Macdonald Polynomials and Double Affine Hecke Algebras

Graduate Algebra and Representation Theory Seminar

<u>Motivation:</u> Schur polynomials

Consider $G = GL_n(\mathbb{C})$.

Let $\rho: G \to GL(V)$ be a finite-dimensional algebraic representation, with <u>character</u> $\chi_\rho(g) = \mathrm{Tr}\,\rho(g)$.

ρ is completely determined by χ_ρ, but we can do better.

χ_ρ is continuous $\Rightarrow$ determined by values on dense set of diagonalizable matrices

$\chi_\rho(hgh^{-1}) = \chi_\rho(g) \Rightarrow$ determined by values on <u>diagonal</u> matrices

So $\rho \longleftrightarrow s_\rho(x_1, \dots, x_n) = \mathrm{Tr}\,\rho\left(\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix}\right)$.

If ρ is irreducible, s_ρ is called a <u>Schur polynomial</u>.

Can do all the usual character theory stuff with just polynomials.

<u>Example:</u> For reps $\rho_1 : GL_n(\mathbb{C}) \to GL(V_1)$
$\rho_2 : GL_n(\mathbb{C}) \to GL(V_2)$,

$$V_1 \otimes V_2 = \bigoplus_\lambda (V_\lambda)^{\oplus c_\lambda} \longleftrightarrow s_{\rho_1} s_{\rho_2} = \sum_\lambda c_\lambda s_\lambda .$$

Macdonald polynomials $E_\mu(x; q, k)$

$q = k = 1$: finite-dim'l reps of $GL_n(\mathbb{C})$

$q = k^{2a}, k \to 1$: "spherical" (∞-dim'l) reps of $GL_n(F)$ ($F = \mathbb{R}, \mathbb{C}, \mathbb{H}$)

$k = 0$: reps of affine Lie algebra $\widehat{\mathfrak{gl}}_n(\mathbb{C})$

$q = 0$: finite-dim'l reps of $GL_n(\mathbb{F}_p)$

"principal series" reps (∞-dim'l) of $GL_n(\mathbb{Q}_p)$

My research is about a further generalization of these. Our goal today is to understand what Macdonald polynomials are and how to work with them.

<u>Outline:</u>

① Weyl group, affine Weyl group, double affine Weyl group

② Double affine Hecke algebra

③ Macdonald polynomials

Weyl group (Type A):

Consider $\mathbb{R}^n$ with standard basis $\varepsilon_1, \dots, \varepsilon_n$.
The finite Weyl group of type A, called W, is the group of permutations of the coordinates. (So $W \approx S_n$)

Since all $w \in W$ fix $(1, \dots, 1)$, we often restrict our attention to the complementary subspace

$$E = \{ v \in \mathbb{R}^n : \sum_{i=1}^{n} v_i = 0 \}.$$

Let $s_i \in W$ swap the i and $i+1$ coordinates:

$$s_1 (v_1, v_2) = (v_2, v_1).$$

For any root $\alpha = \varepsilon_i - \varepsilon_j$, let $s_\alpha \in W$ be the reflection through the hyperplane

$$h^\alpha = \{ v \in E : (\alpha, v) = 0 \}. \qquad (v_i - v_j = 0)$$

s_α swaps i and j coordinates.

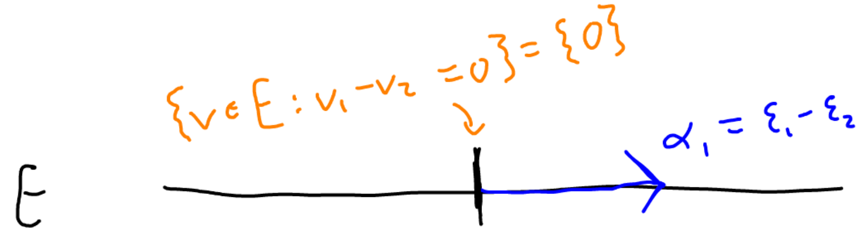
We have $s_i = s_{\alpha_i}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.

$W = \langle s_1, \dots, s_{n-1} \rangle$, with relations

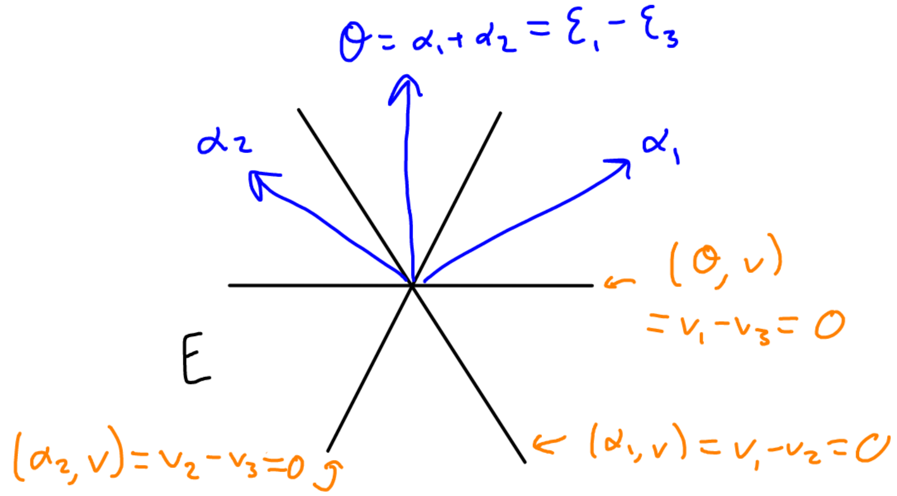
$s_i^2 = 1$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$.

Examples:

$n=2$ $\quad W = \langle s_1 \rangle$



$n=3$ $\quad W = \langle s_1, s_2 \rangle$



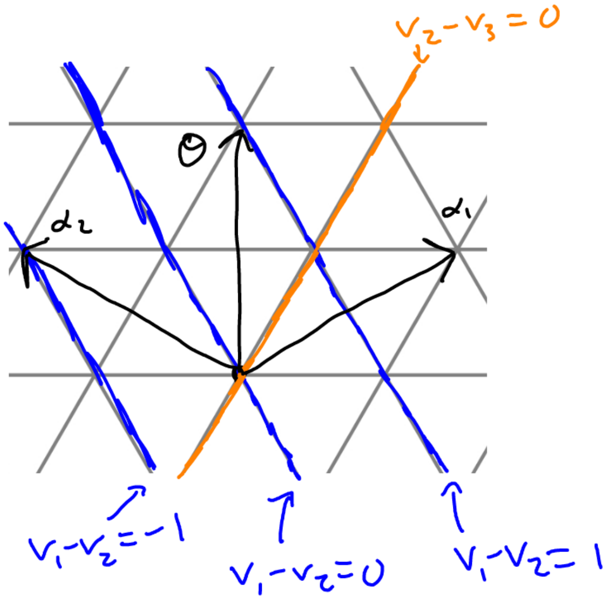
Let $Q = \mathrm{span}_{\mathbb{Z}}\{\alpha_1, \dots, \alpha_{n-1}\}$, the root lattice.

W acts on Q.

Example: With $n=3$, we have

$$s_{\alpha_1}(\alpha_2) = s_{\alpha_1}(0,1,-1) = (1,0,-1) = \alpha_1 + \alpha_2.$$

The affine Weyl group $\widetilde{W}$ is the group generated by the reflections in the affine hyperplanes $\{v \in E : v_i - v_j = s\}$ $(s \in \mathbb{Z})$.



Algebraically, we have

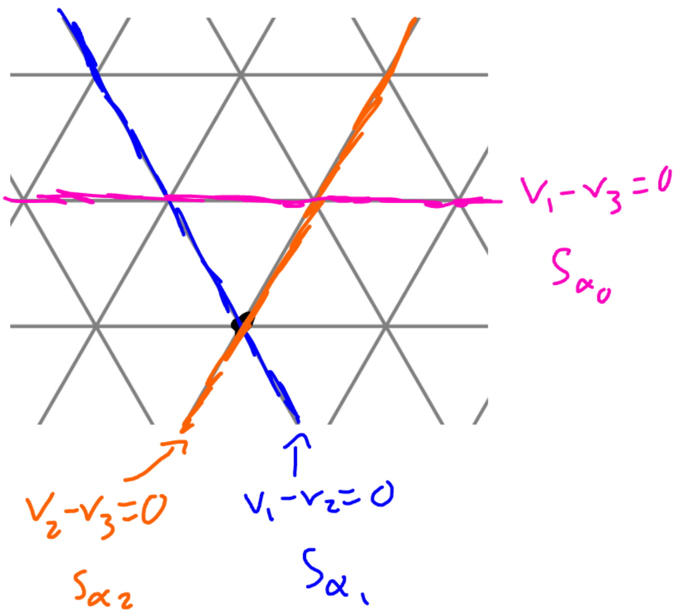
$$\widetilde{W} \cong W \ltimes \tau(Q),$$

where $\tau(\lambda) \longleftrightarrow$ translation by $\lambda \in Q$, and

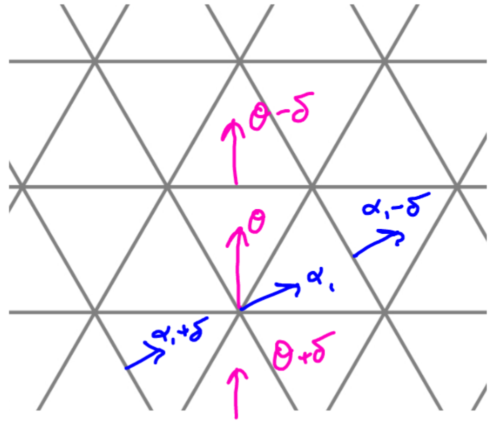
$$w\,\tau(\lambda)\,w^{-1} = \tau(w\lambda), \qquad \tau(\lambda+\mu) = \tau(\lambda)\tau(\mu).$$

action of W on Q $\qquad$ $\tau(Q)$ is abelian

$\widetilde{W}$ can be generated by the reflections around a single alcove: $s_{\alpha_1}, \dots, s_{\alpha_{n-1}}, s_{\alpha_0}$.
(α_0 corresponds to the hyperplane $\{v \in E : v_1 - v_n = 1\}$)

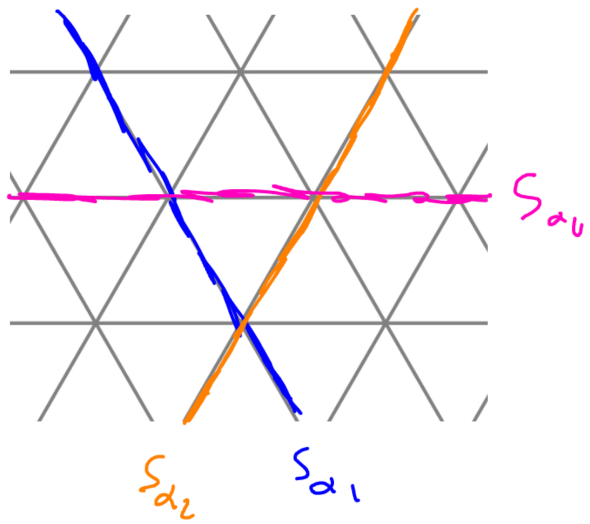


Let $\tilde{Q} = Q \oplus \mathbb{Z}\delta$. The <u>affine roots</u> $(\varepsilon_i - \varepsilon_j) + s\delta$ $(s \in \mathbb{Z})$ index the reflecting hyperplanes.



Let $\alpha_0 = \delta - \theta$. We have
$$\tilde{Q} = \mathrm{span}_{\mathbb{Z}}\{\alpha_0, \alpha_1, \dots, \alpha_{n-1}\}.$$

$\tilde{W}$ acts on the affine roots, and therefore $\tilde{Q}$, by permuting the corresponding hyperplanes.



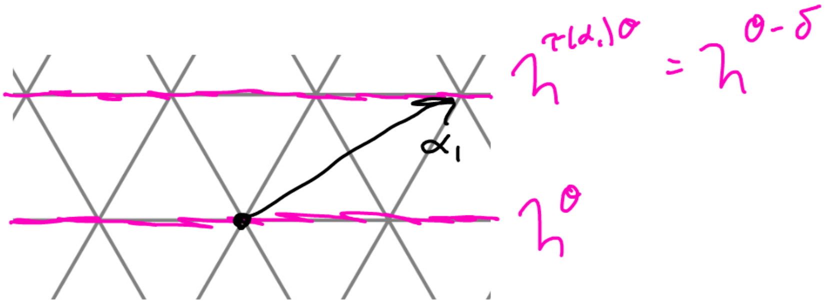
<u>Example</u>:
$$s_0 : \alpha_1 \longmapsto -\alpha_2 + \delta,$$
$$\alpha_1 - \delta \longmapsto -\alpha_2 = (-\alpha_2 + \delta) - \delta.$$ More generally,
$$\tilde{w} : \lambda + s\delta \longmapsto \tilde{w}(\lambda) + s\delta, \quad \text{so } \tilde{w}\delta = \delta.$$

More concretely, for $i \neq 0$, s_i acts on $\tilde{Q}$ the same way it acts on Q (by permutation, ignoring δ).

For s_0, we have $s_0\delta = \delta$ and
$$s_0\lambda = \lambda + (\lambda_1 - \lambda_n)\alpha_0 = \lambda - \frac{2(\alpha_0, \lambda)}{(\alpha_0, \alpha_0)}\alpha_0.$$

We can also calculate, for $\mu \in Q$ and $\lambda + s\delta \in \widetilde{Q}$,

$$\tau(\mu)(\lambda + s\delta) = \lambda + s\delta - (\lambda, \mu)\delta.$$



We can now define the double affine Weyl group:

$$\widetilde{\widetilde{W}} = \widetilde{W} \ltimes \eta(\widetilde{Q}), \text{ where } \eta(\widetilde{Q}) = \{\eta(\lambda + s\delta) : \lambda + s\delta \in \widetilde{Q}\},$$

abelian

$$\widetilde{w}\, \eta(\lambda + s\delta)\, \widetilde{w}^{-1} = \eta(\widetilde{w}(\lambda + s\delta)).$$

(And of course $\eta(\widetilde{\lambda} + \widetilde{\mu}) = \eta(\widetilde{\lambda})\eta(\widetilde{\mu})$.)

Let $q = \eta(\delta)$. q is central: for $\widetilde{w} \in \widetilde{W}$,

$$\widetilde{w}\, q\, \widetilde{w}^{-1} = \widetilde{w}\, \eta(\delta)\, \widetilde{w}^{-1} = \eta(\widetilde{w}\delta) = \eta(\delta) = q.$$

We can write

$$\tilde{\tilde{W}} = \tilde{W} \ltimes n(\tilde{Q}) = (W \ltimes \tau(Q)) \ltimes (n(Q) \oplus \langle q \rangle).$$

Further,

$$\begin{aligned}
\tau(\lambda) n(\mu) \tau(-\lambda) n(-\mu) &= n(\tau(\lambda)\mu) n(-\mu) \\
&= n(\mu - (\lambda,\mu)\delta) n(-\mu) \\
&= \cancel{n(\mu)}\, n(-(\lambda,\mu)\delta)\, \cancel{n(-\mu)} \\
&= n(-(\lambda,\mu)\delta) \\
&= q^{-(\lambda,\mu)}.
\end{aligned}$$

Same calculation shows

$$n(\mu)\tau(\lambda) n(-\mu) \tau(-\lambda) = q^{(\lambda,\mu)}.$$

$\tilde{\tilde{W}}$ consists of W and a "Heisenberg group": a central element q and 2 copies of Q with commutator: $\tau(Q) \times n(Q) \longrightarrow \langle q \rangle$.

Further, writing $\tau(\delta) = q^{-1}$, W has an involution defined by

$$s_i \longmapsto s_i \quad (i \neq 0)$$

$$n(\tilde{\lambda}) \longleftrightarrow \tau(\tilde{\lambda}) \quad (\tilde{\lambda} \in \tilde{Q})$$

$$\Big(\text{In particular, } \underset{n(\delta)}{\underset{\shortparallel}{q}} \longleftrightarrow \underset{\tau(\delta)}{\underset{\shortparallel}{q^{-1}}}.\Big)$$

Hecke Algebras

Let k be a new parameter.

We construct the (finite) <u>Hecke algebra</u> H as a deformation of $\mathbb{C}[W]$.

H	$\mathbb{C}[W]$
Algebra over $\mathbb{C}(k)$	Algebra over $\mathbb{C}$
Generators: $T_1, \dots, T_{n-1}$	Generators: $s_1, \dots, s_{n-1}$
Relations:	Relations:
$T_i^2 = (k-k^{-1})T_i + 1$	$s_i^2 = 1$
$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$	$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

$k \to 1$ recovers $\mathbb{C}[W]$. H arises in rep theory of $GL_n(\mathbb{F}_p)$, with $k = p^{-1}$.

We can also define the affine Hecke algebra (AHA):

$\tilde{H}$	$\mathbb{C}[\tilde{W}]$
Alg. over $\mathbb{C}(k)$	Alg. over $\mathbb{C}$
Generators: $T_0, \dots, T_{n-1}$	Generators: $s_0, s_1, \dots, s_{n-1}$
Relations:	Relations:
$T_i^2 = (k-k^{-1})T_i + 1$	$s_i^2 = 1$
$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$	$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
(letting $T_n = T_0$)	(letting $s_n = s_0$)
Alternate generators:	Alternate generators:
$T_1, \dots, T_{n-1}, Y^\mu$ $(\mu \in Q)$	$s_1, \dots, s_{n-1}, Y(\mu)$ $(\mu \in Q)$

New relations: for $i \neq 0$,

$$T_i Y^\mu = Y^{s_i \mu} T_i + (k - k^{-1}) \frac{Y^\mu - Y^{s_i \mu}}{1 - Y^{-\alpha_i}},$$

$$Y^{\mu+\lambda} = Y^\mu Y^\lambda.$$

New relations:

$$s_i \tau(\mu) = \tau(s_i \mu) s_i$$

$$\tau(\mu)\tau(\lambda) = \tau(\mu+\lambda)$$

$\tilde{H} \longleftrightarrow$ rep theory of $GL_n(\mathbb{Q}_p)$.

Finally, we have the <u>double affine Hecke algebra</u> $\tilde{\tilde{H}}$, generalizing $\mathbb{C}[\tilde{\tilde{W}}]$:

$\tilde{\tilde{H}}$	$\mathbb{C}[\tilde{\tilde{W}}]$
Generators: $T_0, \dots, T_{n-1}$, X^λ $(\lambda \in \tilde{Q})$	Generators: $s_0, \dots, s_{n-1}$, $n(\lambda)$ $(\lambda \in \tilde{Q})$
Relations: same as $\tilde{H}$, plus $T_i X^\lambda = X^{s_i\lambda} T_i + (k-k^{-1}) \frac{X^\lambda - X^{s_i\lambda}}{1 - X^{\alpha_i}}$	Relations: same as $\mathbb{C}[\tilde{W}]$, plus $s_i n(\lambda) = n(s_i\lambda) s_i$.
Let $q = X^\delta$. q is central.	Let $q = n(\delta)$. q is central.
Also let $Y^\delta = q^{-1}$.	Also let $\tau(\delta) = q^{-1}$.
We have an automorphism ε with	Automorphism ε with
$\varepsilon: T_i \longleftrightarrow T_i^{-1}$ $(i \neq 0)$	$\varepsilon: s_i \longleftrightarrow s_i$ $(i \neq 0)$
$\varepsilon: X^\lambda \longleftrightarrow Y^\lambda$	$\varepsilon: n(\lambda) \longleftrightarrow \tau(\lambda)$
$\varepsilon: q \longleftrightarrow q^{-1}$	$\varepsilon: q \longleftrightarrow q^{-1}$
$\varepsilon: k \longleftrightarrow k^{-1}$	

Let $T_0^\vee = \varepsilon(T_0)^{-1}$, $T_i^\vee = T_i \; (= \varepsilon(T_i)^{-1})$.

The $T_i^\vee$ are "T_i but for the Y's."

Recall that for $i=0,\dots,n-1$,

$$T_i X^\mu = X^{s_i\mu} T_i + (k-k^{-1})\left(\frac{X^\mu - X^{s_i\mu}}{1-X^{\alpha_i}}\right).$$

Applying ε, we can show

$$T_i^\vee Y^\mu = Y^\mu T_i^\vee + (k-k^{-1})\left(\frac{Y^\mu - Y^{s_i\mu}}{1-Y^{-\alpha_i}}\right)$$

for $i=0,\dots,n-1$.

(Previously only knew it for $i \neq 0$, where $T_i^\vee = T_i$.)

Macdonald Polynomials

We will construct a representation of $\widetilde{\mathbb{H}}$ on the space of Laurent polynomials

$$\mathbb{F}[Q] = \operatorname{span}_{\mathbb{F}}\{x^\mu : \mu \in Q\}, \text{ where } \mathbb{F} = \mathbb{C}(k,q).$$

("polynomials": $x^\mu = x_1^{\mu_1} x_2^{\mu_2} \cdots x_n^{\mu_n}$)

Macdonald polys will be simultaneous eigenfunctions of the commuting subalgebra $\{Y^\mu : \mu \in Q\}$.

Start with rep of AHA $\tilde{H}$ on $\mathbb{F}$ by

$$T_i . 1 = k \qquad (i = 0, \dots, n-1).$$

Then construct the induced module

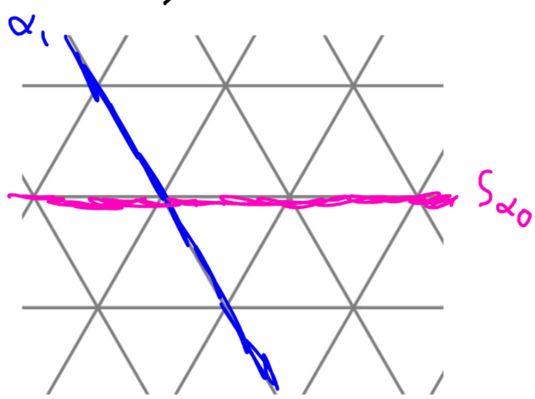
$$\begin{aligned}\mathrm{Ind}_{\tilde{H}}^{\tilde{\tilde{H}}} \mathbb{F} &= \tilde{\tilde{H}} \otimes_{\tilde{H}} \mathbb{F} \\ &= \langle X^\mu, T_0, \dots, T_{n-1} \rangle \otimes_{\langle T_0, \dots, T_{n-1} \rangle} \mathbb{F} \\ &\cong \langle X^\mu \rangle \otimes \mathbb{F} \\ &= \mathrm{span}_{\mathbb{F}} \{X^\mu : \mu \in Q\} = \mathbb{F}[Q].\end{aligned}$$

Concretely, we have:

- $X^\mu \longmapsto$ multiplication by x^μ
- $q \longmapsto$ the scalar q
- T_i acts using the T-X relation from above:

$$T_i x^\lambda = k x^{s_i \lambda} + (k - k^{-1}) \frac{x^\lambda - x^{s_i \lambda}}{1 - x^{\alpha_i}}$$

(where $s_i \lambda$ is the action of $\tilde{W}$ on $\tilde{Q}$, and $x^\delta := q$).



Example: Let $n=3$. Calculate $T_0 x^{\alpha_1}$.

$s_0 \alpha_1 = -\alpha_2 + \delta$, so

$$T_0 x^{\alpha_1} = k\, x^{-\alpha_2+\delta} + (k-k^{-1}) \frac{x^{\alpha_1} - x^{-\alpha_2+\delta}}{1-x^{\delta-\alpha_1-\alpha_2}}$$

$$= q k\, x^{-\alpha_2} + (k-k^{-1}) \frac{x^{\alpha_1} - q x^{-\alpha_2}}{1 - q x^{-\alpha_1-\alpha_2}} \qquad x^\delta = q$$

$$= q k\, x^{-\alpha_2} + (k-k^{-1}) x^{\alpha_1} \frac{1 - q x^{-\alpha_1-\alpha_2}}{1 - q x^{-\alpha_1-\alpha_2}} \qquad \text{Factor out } x^{\alpha_1}$$

$$= q k\, x^{-\alpha_2} + (k-k^{-1}) x^{\alpha_1}. \qquad \text{cancel}$$

<u>Claim:</u> The action of $\{Y^\mu : \mu \in Q\}$ on $\mathbb{F}[Q]$ is simultaneously diagonalizable.
The eigenfunctions are called (nonsymmetric) <u>Macdonald polynomials.</u>

<u>Proof idea:</u> construct the basis of eigenfunctions via a recursion.

For $i=0,\dots,n-1$, let

$$S_i = T_i^\vee + \frac{(k^{-1}-k)}{1-Y^{-\alpha_i}},$$

where $T_i^\vee = \varepsilon(T_i)^{-1}$. ($T_i^\vee = T_i$ for $i \neq 0$.)

Explicitly check $\quad Y^\mu S_i = S_i Y^{s_i \mu}.$

The point:

① If $f \in \mathbb{F}[Q]$ is an eigenvector of all Y^{μ}, say $Y^{\mu} f = \gamma(\mu) f$, then

$$Y^{\mu} S_i f = S_i Y^{s_i \mu} f = \gamma(s_i \mu) S_i f.$$

The S_i intertwine eigenspaces for the Y's!

② $Y^{\mu} = g_{\mu}(T_0, \ldots, T_{n-1})$, so $Y^{\mu}.1$ is a scalar.

③ Applying the S_i's to 1, we get a bunch of eigenvectors. Can show that the polynomials

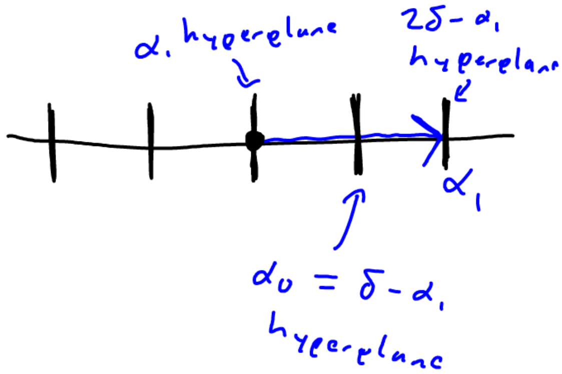
$$S_{i_1} \cdots S_{i_\ell} 1 \quad \text{span} \quad \mathbb{F}[Q].$$

In fact, letting $\mu = s_{i_1} \cdots s_{i_\ell} 0$, affine action $\tilde{W} \curvearrowright Q$

$$S_{i_1} \cdots S_{i_\ell} 1 \sim x^{\mu} + \text{lower terms} =: E_{\mu}$$

We call E_{μ} a (nonsymmetric) Macdonald polynomial.

Example: $n=2$, calculate $S_0 1$. Note $s_0 0 = \alpha_1$. So



$$E_{\alpha_1} \sim S_0 1 = \left(T_0^{\vee} + \frac{(k^{-1}-k)}{1-Y^{-\alpha_0}}\right) 1$$

$$= \left(T_1^{-1} x^{-\alpha_1} + \frac{(k^{-1}-k)}{1-Y^{-\alpha_0}}\right) 1.$$

$$\frac{(k^{-1}-k)}{1-Y^{-\alpha_0}} \mathbb{1} \text{ is some scalar.}$$

$$T_1^{-1} x^{-\alpha_1} \mathbb{1} = T_1^{-1} x^{-\alpha_1}$$

$$T_i x^{\mu} = (k-k^{-1})\left(\frac{x^{\mu}-x^{-\mu}}{1-x^{\alpha_i}}\right) + k x^{s_i \mu}$$

$$= (T_1 - (k-k^{-1})) x^{-\alpha_1}$$

$$= T_1 x^{-\alpha_1} - (k-k^{-1}) x^{-\alpha_1}$$

$$= (k-k^{-1})\left(\frac{x^{-\alpha_1} - x^{s_1(-\alpha_1)}}{1-x^{\alpha_1}}\right) + k x^{\alpha_1} - (k-k^{-1}) x^{-\alpha_1}$$

$$= (k-k^{-1})\left(\frac{x^{-\alpha_1} - x^{\alpha_1}}{1-x^{\alpha_1}}\right) + k x^{\alpha_1} - (k-k^{-1}) x^{-\alpha_1}$$

$$= (k-k^{-1}) x^{-\alpha_1}\left(\frac{1-x^{2\alpha_1}}{1-x^{\alpha_1}}\right) + k x^{\alpha_1} - (k-k^{-1}) x^{-\alpha_1}$$

$$= (k-k^{-1}) x^{-\alpha_1} (1 + x^{\alpha_1}) + k x^{\alpha_1} - (k-k^{-1}) x^{-\alpha_1}$$

$$= \cancel{(k-k^{-1}) x^{-\alpha_1}} + (k-k^{-1}) + k x^{\alpha_1} - \cancel{(k-k^{-1}) x^{-\alpha_1}}.$$

So overall

$$E_{\alpha_1} \sim S_0 \mathbb{1} = k x^{\alpha_1} + (\text{scalar}) \cdot \mathbb{1}.$$

<u>Ram-Yip formula:</u>

Nice formula for $S_{i_1} \cdots S_{i_\ell}$,

$$S_{i_1} \cdots S_{i_\ell} = \sum_{\substack{\text{ordered} \\ \text{sublists } p \\ \text{of } [i_1,\dots,i_\ell]}} X^{wt(p)} T_{\varphi(e)} f_p(Y).$$

Apply to $\mathbb{1}$: $T_{\varphi(e)} f_p(Y)$ acts as a scalar, giving formula for the appropriate Macdonald polynomial.

$$E_\mu \sim S_{i_1} \cdots S_{i_\ell} 1 = \sum_p c_p X^{wt(p)},$$

where c_p and $wt(p)$ are known explicitly.

My recent work: I did this for a more general family of polynomials and I am studying its applications.

Conclusions:

- DAHA is a natural object
- Macdonald polynomials have nice, concrete recursion coming from the DAHA. Use this for everything.