[x_\alpha, x_\beta] + \beta(h)[x_\alpha, x_\beta] = (\alpha + \beta)(h)[x_\alpha, x_\beta]$$

□

Lemma

$(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *if* $\alpha \neq -\beta$

Proof.

Take an h for which α and $-\beta$ disagree. Take $x_\alpha \in \mathfrak{g}_\alpha, x_\beta \in \mathfrak{g}_\beta$.

$$([x_\alpha, h], x_\beta) = (x_\alpha, [h, x_\beta])$$

by $\mathfrak{g}$ associativity of the killing form. Since $[x_\alpha, h] = -[h, x_\alpha] = -\alpha(h)x_\alpha$ and $[h, x_\beta] = \beta(h)x_\beta$, this is equivalent to

$$-\alpha(h)(x_\alpha, x_\beta) = \beta(h)(x_\alpha, x_\beta)$$

By our assumption on h, this forces $(x_\alpha, x_\beta) = 0$. □

By the non-degeneracy of $(\cdot,\cdot)$, this forces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ to pair non-degenerately and $(\cdot,\cdot)$ to be non-degenerate on $\mathfrak{h}$. In particular, $\alpha \in \Phi \iff -\alpha \in \Phi$.

This non-degeneracy on $\mathfrak{h}$ allows us to naturally pair $\mathfrak{h}$ with $\mathfrak{h}^*$ (isomorphically) via

$$h \to (h, \cdot)$$

For $\alpha \in \Phi$, let t_α be the corresponding element of h in this association (so $(t_\alpha, h) = \alpha(h))$).

We can also lift $(\cdot,\cdot)$ to a form on $\mathfrak{h}^*$ via $(\alpha, \beta) = (t_\alpha, t_\beta)$

Lemma

For $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}$, we have

$$[e_\alpha, f_\alpha] = (e_\alpha, f_\alpha)t_\alpha$$

Proof.

$$(h, [e_\alpha, f_\alpha]) = ([h, e_\alpha], f_\alpha) = \alpha(h)(e_\alpha, f_\alpha) = (h, (e_\alpha, f_\alpha)t_\alpha)$$

□

Let $h_\alpha = \frac{2t_\alpha}{(t_\alpha, t_\alpha)}$ (one can show this denominator isn't 0). Choose e_α arbitrarily and choose f_α such that $(e_\alpha, f_\alpha) = \frac{2}{(t_\alpha, t_\alpha)}$
Then from this, we can see that $e_\alpha, h_\alpha, f_\alpha$ forms an $\mathfrak{sl}(2)$ (with the same relations as e, h, f from earlier):

$$[e_\alpha, f_\alpha] = \frac{2t_\alpha}{(t_\alpha, t_\alpha)} = h_\alpha$$

$$[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha = (t_\alpha, h_\alpha)e_\alpha = \frac{2(t_\alpha, t_\alpha)}{(t_\alpha, t_\alpha)}e_\alpha = 2e_\alpha$$

(note we've shown here that $\alpha(h_\alpha) = 2$).

$$[h_\alpha, f_\alpha] = -\alpha(h_\alpha)f_\alpha = -2f_\alpha$$

Thus the span of $\{h_\alpha, e_\alpha, f_\alpha\}$ form an $\mathfrak{s}\ell(2)$ subalgebra (we'll call it $\mathfrak{s}\ell(2)_\alpha$)
Now for each $\alpha \in \Phi$ and choice of associated $\mathfrak{s}\ell(2)_\alpha$, we can view $\mathfrak{g}$ as an $\mathfrak{s}\ell(2)_\alpha$ module via adjoint.
Since $\mathfrak{g}$ is finite dimensional and $\mathfrak{s}\ell(2)_\alpha$ simple, we know it decomposes uniquely into a direct sum of irreducible modules of the type we described earlier (and thus every submodule has a complement).

Lemma

We have shown if $\alpha \in \Phi$, then $-\alpha \in \Phi$. There are no other multiples of α in Φ. Furthermore, the root space $\mathfrak{g}_\alpha$ is 1-dimensional.

Let's consider the $\mathfrak{sl}(2)_\alpha$ submodule

$$W_\alpha = \bigoplus_{c \in \mathbb{C}} \mathfrak{g}_{c\alpha}$$

This is an $\mathfrak{sl}(2)$ submodule (why?) and note h_α scales vectors in $\mathfrak{g}_{c\alpha}$ by $c\alpha(h_\alpha) = 2c$. Since we know h_α acts integrally on finite dimensional modules, we have $c \in \mathbb{Z}/2$

Now note the following is an $\mathfrak{sl}(2)_\alpha$ submodule of W_α:

$$W_{\alpha,0} = \mathfrak{h} \oplus \mathbb{C}e_\alpha \oplus \mathbb{C}f_\alpha$$

The lemma is equivalent to showing $W_{\alpha,0} = W_\alpha$

Suppose not. Then there is some complement $W_{\alpha,1} \subset W_\alpha$. Note

$$W_{\alpha,1} \subset \bigoplus_{c \in \mathbb{Z}/2 \backslash 0} \mathfrak{g}_{c\alpha}$$

Thus h_α has no 0 weight on $W_{\alpha,1}$. This forces all weights in $W_{\alpha,1}$ to be odd, since all irreducible submodules of $W_{\alpha,1}$ with even weights would contain a 0 weight. Thus

$$W_{\alpha,1} \subset \bigoplus_{c \in \mathbb{Z}+\frac{1}{2}} \mathfrak{g}_{c\alpha}$$

This immediately shows that $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ have dimension 1, and there are no other integer multiples of α as roots.
In particular, α and 2α cannot both be roots. Hence $\alpha/2$ cannot be a root either.

So we have

$$W_{\alpha,1} \subset \bigoplus_{c \in \mathbb{Z}+\frac{1}{2} \setminus \{\frac{1}{2}\}} \mathfrak{g}_{c\alpha}$$

So $W_{\alpha,1}$ does not have the weight 1. Thus it cannot contain any odd weights, as all its irreducible submodules with odd weights would contain the weight 1.
So $W_{\alpha,1}$ is a finite-dimensional $\mathfrak{s}\ell(2)_\alpha$ submodule with no even or odd weights. So it must be 0.

Lemma

If $\alpha, \beta \in \Phi$, then $\beta(h_\alpha) \in \mathbb{Z}$ and $\beta - \beta(h_\alpha)\alpha \in \Phi$.

If $\alpha = \pm\beta$ this is clear. Assume otherwise.
Consider the $\mathfrak{sl}(2)\alpha$ submodule of $\mathfrak{g}$:

$$W_\alpha^\beta = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha}$$

Since $\beta \neq \pm\alpha$ and no other multiples of α are roots, $\mathfrak{g}_0 = \mathfrak{h}$ is not among these spaces. Thus they are all root spaces - 1 dimensional. The weight of h_α on $\beta + i\alpha$ is $\beta(h_a) + i\alpha(h_\alpha) = \beta(h_\alpha) + 2i$. So all weight spaces are 1-dimensional and all weights are the same parity. This means it is impossible for W_α^β to be the sum of 2 or more irreducibles so W_α^β is irreducible.

Let q be largest such that $\beta + r\alpha \in \Phi$. Then the highest weight of W_α^β is $\beta(h_\alpha) + 2r$. So the lowest weight is $-\beta(h_\alpha) - 2r = \beta(h_\alpha) - 2(\beta(h_\alpha) + r)$ and all integers of the same parity in between are weights. Thus

$$\beta + i\alpha \in \Phi \iff -\beta(h_\alpha) - r \leq i \leq r$$

In particular $\beta - \beta(h_\alpha)\alpha \in \Phi$
Note that we have also shown that the action of $\mathfrak{s}\ell(2)_\alpha$ on $\mathfrak{g}$ decomposes into irreducibles as follows:

$$\mathfrak{g} = \mathrm{Ker}(\alpha) \oplus \mathfrak{s}\ell(2)_\alpha \oplus \bigoplus_{\beta \in \Phi/\mathbb{C}\alpha} \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha}$$

Note $\beta - \beta(h_\alpha)\alpha = \beta - \frac{2(t_\beta, t_\alpha)}{(t_\alpha, t_\alpha)}\alpha = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$.
If these were vectors in Euclidean space and $(\cdot,\cdot)$ was dot product, this would mean the reflection of β across the hyperplane orthogonal to α is in Φ.
To get this realization, need to show the following:

Lemma

1. *All Φ lies in an $\mathbb{R}$ vector subspace of $\mathfrak{h}^*$ of the same dimension. Call this space E.*
2. *$(\cdot,\cdot)$ is non-degenerate and positive-definite on E*

Since Φ spans $\mathfrak{h}^*$, choose a basis $\{\alpha_1, \dots \alpha_n\} \in \Phi$ for $\mathfrak{h}^*$. We show that all roots $\beta \in \Phi$ are in the $\mathbb{R}$ span of the $\{\alpha_i\}$.
We know

$$\beta = \sum_i c_i \alpha_i$$

for $c_i \in \mathbb{C}$.
So for all α_j we have

$$\frac{2(\beta, \alpha_j)}{(\alpha_j, \alpha_j)} = \sum_i c_i \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

Since the α_j span, treting c_i as free variables, this set of equations has a unique solution (the actual c_i). And since all coefficients are integers, the solutions are rational; in particular real. So $\Phi \subset E = \mathbb{R}\{\alpha_1, \dots \alpha_n\}$.

Next, we show $(\gamma, \gamma) > 0$ for all $\gamma \in E, \gamma \neq 0$.
Note $(\gamma, \gamma) = (t_\gamma, t_\gamma) = \mathrm{tr}(\mathrm{ad} t_\gamma)^2$. Since all root spaces $\mathfrak{g}_\alpha$ are one dimensional and $\mathrm{ad} t_\gamma$ kills $\mathfrak{h}$, we have

$$(t_\gamma, t_\gamma) = \sum_{\alpha \in \Phi} (\alpha(t_\gamma))^2 = \sum_{\alpha \in \Phi} (\alpha, \gamma)^2 = \sum_{\alpha \in \Phi} \left(\frac{2(\gamma, \alpha)}{(\alpha, \alpha)} \right)^2 (\alpha, \alpha)^2$$

Since $\gamma \in E$ and $\frac{2(a_i, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$ for all α_i in the basis and $\alpha \in \Phi$, we know $\frac{2(\gamma, \alpha)}{(\alpha, \alpha)} \in \mathbb{R}$.

All that remains to be shown is that $(\beta,\beta)\in\mathbb{R}$ for $\beta\in\Phi$. We use a similar idea:

$$(\beta,\beta)=\sum_{\alpha\in\Phi}(\alpha,\beta)^2=\sum_{\alpha\in\Phi}\left(\frac{2(\alpha,\beta)}{(\beta,\beta)}\right)^2(\beta,\beta)^2$$

and divide through by $(\beta,\beta)^2$ to get

$$\frac{1}{(\beta,\beta)}=\sum_{\alpha\in\Phi}\left(\frac{2(\alpha,\beta)}{(\beta,\beta)}\right)^2\in\mathbb{Z}\subset\mathbb{R}$$

So $(\beta,\beta)\in\mathbb{R}$ for all $\beta\in\Phi$ (actually the inverse of a positive integer, from this argument).

Thus $\sum_{\alpha\in\Phi}\left(\frac{2(\gamma,\alpha)}{(\alpha,\alpha)}\right)^2(\alpha,\alpha)^2$ is the sum of squares of real numbers; hence non-negative. And since Φ spans, not all terms are 0.

We summarize as follows: Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a maximal abelian semisimple subspace, and $(\cdot, \cdot)$ the killing form of $\mathfrak{g}$. Then there is a finite subset $\Phi \in \mathfrak{h}^*$ such that we can write

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

with:

1. $h \in \mathfrak{h}$ acts on $\mathfrak{g}_\alpha$ (via bracket) with eigenvalue $\alpha(h)$
2. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$
3. All $\mathfrak{g}_\alpha$ are 1-dimensional
4. For every $\alpha \in \Phi$, we have $-\alpha \in \Phi$ and no other multiples
5. $(\cdot, \cdot)$ restricts non-degenerately to $\mathfrak{h}$, so we can equip $\mathfrak{h}^*$ with a form corresponding to $(\cdot, \cdot)$ that we label the same way.
6. Φ spans $\mathfrak{h}^*$ and an $\mathbb{R}$-subspace E of $\mathfrak{h}^*$ of the same dimension. On this subspace, $(\cdot, \cdot)$ is positive definite and this subspace can therefore be realized as Euclidean space with $(\cdot, \cdot)$ being dot product.
7. For all $\alpha, \beta \in \Phi$, $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$
8. On E, define s_α ($\alpha \in \Phi$) to be the map $\gamma \to \gamma - \frac{2(\gamma,\alpha)}{(\alpha,\alpha)}\alpha$; in other words, reflection across the hyperplane orthogonal to α. Φ is closed under s_α for all $\alpha \in \Phi$.

A subset $\Phi \subset E$ of Euclidean space with the properties

1. Φ spans E and is finite.
2. Φ is closed under s_α for all $\alpha \in \Phi$.
3. For all $\alpha, \beta \in \Phi$, $\frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$
4. For every $\alpha \in \Phi$, we have $-\alpha \in \Phi$ and no other multiples are in Φ

is called a root system. These have beautiful structures that hint at the beauty of Lie theory as a whole. A root system is decomposable if we have $\Phi = \Phi_1 \cup \Phi_2, \Phi_1 \cap \Phi_2 =$ and $(\Phi_1, \Phi_2) = 0$. Indecomposable otherwise. Simple Lie algebras will have indecomposable root systems; the idea being otherwise you could separate the root space decomposition based on this partition of Φ and each would be an ideal in $\mathfrak{g}$.
With a bit of work, one can also show the reverse direction - for any simple root system there is an associated simple Lie algebra.

Last time, we saw that any finite dimensional module of a simple Lie algebra is diagonalized by $\mathfrak{h}$:

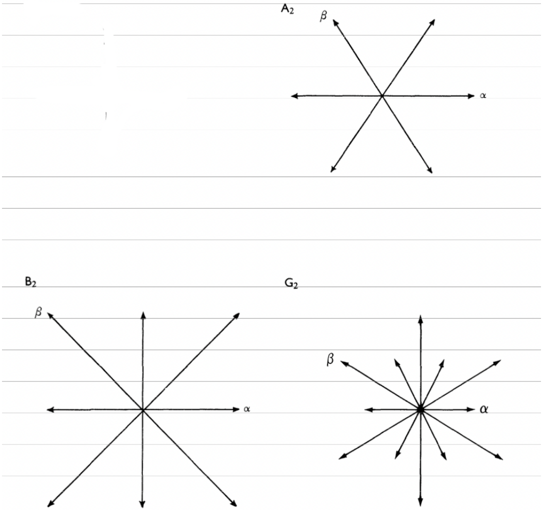
$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$$

and note

$$h \cdot (x_\alpha \cdot v_\lambda) = (x_\alpha \cdot h \cdot v_\lambda) + [h, x_\alpha] \cdot v_\lambda = \lambda(h) x_\alpha \cdot v_\lambda + \alpha(h) x_a \cdot v_\lambda$$

So $x_\alpha \cdot V_\lambda \subset V_{\lambda+\alpha}$.
Since V_λ is also a module for the subalgebra $\mathfrak{sl}(2)_\alpha$ for each $\alpha \in \Phi$, we must have $\lambda(\mathfrak{h}_\alpha) \in \mathbb{Z}$ for all weights λ in V and $\alpha \in \Phi$.

Here are the 2-dimensional indecomposable root systems:



Let θ be the angle between roots α and β. Then since $\frac{2(\alpha,\beta)}{(\beta,\beta)} = 2\frac{||\alpha||}{||\beta||}\cos(\theta)$, we have

$$\frac{2(\alpha,\beta)}{(\beta,\beta)}\frac{2(\beta,\alpha)}{(\alpha,\alpha)} = 4\cos^2(\theta)$$

In particular, $4\cos^2(\theta) \in \mathbb{Z}$. So θ must be related (in the pre-calculus sense) to $0, \pi/6, \pi/3$, or $\pi/4$.
Furthermore, if α and β are non-proportional, this forces $\frac{2(\alpha,\beta)}{(\beta,\beta)}\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \{0, 1, 2, 3\}$ (and only 0 if they are orthogonal).

Lemma

For non-proportional root α, β, if $(\alpha, \beta) < 0$ then $\alpha + \beta \in \Phi$. If $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi$

Proof.

In both cases, we have $\frac{2(\alpha,\beta)}{(\beta,\beta)}\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \{1, 2, 3\}$. Since both are integral, one must be ± 1. In the first case, one must be -1 and in the second case, one must be 1. WLOG, let this be $fr\alpha\beta$.
Then $s_\beta(\alpha) = \alpha - fr\alpha\beta\beta = \alpha \pm \beta$ and the conclusion follows. □

Now cut Φ by some arbitrary hyperplane that does not intersect any root, and let γ be a vector orthogonal to it. Let Φ^+ be the roots acute with γ (the positive roots) and Φ^- the roots obtuse with γ (the negative roots).
Let $\Delta = \{\alpha_i\}$ be a minimal set of positive roots such that every positive root is a non-negative integral combination of the α_i. We call these simple roots.

Lemma

Δ is linearly independent. Thus it is a basis for E for which every root in Φ has either all coeffients non-negative or non-positive (based on whether it's in Φ^+ or Φ^-)

We show that vectors in Δ are all mutually non-acute. Then if we had a dependence relation, we could write $\sum c_i\alpha_i = \sum c_j\alpha_j$, all coefficients non-negative and distinct simple roots on both sides. But by assumption

$$(\sum c_i\alpha_i, \sum c_i\alpha_i) = (\sum c_i\alpha_i, \sum c_j\alpha_j) \leq 0$$

so $\sum c_j\alpha_j = 0$ by positive definiteness (in otherwords, this expression says a non-negative sum of simple roots is 0). But such a sum must have a positive inner product with γ as all $(\gamma, \alpha_i) > 0$, leading to a contradiction. So we just need to show that all $(\alpha_i, \alpha_j) \leq 0$.

Suppose otherwise - $(\alpha_i, \alpha_j) > 0$. Then we know $\alpha_i - \alpha_j$ and $\alpha_j - \alpha_i$ are roots. Suppose $\alpha_i - \alpha_j$ is positive without loss of generality. Then by assumption $\alpha_i - \alpha_j$ is a non-negative integral combination of simple roots $\alpha_i - \alpha_j = \sum c_k \alpha_k$. So any time we see α_i, we can replace it with $\alpha_j + \sum c_k \alpha_k$, so the $\alpha_i \in \Delta$ is not needed. This contradicts minimality.

This also shows that $\mathbb{Z}\Delta = \mathbb{Z}\Phi$. Since the former is linearly independent, this means the roots form a lattice.
The matrix $A = \left[\frac{2(\alpha_i,\alpha_j)}{(\alpha_j,\alpha_j)}\right]$ is called a *Cartan Matrix* for this Root system (some define it to be the transpose of this). By construction, we will have

- 2s on the diagonal all off-diagonal entries in $\{0, -1, -2, -3\}$
- 0s symmetric
- If $a_{i,j} \in \{-2, -3\}$, $a_{j,i} = -1$
- A positive definite

To an $n \times n$ Cartan matrix, we associate a graph on n nodes called the *Dynkin Diagram* as follows:

- If $a_{i,j} = 0$, no edges between nodes i and j.
- If $a_{i,j} = a_{j,i} = -1$, draw 1 edge between nodes i and j
- If $a_{i,j} - n < -1$, draw n edges between nodes i and j, and an arrow from node i to node j.

The Dynkin Diagrams of all indecomposable root systems are as follows: