Level 1 (untwisted simply laced) affine modules:

$$S(\hat{\mathfrak{h}}^{-}_{\mathbb{Z}}) \otimes \mathbb{F}\{\widetilde{M}\}$$

where M is a coset of the (associated finite-type) root lattice in its weight lattice

Simple Lie algebras:

$\mathbb{C}$-vector space with bilinear mult $[\cdot,\cdot]$:

$[x,x]=0$ ✓

$[x,[y,z]]-[y,[x,z]]=[[x,y],z]$ ✓

Ideal: $[I,g]\subseteq I$ ✓

Simple: No non-trivial ideals and $\dim(g)>1$

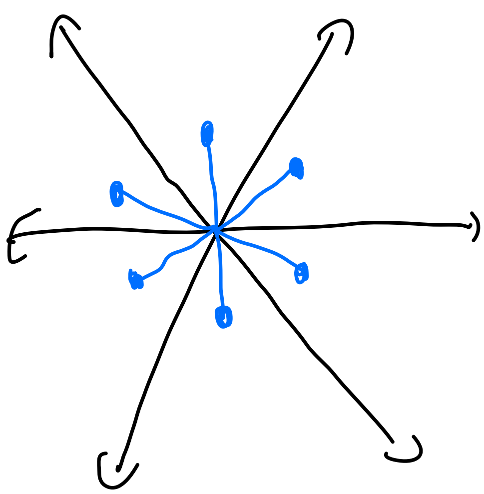
All fin. dim. simple Lie algebras have "root space decomp":

$$g=\overset{\text{"}\mathcal{H}\text{"}}{H}\oplus\coprod_{\alpha\in\Phi\subset H^*}X_\alpha^{\mathbb{C}}$$

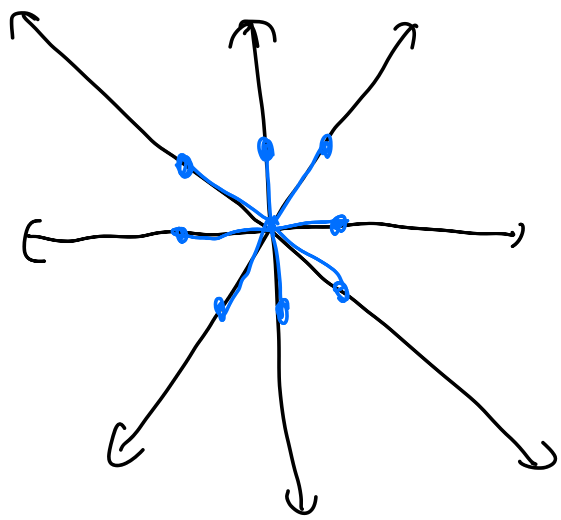
1. $[H,H]=0$ ("$H=X_0$")
2. $[h,x_\alpha]=\alpha(h)x_\alpha$
3. $[X_\alpha,X_\beta]\subseteq X_{\alpha+\beta}$

Φ: finite, span H, has a natural form $(\cdot,\cdot)$, can embed into $\mathbb{R}^n$ with this form.

Here, Φ becomes set of vectors orthogonal to walls of a finite reflection group:



D_3 (A_2)



D_4 (B_2)

Key properties:

1. Φ closed under reflection gp (W)

2. $\underbrace{\dfrac{2(\alpha,\beta)}{(\beta,\beta)}} \in \mathbb{Z}$ $\qquad \overset{\in H^*}{\alpha} \to \alpha^\vee = \dfrac{2\alpha}{(\alpha\alpha)} \in H$

$\Rightarrow$ $\mathbb{Z}\Phi$ forms a lattice L, $\mathbb{Z}\Phi^\vee$ forms $L^\vee$

$\Rightarrow$ If $\alpha^\vee = \dfrac{2\alpha}{(\alpha,\alpha)}$, $\underline{(\Phi, \Phi^\vee)} \subseteq \mathbb{Z}$

(while $\Phi \subset H^*$, $\Phi^\vee \subset H$)

Simply-laced Lie algebras - All $\alpha \in \Phi$ same length, can set to 2 so $\alpha^\vee = \alpha$. $\Big\}$

(Jump)

Base: $\Delta \subset \Phi$, basis for H,

$$\Phi \subset \underbrace{\mathbb{Z}^{\geq 0}\Delta}_{\Phi^+} \cup \underbrace{\mathbb{Z}^{\leq 0}\Delta}_{\Phi^-}$$

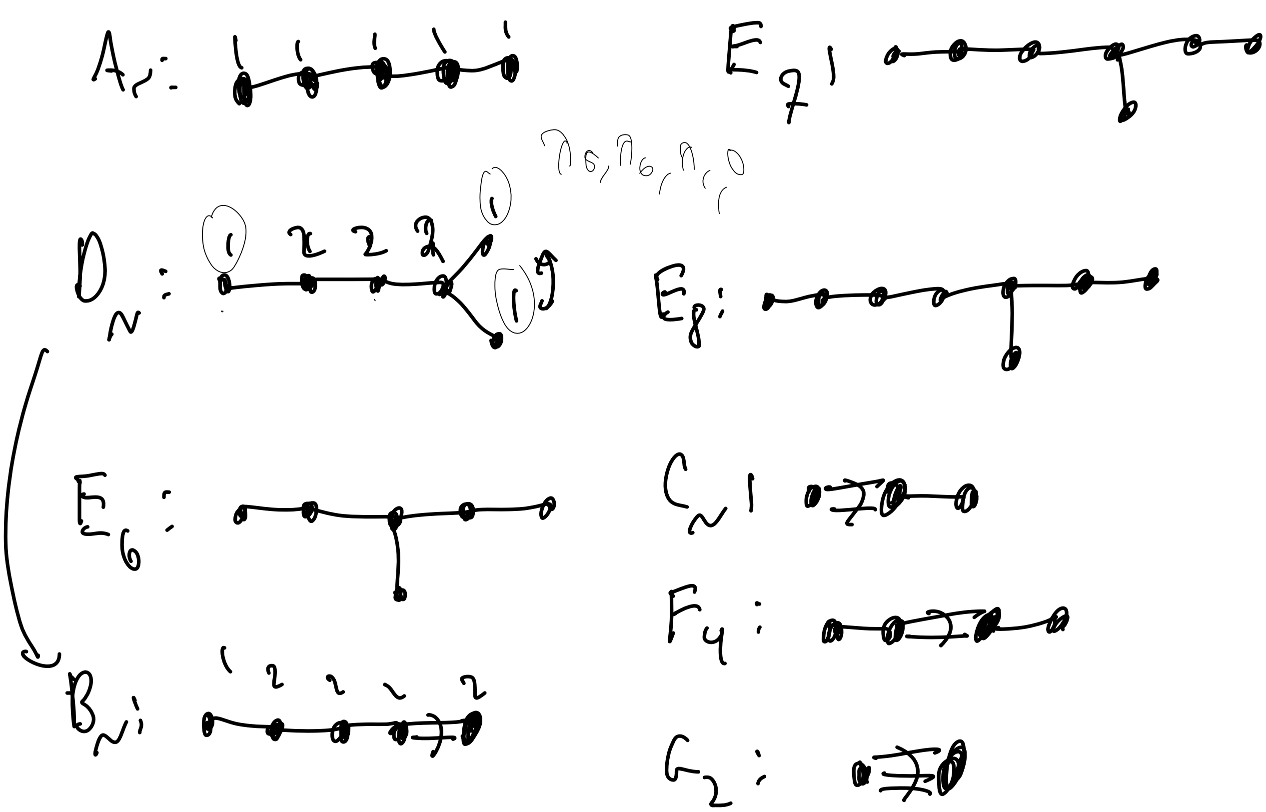
Ordering (on $\mathfrak{h}^*$): $\gamma_1 \geq \gamma_2$ if

$$\gamma_1 - \gamma_2 \in \mathbb{Z}^{\geq 0}\Delta$$

Φ has unique maximal elt: θ

$$\theta \geq \alpha \text{ for all } \alpha \in \Phi$$

Dynkin diagram: One node for each $\alpha_i \in \Delta$; $(\alpha_i, \alpha_j^\vee)$ edges from i to j:



Modules: $\mathfrak{g} \cdot V \subset V$

$$\underbrace{(x \cdot y - y \cdot x) \cdot v = [x, y] \cdot v}$$

H acts diagonally on fin. dim. modules:

$$V = \bigsqcup_{\lambda \in \Lambda \subset H^*} V_\lambda$$

$$h \cdot v_\lambda = \lambda(h)\, v_\lambda$$

$$\lambda^*(\Delta) \geq 0$$

Key properties (fin dim, irreducible)

1. $\Lambda(\Phi^\vee) \subseteq \mathbb{Z}$
2. $\exists !\ \underline{\lambda}^*$ maximal in Λ

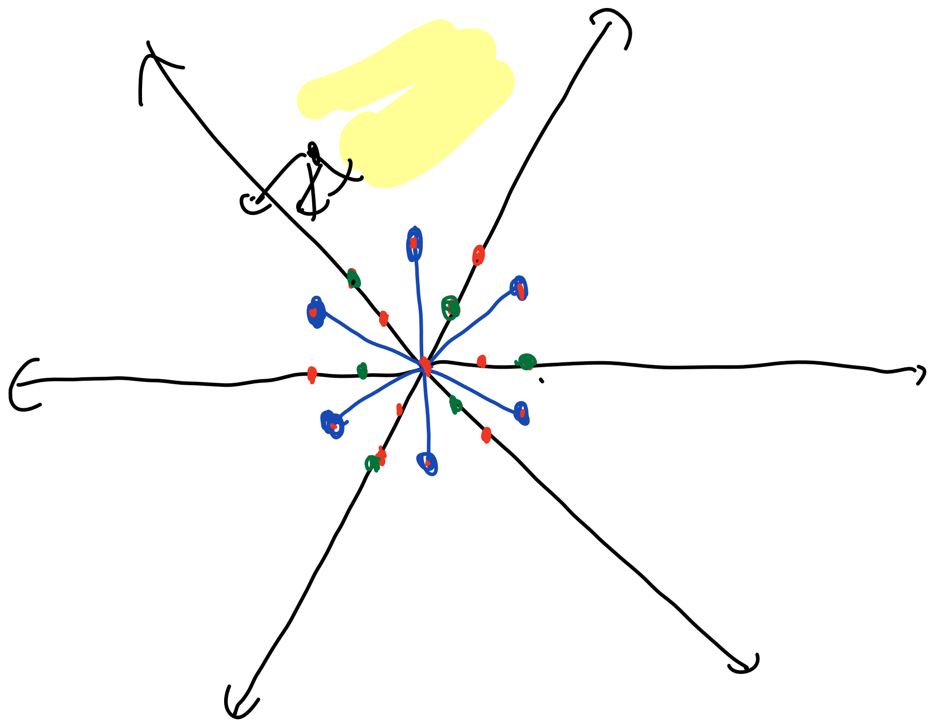
$\forall \lambda \in \Lambda,\ \lambda^* - \lambda \in \mathbb{Z}^{\geq 0} \Delta$

Lies in closure of fund chamber

3. $\underline{\Lambda \text{ closed under } W}$

"$\mathbb{Z}$-dual"

Weight lattice $\underline{P} = (\mathbb{Z}^\vee)^* \subseteq H^*$



$(\Phi, \Phi^\vee) \subseteq \mathbb{Z}$

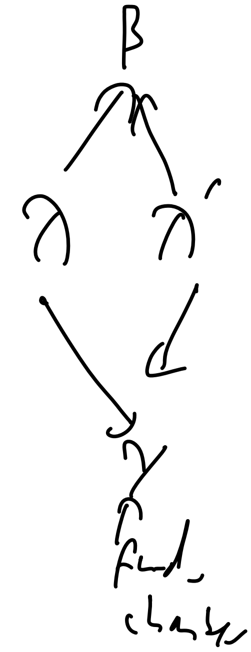
$\mathbb{Z}[\Phi]$ subset of H^* that is integral on $\Phi^\vee$

$\uparrow$ $\uparrow$

Cosets of L in P:

Each contains some $\leq$-minimal weight in the fundamental chamber (miniscule)

Uniqueness: Suppose λ, λ' miniscule and $\lambda - \lambda' \in \mathbb{Z}\Phi$



$$\underbrace{\lambda + \sum c_i \alpha_i}_{\beta} = \lambda' + \sum c_j \alpha_j$$

$$(\beta - \sum c_i \alpha_i - \sum c_j \alpha_j, \alpha_k)$$

$$= (\lambda - \sum c_j \alpha_j, \alpha_k) \geq 0$$

$$(\beta - \sum c_i \alpha_i - \sum c_j \alpha_j, \alpha_j)$$

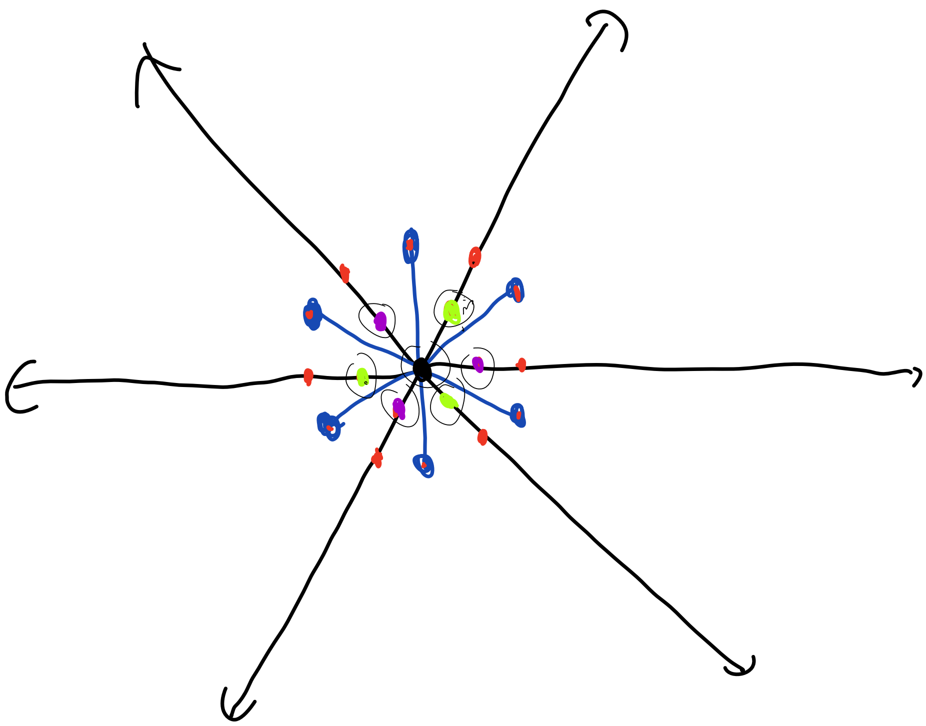
$$= (\lambda' - \sum c_i \alpha_i, \alpha_j) \geq 0$$

$$(\beta - \textstyle\sum c_i \alpha_i - \sum c_j \alpha_j, \alpha_i)$$

$$= (\lambda - \textstyle\sum c_j \alpha_j, \alpha_i) \geq 0$$

So $\lambda - \sum c_j \alpha_j$ $(= \lambda' - \sum c_i \alpha_i)$ is in fund. chamber.

Miniscule modules:



1. All weights reflections of highest weight
2. All spaces 1-dimensional
3. Easy to construct.

Cosets of L in $P^{\cong (L^\vee)^*}$ $\Longleftrightarrow$ Miniscule weights

Let $\theta^\vee = \sum_{\alpha_i \in \Delta} a_i \alpha_i^\vee$, λ_i : weight with

$\lambda_i(\alpha_i^\vee) = 1$, $\lambda_i(\alpha_{j \neq i}^\vee) = 0$

Prop: All Non-zero miniscule weights are of the form λ_i where $a_i = 1$ and all such are miniscule weights.

Proof: λ_i are miniscule:

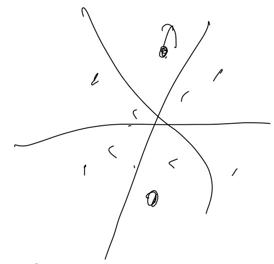
λ_i is in fund. chamber so wts
if $\alpha \in \Phi^+$, $\lambda_i - \alpha$ is not in fund. chamber

$$\begin{aligned}(\underline{\lambda_i - \alpha})(\alpha^\vee) &= \lambda_i(\alpha^\vee) - 2 \\ \lambda\ \theta^\vee \quad &\leq \lambda_i(\theta^\vee) - 2 \\ &\leq 1 - 2 \leq -1\end{aligned}$$

Non-zero miniscules are of the form λ_i:

Suppose $\lambda \neq 0$ is in fund. chamber.

If not λ_i, then $\lambda(\theta^\vee) \geq 2$



$\lambda(\theta^\vee) \geq 2$ $\quad \lambda \geq 2\theta$

$\lambda - \theta$

λ

$\lambda - \theta$

$\lambda - \theta$

Consider V_λ - module of highest weight λ

Has weight $s_\theta(\lambda) = \lambda - \lambda(\theta^\vee)\theta$

$= \lambda - k\theta \quad (k \geq 1)$

$\Rightarrow$ $\lambda - \theta$ is a weight of V_λ

But not reflection of λ

reflecting $\lambda - \theta$ to fund chamber will give a weight 2λ in fund chamber - V_λ not miniscule

Thus we have:

Non-zero minuscules $\longleftrightarrow$ cosets of L in P
(except L)

* 1s on Dynkin Diagram (coefficients of θ)
of coroot system
(root system if simply-laced).

$$e_\alpha \cdot e_\beta = \kappa^{(\alpha,\beta)} e_\beta \cdot e_\alpha$$

$$1 \to \hat{L} \to L \to \langle 1, \kappa \rangle \to 1$$

$(e_\alpha, \kappa e_\alpha) \to \alpha \in L$

Centralize

$$e_\alpha \cdot e_\beta = \{e_{\alpha+\beta}, \kappa e_{\alpha+\beta}\}$$

Lattice realization: Assume Φ simply-laced.

$L\ (= \mathbb{Z}\Phi)$ has a central extension $\hat{L}$ - a group s.t. $\hat{L} \to L$ has kernel $\{1, \kappa\}$ ← commutes

(think of basis are $\{e_\alpha, \kappa e_\alpha\}_{\alpha \in L}$), $\overline{ab} = \bar{a} + \bar{b}$

$ab = ba\kappa^{(\bar{a}, \bar{b})}$

"Double basis" for g: $H\ (= L \otimes \mathbb{C})$, $\{X_a\}_{a \in \hat{\Phi}}$

$\underline{X_a = -X_{\kappa a}}$

$$g = H + \sum_{\alpha \in \Phi} X_\alpha$$

$$[X_a, X_b] = X_{ab} \quad \text{if } ab \in \hat{\Phi}$$
$$= \bar{a} \quad \text{if } ab = 1$$
$$= -\bar{a} \quad \text{if } ab = \kappa$$
$$= 0 \quad \text{else}$$

$$[X_\alpha, X_\beta] \overset{\pm}{=} X_{\alpha+\beta}$$
$$[X_\alpha, X_{-\alpha}] = \mathbb{C}\alpha^\vee$$

$$[h, X_a] = \bar{a}(h) X_a$$

Can form $\hat{L}$ module $\hat{P}$ (just a set with 2 elts for each elt of P) that mimics how L shifts P.

$e_q \cdot v_n \in \{ v_{n+\cancel{q}\,q},\ z\, v_{n+q} \}$

"$-v_{n+q}$"

$$ab \cdot v_n = z^{(\bar{a}, \bar{b})}\, ba \cdot v_n$$

Miniscule Module construction:

$$x_a \cdot v_n = \begin{cases} a \cdot v_n & \text{if } \bar{a} + n \in \Omega \\ 0 & \text{else} \end{cases}$$

$\{ v_n \}$

Affine Lie algebras:
Let g be a finite, simple Lie algebra

Untwisted affinization:

$$\tilde{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$$

$$[x \otimes t^i, y \otimes t^j] = [x, y] \otimes t^{i+j} + \delta_{i,-j} (x, y) c$$

$$[c, \tilde{g}] = 0$$

$$[d, x \otimes t^k] = k\, x \otimes t^k$$

$$\tilde{H} = H \otimes 1 \oplus \mathbb{C}c \oplus \mathbb{C}d \qquad \Delta$$

$$\alpha_1^\vee = \alpha_1^\vee \otimes 1$$
$$\vdots$$
$$\alpha_r^\vee = \alpha_r^\vee \otimes 1$$
$$\alpha_0^\vee = -\theta^\vee \otimes 1 + c$$

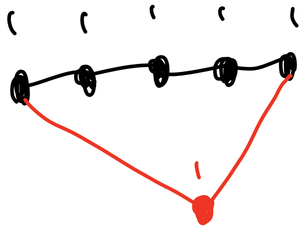
(d outside of coroot span, but necessary)

$h \otimes t^k : k(\alpha_0 + \theta) = k\delta$ "coroot-analogue of c"

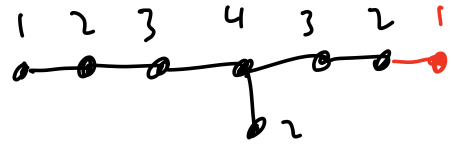
$x_\alpha \otimes t^k : \alpha + k\delta \qquad (\delta, \alpha_i) = 0$

Affine Dynkin: (α_0 "$:= -\theta$")

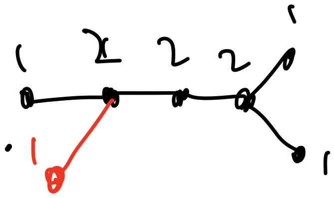
A_r:



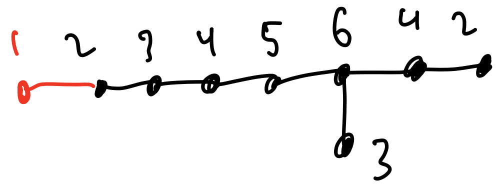
E_7:



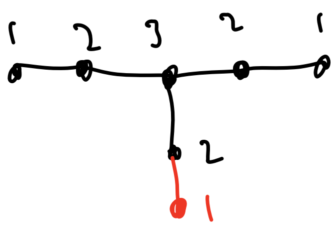
D_n:



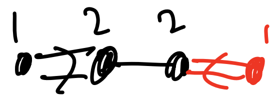
E_8:



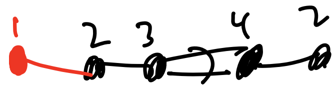
E_6:



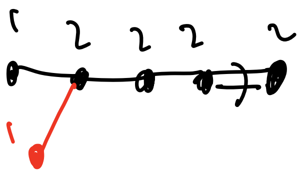
C_n:



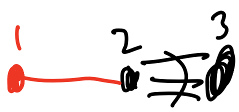
F_4:



B_n:



G_2:



Notice 1s on affine Dynkin are "equivalent" to affine node.

$\tilde{\Phi}$: affine root system

$\tilde{\Delta}$: base

Have $\tilde{\Phi} \subseteq \underbrace{\overbrace{\mathbb{Z}^{\geq 0}\tilde{\Delta}}}_{\tilde{\Phi}^+} \cup \underbrace{\overbrace{\mathbb{Z}^{\leq 0}\tilde{\Delta}}}_{\tilde{\Phi}^-}$

$\tilde{P} \subseteq \tilde{H}^*$: affine weights

There is still geometric notion of fundamental chamber, but for now say $\lambda \in \tilde{P}$ <u>dominant</u> if $\lambda(\tilde{\Delta}) \underset{?}{\subseteq} \mathbb{Z}^{\geq 0}$

Standard modules: generated by a dominant weight vector + irreducible.

$\alpha_0^\vee + \theta^\vee$

Level-1: if $\lambda^*(c) = 1$ $\quad \lambda^* \geq 0$

$\lambda^*(\alpha_0^\vee + \sum a_i^\vee \alpha_i^\vee) = 1$

So level 1 modules $\Longleftrightarrow$ 1s on (coroot) affine Dynkin

$\Updownarrow$

Cosets of (finite) coroot lattice in weight lattice $\Longleftrightarrow$ Miniscule weights or corresponding finite

$(h \otimes t^n / n \neq 0,\ h \otimes 1,\ c,\ d$

$h + \sum_{\alpha \in R} x_\alpha \qquad x_\alpha \otimes t^n$

$\tilde{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$

Borcs: So what was the structure of a level 1 affine module?

$h \otimes t^{--}$

$S(\hat{h}^-_{\mathbb{R}})$: polynomials in $\{h \otimes t^{-k}\}_{h \in H,\, k>0}$

$(h_1 \otimes t^{-2})(h_2 \otimes t^{-1})$

$h \otimes t^{-3} \cdot h' \otimes t^{-2} \to h \otimes t^{-3}\, h \otimes t^{-2}$

$h \otimes t^2 \cdot p(h_i \otimes t^{-k_i}) \to \dfrac{\partial}{\partial(h \otimes t^{-2})} \, p(h_i \otimes t^{-k_i})$

$\underbrace{d \text{ acts as } + \text{degree of polynomial}}$
c acts as 1

$h \otimes t^k$ $(k>0)$ acts as $k \dfrac{\partial}{\partial(h \otimes t^{-k})}$

$h \otimes 1$ acts as 0

$S(\hat{h}^-_{\mathbb{R}})$ is an irreducible $\{H \otimes t^k\}_{k \neq 0} \cup \{c\}$ module.

$$\underbrace{\{\mathfrak{h} \otimes t^k\}_{k \neq 0} \oplus \mathbb{C}c \oplus \mathbb{C}d}$$

$$[h \otimes t^k, h' \otimes t^j] = \delta_{k,-j}\, k\, (h, h')\, c$$

These actions make $\underline{S(\hat{\mathfrak{h}}_{\mathbb{R}}^{-})}$ into a irreducible module.

$\mathbb{F}\{M\}$: M a coset of $\hat{L}$ in $\hat{P}$

($\hat{P}$ is a set that $\hat{L}$ acts on)

$\mathbb{F}\{M\}$: vector space with ~~basis M~~, double basis $\hat{m}$ where $\kappa e_m = -e_m$

$\mathbb{F}\{M\}$: vector space over M
(coset of L in P), thought of
as $\mathbb{F}[\hat{M}_{\cap \hat{P}}] / K = -1$

$h \otimes 1 \cdot e_M = m(h)\, e_M$

$h \otimes t^n \cdot \mathbb{F}\{M\} = 0$, $h \otimes 1 \cdot S(\hat{h}^-_{\mathbb{Z}})$ as 0

$S(\hat{h}^-_{\mathbb{Z}}) \otimes \mathbb{F}\{M\}$ is a module
for $\{H \otimes t^n\}_{n \in \mathbb{Z}} \cup \{\mathbb{C}c\} \cup \{\mathbb{C}d\}$

$g \circ (v \otimes w) = g \cdot v \otimes w - v \otimes g \cdot w$

$S(\mathfrak{h}^-_{\mathbb{Z}}) \otimes F(M)$ is a module for

$\{H \otimes t^u\}_{u \in \mathbb{Z}}, \mathbb{C}c, \mathbb{C}d$

$\underline{x_a \otimes t^n}$

$$\sum_{N \in \mathbb{Z}} \underline{x_a \otimes t^n}\, z^{-N} =$$

action $S(\mathfrak{h}^-_{\mathbb{Z}})$

$$\exp\Big(\sum_{n<0} \frac{\overbrace{\bar{a} \otimes t^n}}{n} z^{-n}\Big) \underbrace{\exp\Big(\sum_{n>0} \frac{-\bar{a} \otimes t^n}{n} z^{-n}\Big) z^{\bar{a} + \frac{(\bar{a},\bar{a})}{2}}}\; a$$

$$\underline{z^{\bar{a}} \cdot e_m = z^{m(\bar{a})} e_m}$$

action $F(M)$

$$" = \exp\Big(\sum_{n \in \mathbb{Z}} \frac{-\bar{a} \otimes t^n}{n} z^{-n}\Big)"$$

And in fact all of $\hat{\mathfrak{g}}$:

$$\sum_{n \in \mathbb{Z}} (X_a \otimes t^n) z^{-n} = :\exp\left(\sum_{n \in \mathbb{Z}\setminus\{0\}} \frac{-\bar{a} \otimes t^n}{n} z^{-n}\right) a z^{a}:$$

$$(z^a \cdot e_\lambda) = e_\lambda z^{\lambda(\bar{a})}$$

$\otimes \, \mathbb{F}\{\Lambda\}$

Taction

$\{a \in \hat{\mathfrak{k}}\}$

Cosets of circul lattice in weight lattice

$\Updownarrow$

level modules

$\mathfrak{h} \otimes t^n$

$n(h)$

$h \otimes l \cdot S(\hat{\zeta}^-_R) \cdot e_m \cdot e_0 \cdot e_{R^*}$

$d \cdot \text{degree} + \frac{(m, m)}{2}$

$(h \otimes f^{-1})(h' \otimes f^{-1}) \otimes e_m$

$d: 5 + \frac{(m,m)}{2}$

$c \cdot S(\hat{\zeta}^-_R) = 1$

$\lambda(c) = 1$

$\varphi(c) = 0$

$(\lambda - \xi\varphi)(c) = 1$