The Utility of Lattice Realizations of Simple Lie algebras

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1 Introduction

Typically the theory of simple Lie algebras is done without a choice of basis that has specified relations. This is likely due to the fact that there doesn't seem to be any "natural" choice of root vectors.

However, one can get a highly natural *double basis* for each root space; that is, a choice of a root vector and its negative, with structure constants that are natural and easy to compute.

This writeup will explore this double basis, along with some examples of how it can be used to simplify the theory of simple Lie algebras.

2 Lattices and Central Extensions

Let L be a non-degenerate, positive-definite, even $\mathbb{Z}$-lattice (where even means $(\alpha, \alpha) \in 2\mathbb{Z}$ for all $\alpha \in L$).

Since

$$(\alpha, \beta) = \frac{(\alpha + \beta, \alpha + \beta) - (\alpha, \alpha) - (\beta, \beta)}{2} \in \mathbb{Z}$$

L will also be integral.

Let $\Phi = \{\alpha \in L, (\alpha, \alpha) = 2\}$

Remark 2.1. One should think of L as the root lattice of a simply-laced root system Φ (normalized to have norm-square 2). If so, the set of vectors of norm 2 in L is precisely Φ (see appendix)

Lemma 2.2. *For all $\alpha, \beta \in \Phi$, $(\alpha, \beta) \in \{-2, -1, 0, 1, 2\}$. Furthermore, $\alpha + \beta \in \Phi \iff (\alpha, \beta) = -1$.*

Proof. The first part follows from Cauchy-Schwarz

$$(\alpha, \beta)^2 \leq (\alpha, \alpha)(\beta, \beta) = 4$$

(with equality only if $\alpha = \pm\beta$).

And note

$$(\alpha + \beta, \alpha + \beta) = 4 + 2(\alpha, \beta)$$

so $(\alpha + \beta, \alpha + \beta) = 2 \iff (\alpha, \beta) = -1$. $\square$

We want a central extension $\hat{L}$ of L

$$1 \to \langle \kappa, \kappa^2 = 1 \rangle \to \hat{L} \bar{\to} L \to 0$$

where the image of κ (also called κ) is central in L and for all $a, b \in \hat{L}$

$$ab = ba\kappa^{(\bar{a}, \bar{b})}$$

To understand the intended structure of $\hat{L}$ better (and see why this does describe a central extension), consider a section

$$e : \hat{L} \to L, \quad \alpha \mapsto e_\alpha$$

such that $\bar{e_\alpha} = \alpha$ for all $\alpha \in L$. So we have

$$\hat{L} = \{e_\alpha\}_{\alpha \in L} \cup \{\kappa e_\alpha\}_{\alpha \in L}$$

Define $\varepsilon(\alpha, \beta) : L \times L \to \mathbb{F}_2$ (with respect to this choice of section) such that

$$e_\alpha e_\beta = \kappa^{\varepsilon(\alpha,\beta)} e_{\alpha+\beta}$$

To be a group, we must have

$$\varepsilon(\alpha + \beta, \gamma) = \varepsilon(\alpha, \beta + \gamma)$$

for all $\alpha, \beta, \gamma \in L$. And to satisfy our intended commutation relations, we must have

$$\varepsilon(\alpha, \beta) = \varepsilon(\beta, \alpha) + (\alpha, \beta)$$

(where (α, β) is taken mod 2). Let us first show that such a central extension is unique (somewhat informally). Let $\{\alpha_i\}$ be a base for L. Observe that every element of $\hat{L}$ can be expressed uniquely in the form

$$\kappa^j (e_{\alpha_1})^{k_1} (e_{\alpha_2})^{k_2} \dots e_{\alpha_n}^{k_n}$$

with $j \in \{0, 1\}$ and each $k_i \in \mathbb{Z}$.

Furthermore, κ and the e_{α_i} generate $\hat{L}$, and any word involving these generators can be expressed in the above form using only the commutation relation. Thus $\hat{L}$ is uniquely determined.

We now show that such a central extension exists. Let $\{\alpha_1, \dots \alpha_n\}$ be a base for L. Define ε such that

1. $\varepsilon(\alpha_i, \alpha_j) = 0$ if $i \geq j$
2. $\varepsilon(\alpha_i, \alpha_j) = (\alpha_i, \alpha_j)$ if $j > i$, (α_i, α_j) taken mod 2
3. $\varepsilon(\cdot, \cdot)$ is bilinear

Now let $\hat{L}$ be the unique unital magma generated by symbols κ and $\{e_\alpha\}_{\alpha \in L}$ with the relations:

1. κ is central and $\kappa^2 = 1$
2. $e_\alpha e_\beta = \kappa^{(\alpha,\beta)} e_{\alpha+\beta}$

And defining $(e_\alpha)^{-1} = e_{-\alpha} \kappa^{(\alpha,-\alpha)}$, one can verify that these relations make $\hat{L}$ into a group.

Now via the map $\bar{e_\alpha} = \alpha, \quad \bar{1} = 0$, we see that $\hat{L}$ is a central extension of L by $\langle \kappa, \kappa^2 = 1 \rangle$.

All that is left to check is the commutation relations. Set

$$c(\alpha, \beta) = \varepsilon(\alpha, \beta) - \varepsilon(\beta, \alpha).$$

This is bilinear as well and on any pair of basis vectors, we have

$$c(\alpha_i, \alpha_j) = (\alpha_i, \alpha_j)$$

by considering the cases $i \geq j, i < j$.

So since $c(\cdot, \cdot)$ and $(\cdot, \cdot)$ are bilinear, we have

$$c(\alpha, \beta) = (\alpha, \beta)$$

where (α, β) is taken mod 2. Finally note

$$e_\alpha e_\beta = e_{\alpha+\beta} \kappa^{\varepsilon(\alpha,\beta)} = e_\beta e_\alpha \kappa^{\varepsilon(\alpha,\beta) - \varepsilon(\beta,\alpha)} = e_\beta e_\alpha \kappa^{c(\alpha,\beta)} = e_\beta e_\alpha^{(\alpha,\beta)}$$

and thus we have proven existence.

3 Realizations of Simply-Laced Lie algebras

Let
$$\hat{\Phi} = \{a \in \hat{L}, \bar{a} \in \Phi\}$$
and
$$H = L \otimes \mathbb{F}$$

For each $a \in \hat{\Phi}$, define the symbol x_a

Define the vector space
$$\mathfrak{g} = H \oplus \sum_{a \in \hat{\Phi}} \mathbb{F} x_a$$
with the only dependence relation $x_a = -x_{\kappa a}$. We equip a bilinear multiplication $[\cdot,\cdot]$ on $\mathfrak{g}$ uniquely defined by:

$$\begin{aligned} [g, g] &= 0 \quad \text{for all } g \in \mathfrak{g} \\ [h, x_a] &= (\bar{a}, h) x_a \quad \text{for } a \in \Phi, h \in H \\ [x_a, x_b] &= x_{ab} \quad \text{if } ab \in \hat{\Phi} \\ [x_a, x_{a^{-1}}] &= \bar{a} \end{aligned}$$

See FLM for a proof that this is really a Lie algebra. One can extend the symmetric bilinear form $(\cdot,\cdot)$ from L to H, then to $\mathfrak{g}$ by setting:
$$(x_a, x_b) = 1 \iff ab = 1$$
and if $\bar{ab} \neq 0$,
$$(x_a, x_b) = 0$$

The resulting form can be checked to be nondegenerate and $\mathfrak{g}$-invariant.

Now, suppose Φ spans H (as will be the case for any root lattice L). View $\mathfrak{g}$ as a weight module for H. Then any ideal $I \subset \mathfrak{g}$ is a submodule of $\mathfrak{g}$ under H, and thus also a weight module.

So we have $I = I \cap H \oplus \sum I \cap \mathbb{C} x_a$. If I is proper, then we have $I \cap H \neq 0$ or some $x_a \in I$. But if $x_a \in I$, then $[x_a, x_{a^{-1}}] = \bar{a} \in I$.

So I must contain some h. Then since Φ spans H, let $\alpha \in \Phi$ be such that $(\alpha, h) \neq 0$, and let a be a section of α. Since
$$[h, x_a] = (\alpha, h) x_a, \quad [h, x_{a^{-1}}] = (-\alpha, h) x_a, \quad [x_a, x_{a^{-1}}] = \alpha$$
we see that the $\mathfrak{s}\ell(2)$ $x_a, x_{a^{-1}}, \alpha \in I$, so I is not solvable. Thus $\mathfrak{g}$ is semisimple.

Remark 3.1. Given a symmetric $n \times n$ Cartan matrix A, one can form a lattice with a base $\{\alpha_i\}$ that has inner product matrix A. The Lie algebra constructed from $\mathfrak{g}$ as in this section will be the semisimple Lie algebra with Cartan matrix A.

4 Automorphisms

This section can be skipped for the sake of constructing the simply-laced simple Lie algebras, but will be important after.

Let σ be an isometry on L. Define:
$$\bar{\sigma} : \hat{L} \to L, \quad a \mapsto \sigma(\bar{a})$$
Then we have another realization of $\hat{L}$ as a central extension of L:
$$1 \to \langle \kappa, \kappa^2 = 1 \rangle, \to \hat{L} \xrightarrow{\bar{\sigma}} L \to 1$$
Now for $\alpha \in L$, let e_α be a section of $\hat{L}$ with respect to the map $\bar{\ }$. Note that $e_{\sigma^{-1}(\alpha)}$ is then a section of $\hat{L}$ with respect to the map $\bar{\sigma}$. And we have
$$e_{\sigma^{-1}(\alpha)} e_{\sigma^{-1}(\beta)} (e_{\sigma^{-1}(\alpha)})^{-1} (e_{\sigma^{-1}(\beta)})^{-1} = \kappa^{(\sigma^{-1}(\alpha), \sigma^{-1}(\beta))} = \kappa^{(\alpha,\beta)} = e_\alpha e_\beta (e_\alpha)^{-1} (e_\beta)^{-1}$$

So they have the same commutator map and are thus equivalent central extensions. So we have an automorphism $\psi : \hat{L} \to \hat{L}$ such that

$$\bar{}\circ\psi = \bar{\sigma}.$$

This map is not unique. To show the extent it is not unique, consider a base $\{\alpha_i\}$ for L and a corresponding section a_i. Then we have a bijection between elements of $(\mathbb{Z}/2\mathbb{Z})^n$ and automorphisms of $\hat{L}$ that preserve $\bar{}$ via

$$\psi_{t_1,\ldots t_n}(a_i) = a_i\kappa^{t_i}, \psi(\kappa) = \kappa$$

and extending.

Lemma 4.1. *Let $\Delta = \{\alpha_i\} \subset \Phi$, a choice of base of L whose inner product matrix is M with section $\{a_i\}$. Let σ be a linear isometry of L such that $\sigma(\Delta) = \Delta$. Then we can find a unique automorphism ψ such that $\psi(a_i) = a_{\sigma(i)}$, $\psi(\kappa) = \kappa$.*

Proof. Let ψ_1 be some automorphism on $\hat{L}$ such that $\bar{\sigma} = \bar{}\circ\psi_1$. Now for each i we have some $t_i \in \mathbb{Z}/2\mathbb{Z}$ we have $\psi_1(a_i) = a_{\sigma(i)}\kappa^{t_i}$. Composing with $\psi_{t_1,\ldots t_n}$ will give us the desired automorphism. □

Definition 4.2. *We call such an automorphism ψ the Δ-preserving lifting of σ to $\hat{L}$*

Theorem 4.3. *Let σ be a linear isometery on L and ψ an automorphism on $\hat{L}$ such that $\bar{}\circ\psi = \bar{\sigma}$. Then ψ lifts to an automorphism Ψ on $\mathfrak{g}$.*

Proof. We define a map $\Psi : \mathfrak{g} \to \mathfrak{g}$ such that:

$$h \mapsto \sigma(h) \text{ for } h \in H$$
$$x_a \mapsto x_{\psi(a)}$$

One can quickly verify that this is an automorphism by using $\bar{\psi(a)} = \sigma(\bar{a})$ and $(\sigma(\alpha), \sigma(\beta)) = (\alpha, \beta)$. Note also that as an isometry, $\sigma(K) = K$, so mapping $\phi(\alpha)$ to $\phi(\sigma(\alpha))$ is well-defined. □

Definition 4.4. *We say a linear isometry σ is a diagram automorphism if $\sigma(\Delta) = \Delta$.*

Definition 4.5. *We say a diagram automorphism σ is an orthogonal folding if for all $\alpha_i \in \Delta$, $\sigma(\alpha_i) = \alpha_i$ or $(\sigma(\alpha_i), \alpha_i) = 0$.*

We next prove a lemma that will allow us to show that all root vectors corresponding to roots fixed by an orthogonal folding σ will be fixed in the corresponding Lie algebra automorphism Ψ.

Lemma 4.6. *Let σ be an orthogonal folding and ψ be the Δ-preserving lifting of σ to $\hat{L}$.*
Then for $a \in \hat{L}$, if $\bar{\psi(a)} = \bar{a}$, then $\psi(a) = a$.

Proof. Let O_1, O_2,...O_k be disjoint subsets of $\{1, 2, ...n\}$ corresponding to orbits of σ on Δ. Express a as:

$$a = \kappa^s \prod_{j=k}^{k} \Big(\prod_{i \in O_j} {a_i}^{c_j} \Big)$$

(the $\prod$ notation might be a bit dubious due to lack of commutativity, but just pick an ordering in each orbit, for instance increasing order). Note we can make the exponent of a_i c_j since all elements of an oribit must have the same coefficient in an α fixed by σ.

$$\psi(a) = \kappa^s \prod_{j=k}^{k} \Big(\prod_{i \in I_j} a_{\sigma(i)}^{c_j} \Big)$$

Now since σ is an orthogonal folding, all the a_i for $i \in I_j$ commute with eachother (for each j). Thus it is clear that $\psi(a) = a$. □

Corollary 4.7. *Let σ be an orthogonal folding, $\alpha \in L$, and $a \in \hat{\Phi}$ a section of α. If for some k, $\sigma^k(\alpha) = \alpha$, then $\psi^k(a) = a$.*

Remark 4.8. The only non-orthogonal folding in the classical theory is the folding of A_{2n}. In the next section, we will show how to realize the non-simply laced simple Lie algebras as fixed point subalgebras of lifted diagram automorphisms; we will not need A_{2n} for this since the orthogonal folding of D_n gives the same fixed point subalgebra – B_n. However, its folding is still interesting to study and relevant because of the -1-eigenspace of this folding.

Lemma 4.9. *Let Φ be the A_{2n} root system with root lattice L and associated Lie algebra $\mathfrak{g}$.*

Let $\Delta = \{\alpha_i\}_{i=1}^{2n} \subset \Phi$ be a base for L such that $(\alpha_i, \alpha_{i+1}) = -1$ and $(\alpha_i, \alpha_j) = 0$ if $|i-j| > 2$. Let σ be the diagram automorphism defined by $\sigma(\alpha_i) = \alpha_{2n+1-i}$.

Let ψ be the Δ-preserving lifting of σ to $\hat{L}$. Then for $a \in \hat{L}$, if $\bar{\psi(a)} = \bar{a}$, we have $\psi(a) = \kappa a$

Proof. This can be proven from scratch, but we will assume the reader is familiar with the structure of A_{2n} and can recognize that the only roots fixed by σ are of the form:

$$\pm(\alpha_j + \alpha_{j+1} + \dots \alpha_{2n-j+1})$$

It will suffice to show that for any $1 \leq j \leq n$, and sections a_i for each α_i,

$$\psi\big((a_j a_{2n-j+1})(a_{j+1} a_{2n-j}) \dots (a_n a_{n+1})\big) = \kappa (a_j a_{2n-j+1})(a_{j+1} a_{2n-j}) \dots (a_n a_{n+1})$$

Now

$$\psi(a_n a_{n+1}) = a_{n+1} a_n = \kappa^{-1} a_n a_{n+1} = \kappa a_n a_{n+1}.$$

(This is where the non-orthogality plays a role). And for $i < n$,

$$\psi(a_i a_{2n-i+1}) = \kappa^0 a_{2n-i+1} a_i = a_i a_{2n-i+1}$$

The conclusion follows. ☐

5 Realizations of non-simply-laced simple Lie algebras

For this section, take L to be the root lattice of some simply-laced indecomposable root system Φ and let $\Delta = \{\alpha_i\}$ be a base for Φ in the usual sense (that is, all roots in Φ can be expressed as a non-negative or non-positive combinations of roots in Δ). Call roots in Δ *simple roots.*

σ will be a diagram automorphism with Δ-preserving lifting ψ. Ψ will be the corresponding automorphism on $\mathfrak{g}$

We will also implicitly use the isomorphism $H \to H^*, \quad h \mapsto (h, \cdot)$ to identify functionals on H (e.g. roots) with vectors in H. Likewise we will identify H^σ with $H^{\sigma,*}$ via this map.

Definition 5.1. *Let*

$$H^\sigma = \{\alpha \in H, \sigma(\alpha) = \alpha\}, \quad n^\sigma = \dim(H^\sigma)$$

Let p denote the order of σ (determined by its order on Δ), p_α the order of σ on $\alpha \in \Phi$, and p_a the order of σ on $a \in \hat{\Phi}$.

Definition 5.2. *For $\alpha \in \Phi$, let*

$$\alpha^\sigma = \frac{\alpha + \sigma(\alpha) + \dots \sigma^{p_\alpha - 1}(\alpha)}{p_\alpha}$$

Note since σ preserves positive roots, we always have $\sigma^\alpha \neq 0$

and for $a \in \hat{L}$, set

$$x_a^\sigma = x_a + \Psi(x_a) + \dots \Psi^{p_a - 1}(x_a)$$

Remark 5.3. By their definitions, we can see that ψ also has order p on $\hat{L}$. Furthermore, the only possible values for p can be checked experimentally to be 2 or 3, so p_α and p_a are always either 1 or p.

Thus for any $a \in \hat{\Phi}$ we have three possibilities:

1. $p_a = 1, \quad p_{\bar{a}} = 1$
2. $p_a = p, \quad p_{\bar{a}} = p$
3. $p_a = p, \quad p_{\bar{a}} = 1$

Now lemma 4.6 has shown that if σ is orthogonal and $p_{\bar{a}} = 1$, we must have $p_a = 1$.

Furthermore, by lemma 4.9, we have that if σ is the non-orthogonal A_{2n} folding and $p_{\bar{a}} = 1$, then $\psi(a) = \kappa a$ (so $p_a = 2$).

This gives us the following lemma:

Lemma 5.4. *If $\sigma(\bar{a}) \neq \bar{a}$, then $p_a = p_{\bar{a}} = p$. In particular,*

$$x_a^\sigma = x_a + \Psi(x_a) + ...\Psi^{p-1}(x_a) = x_a + x_{\psi(a)} + \dots x_{\psi^{p-1}(a)}$$

and all summands are linearly independent.

If $\sigma(\bar{a}) = \bar{a}$ and σ is orthogonal, we have

$$x_a^\sigma = x_a$$

If on the other hand $\sigma(\bar{a}) = \bar{a}$ and σ is the A_{2n} folding, we have

$$x_a^\sigma = x_a + x_{\kappa a} = x_a - x_a = 0$$

Now let $\mathfrak{g}^\Psi$ be the fixed point subalgebra of Ψ. We have

$$\mathfrak{g}^\Psi = H^\sigma \oplus \sum_{a \in \hat{\Phi}} x_a^\sigma$$

What are the roots of $\mathfrak{g}^\Psi$? For $h \in H^\sigma$, we have

$$[h, x_a^\sigma] = [h, \sum_{i=1}^{p_a} x_{\psi^i(a)}] = \sum_{i=1}^{p_a} (h, \sigma^i(\bar{a})) x_{\psi^i(a)} = (h, \bar{a}) x_a^\sigma = (h, \bar{a}^\sigma) x_a^\sigma$$

since

$$(h, \sigma(\alpha)) = (\sigma^{-1}(h), \alpha) = (h, \alpha)$$

and if $x_a^\sigma \neq 0$, $p_a = p$.

Set

$$\begin{aligned}
\Delta^\sigma &= \{\alpha^\sigma\}_{\alpha \in \Delta} \\
\Phi^\sigma &= \{\alpha^\sigma\}_{\alpha \in \Phi} \\
\Phi^{\sigma,o} &= \{\alpha^\sigma\}_{\alpha \in \Phi, \sigma(\alpha) \neq \alpha} \\
\Phi^{\sigma,f} &= \{\alpha\}_{\alpha \in \Phi, \sigma(\alpha) = \alpha} = \Phi^\sigma \setminus \Phi^{\sigma,o}
\end{aligned}$$

and let Φ^Ψ be the root system of $\mathfrak{g}^\Psi$

Since σ preserves Δ and Δ is a basis for H, Δ^σ is a basis for H^σ (recall the identification of H with H^*).

Let $\{O_j\}$ be the set of orbits of σ on Δ. If α and β are in the same orbit, we have $\alpha^\sigma = \beta^\sigma$, so let $\alpha_{O_j} = \alpha_J \sigma$ for some $\alpha_J \in O_j$.

If σ is orthogonal, $\Phi^\Psi = \Phi^\sigma$ since for each $\alpha \in \Phi$ with section $a \in \hat{L}$, x_a^σ is a non-zero vector with root α^σ.

If σ is the nonorthogonal folding of A_{2n}, we have $\Phi^\sigma = \Phi^\Psi$ since we must exclude the roots fixed by σ. Note we still have $\Delta^\sigma \subset \Phi^\Psi$ for this folding.

In particular, All these together show:

Lemma 5.5. Δ^σ *is a base for the root system of* $\mathfrak{g}^\Psi$

Proof. We have shown that Φ^Ψ always contains Δ^σ and thus spans H^σ. Furthermore, since every vector in Φ is a non-negative or non-positive combination of vectors in Δ, one can show that every vector in Φ^σ is a non-negative or non-positive combination of vectors in Δ^σ. □

Theorem 5.6. $\mathfrak{g}^\Psi$ *is semisimple*

Proof. Let I be a non-zero ideal of $\mathfrak{g}^\Psi$.

Since H^σ acts ad-semisimply on $\mathfrak{g}^\Psi$, it acts semisimply on any $\mathfrak{g}^\Psi$ module. In particular, I is H^σ graded.

Now we can show that I intersects H^σ. If not, it contains a vector x lying in some $\alpha^\sigma \in \Phi^\sigma$ root space.

We have not yet shown that each root space is 1-dimensional, so it is possible to have representatives from distinct root orbits $\{\beta_1, \beta_2, \dots \beta_k\}$ with all $\beta_i^\sigma = \alpha^\sigma$.

Write $x = \sum c_i x^\sigma_{b_i}$ where for each i, $\bar{b_i} = \beta_i$.

Then if j is such that $c_j \neq 0$, we have

$$[x, x^\sigma_{b_j^{-1}}] = p_\alpha \alpha^\sigma + r$$

where r lies in a sum of root spaces. Since H^σ acts semisimply, we can thus conclude $\alpha^\sigma \in I$.

Let $h \neq 0$ be a non-zero vector in $I \cap H^\sigma$. Since the root system spans H^σ, we have some root β^σ with $(h, \beta^\sigma) \neq 0$, so if b is such that $\bar{b} = \beta$,

$$[h, x^\sigma_b] = (h, \beta^\sigma) x^\sigma_b \neq 0$$
$$[h, x^\sigma_{b^{-1}}] = (h, -\beta^\sigma) x^\sigma_b \neq 0$$

$[h, x^\sigma_{b^{-1}}]$

Now since $[x^\sigma_b, x^\sigma_{b^{-1}}]$ is a non-zero multiple of β^σ, we have

$$\{\beta^\sigma, x^\sigma_b, x^\sigma_{b^{-1}} \in I$$

so I cannot be solvable. Thus $\mathfrak{g}$ is semisimple. □

Thus, Φ^Ψ is a crystallographic root system and each root space is 1-dimensional.

We know it has base Δ^σ, so we can determine what this root system is.

First we will try to understand the simple coroots. Let

$$h_{O_i} = \frac{2\alpha_{O_i}}{(\alpha_{O_i}, \alpha_{O_i})}$$

Lemma 5.7. *If σ is an orthogonal folding, we have*

$$h_{O_i} = \alpha_i + \sigma(\alpha_i) + \dots \sigma^{p_\alpha - 1}(\alpha_i)$$

Proof. If $\alpha_{O_i} \in \Phi^{\sigma,f}$, then α_{O_i} has norm-square 2, so $h_{O_i} = \alpha_{O_i}$

On the other hand, if $\alpha_{O_i} \in \Phi^{\sigma,o}$, then since σ is orthogonal, we have

$$(\alpha_{O_i}, \alpha_{O_i}) = \frac{2p_\alpha}{p_\alpha^2} = \frac{2}{p_\alpha}$$

so

$$h_{O_i} = p_\alpha \alpha_{O_i} = \alpha_i + \sigma(\alpha_i) + \dots \sigma^{p_\alpha - 1}$$

□

The Cartan matrix A^σ of $\mathfrak{g}^\Psi$ is determined by

$$A_{i,j} = \alpha_{O_j}(h_{O_i})$$

Remark 5.8. We know this is a Cartan matrix for a semisimple Lie algebra, so we know in particular that the $A_{i,j}$ are integral.

These are the Dynkin Diagrams that result:

Next, we consider what happens for $\Phi = A_{2n}$. Label the simple roots $\{\alpha_i\}$ as before such that $(\alpha_i, \alpha_{i+1}) = -1$ and $(\alpha_i, \alpha_j) = 0$ if $|i-j| > 2$.

For $1 \leq i \leq n$, let $O_i = \{\alpha_i, \alpha_{2n+1-i}\}$. Then O_i contains an orbit of non-orthogonal roots for each $i < n$, but the roots in O_n are not orthogonal.

Accordingly, we have $h_{O_i} = \alpha_i + \alpha_{2n+1-i}$ for $i < n$ and $h_{O_n} = 2(\alpha_n + \alpha_{n+1})$.

We get the resulting Dynkin diagram:

Thus $\mathfrak{g}^\Psi$ is B_n.

Remark 5.9. Recall that for this folding only, $\Phi^\Psi = \Phi^{\sigma,o}$, whereas for all the orthogonal foldings, $\Phi^\Psi = \Phi^\sigma = \Phi^{\sigma,o} \cup \Phi^{\sigma,f}$.

Let's investigate the "excluded roots" $\Phi^{\sigma,f}$. These are of the form

$$\beta = \pm(\alpha_j + \alpha_{j+1} + \dots \alpha_{2n-j} + \alpha_{2n-j+1}).$$

For this choice of sign and j, consider the root

$$\beta' = \pm(\alpha_j + \alpha_{j+1} + \dots \alpha_n)$$

We have $2\beta'^\sigma = \beta^\sigma$. And $\beta'^\sigma \in \Phi^{\sigma,o}$ has norm-square $1/2$; i.e. it has the same norm as the short simple root α_{O_n}. In this way, each vector in $\Phi^{\sigma,f}$ is twice a short root in $\Phi^{\sigma,o} = B_n$.

Thus Φ^σ is actually the BC_n non-reduced root system.

For the sake of twisted affine Lie algebras, it will be helpful to consider the other eigenspaces of Ψ on $\mathfrak{g}$

Let $\mathfrak{g}^{\Psi,\omega}$ be the eigenspace of Ψ on $\mathfrak{g}$ with eigenvalue ω (ω a p-th root of unity except 1).

Let

$$x_a^{\sigma,\omega} = x_a + \omega\Psi(x_a) + \dots \omega^{p-1}\Psi^{p-1}(x_a)$$

and for each i, let

$$\alpha_{O_i}^\omega = \beta + \omega\sigma(\beta) + \dots \omega^{p-1}\sigma^{p-1}(beta)$$

for some choice of β in each α_{O_i}.

Then $\mathfrak{g}^{\Psi,\omega}$ will be spanned by these $x_a^{\sigma,\omega}$ and $\alpha_{O_i}^\omega$.

Now observe that $\alpha_{O_i}^\omega \neq 0$ precisely when $\alpha_{O_i} \in \Phi^{\sigma,o}$.

Likewise, we have $x_a^{\sigma,\omega} \neq 0$ if $\bar{a}$ is not fixed by σ. If σ is orthogonal, then by lemma 4.6 (along with the fact that $1 + \omega + \dots \omega^{p-1} = 0$), we have $x_a^{\sigma,\omega} = 0$ if $\bar{a}$ is fixed by σ.

On the other hand, if σ is the non-orthogonal folding of A_{2n}, by lemma 4.9, $x_a^{\sigma,-1} \neq 0$ if $\bar{a}$ is fixed by σ.

Since $\mathfrak{g}^{\Psi,\omega}$ is a $\mathfrak{g}^\Psi$ module, we can view these as root vectors of H^σ acting on $\mathfrak{g}^{\Psi,\omega}$. By a counting argument, all non-zero roots of $\mathfrak{g}^{\Psi,\omega}$ have dimension 1.

6 Weyl Group

In section 2, we showed how any automorphism of a simply-laced root system Φ could lift non-uniquely to an automorphism of the associated Lie algebra. We also identified special automorphisms known as Diagram automorphisms.

There is another important subgroup of automorphisms knows as the *Weyl Group* W. This is the group generated by the reflections:

$$s_\alpha : \Phi \to \Phi, \quad x \mapsto x - \frac{2(\alpha, x)}{(\alpha, \alpha)}\alpha$$

for each $\alpha \in L$.

The Weyl group is actually generated by the simple reflections $s_i = s_{\alpha_i}$. Note $s_i(\alpha_j) = \alpha_j - A_{i,j}\alpha_i$ where A is the Cartan matrix.

The Weyl group can also be defined analogously for non-simply laced root systems. Here we have shown how to realize all non-simply laced root systems in the form Φ^σ where Φ is simply-laced and σ is orthogonal.

Lemma 6.1. *Let $s_{O_i} = \prod \alpha \in O_i s_\alpha$. Note by orthogonality the order of this product does not matter. Then s_{O_i} commutes with σ and*

$$s_{O_i}(\alpha_{O_j}) = \alpha_{O_j} - \alpha_{O_j}(h_{O_i})\alpha_{O_i} = \alpha_{O_j} - A^\sigma_{i,j}\alpha_{O_i}$$

Proof. We have

$$s_{O_i}(x) = x - \sum_{\alpha \in O_i} \frac{2(\alpha, x)}{(\alpha, \alpha)}\alpha = x - \sum_{\alpha \in O_i} (\alpha, x)\alpha$$

And since σ permutes the roots in O_i and preserves the inner product, we have

$$\begin{aligned} s_{O_i}(\sigma(x)) &= \sigma(x) - \sum_{\alpha \in O_i} (\alpha, \sigma(x))\alpha \\ &= \sigma(x) - \sum_{\alpha \in O_i} (\alpha, x)\sigma(\alpha) \\ &= \sigma(s_{O_i}(x)) \end{aligned}$$

so σ commutes with each s_{O_i}.

Now

$$s_{O_i}(\alpha_{O_j}) = \alpha_{O_j} - \sum_{t=0}^{p-1} \sum_{s=0}^{|O_i|-1} \frac{(\alpha_{\sigma^s(i)}, \alpha_{\sigma^t(j)})}{p} \alpha_{\sigma^s(i)}$$

for some $i \in O_i, j \in O_j$. We have

$$\begin{aligned} \sum_{t=0}^{p-1} \sum_{s=0}^{|O_i|-1} \frac{(\alpha_{\sigma^s(i)}, \alpha_{\sigma^t(j)})}{p} \alpha_{\sigma^s(i)} &= \sum_{t=0}^{p-1} \sum_{s=0}^{|O_i|-1} \frac{(\alpha_{\sigma(i)}, \alpha_{\sigma^{t-s}(j)})}{p} \alpha_{\sigma^s(i)} \\ &= \sum_{t=0}^{p-1} \sum_{s=0}^{|O_i|-1} \frac{(\alpha_i, \alpha_{\sigma^t(j)})}{p} \alpha_{\sigma^s(i)} \\ &= \sum_{s=0}^{|O_i|-1} (\alpha_i, \alpha_{O_j})\alpha_{\sigma^s(i)} \\ &= \sum_{s=0}^{|O_i|-1} (\alpha_{O_i}, \alpha_{O_j})\alpha_{\sigma^s(i)} \\ &= (\alpha_{O_i}, \alpha_{O_j})h_{O_i} \\ &= (\alpha_{O_j}, h_{O_i})\alpha_{O_i} \\ &= -A_{i,j}\alpha_{O_i} \end{aligned}$$

□

This shows that the simple reflections of the Weyl group W^σ of Φ^σ are the s_{O_i}, so these generate W^σ. Therefore $W^\sigma \subset W$ and W^σ commutes with σ.

Lemma 6.2. *Let σ be an orthogonal folding of Φ and $\alpha \in \Phi$. Then there exists a $w \in W^\sigma$ such that $(w\alpha)^\sigma = w(\alpha^\sigma) = \alpha_i^\sigma$ for some $\alpha_i \in \Delta$.*

Furthermore, σ will have the same order on α as it does on α_i and the coroot $h^\sigma_\alpha = \alpha + \sigma(\alpha) + \dots \sigma^{p_\alpha - 1}(\alpha)$

Proof. Since Φ^σ is a root system with Weyl group W^σ and simple roots of the form α_i^σ (also expressed α_{O_i} here), the first statement is a standard result about root systems.

Now we have

$$w\alpha + \sigma(w\alpha) + \dots \sigma^{p-1}(w\alpha) = \alpha_i + \sigma(\alpha_i) + \dots \sigma^{p-1}(\alpha_i)$$

Since $W^\sigma \subset W$, $w\alpha$ must be a root, and since σ preserves positivity, $w\alpha$ must be positive. Then comparing heights on both sides of the equation, we see that $w\alpha$ must be simple.

Now if α_i, α_j are simple and $\alpha_i^\sigma = \alpha_j^\sigma$, we must have α_i and α_j in the same σ orbit since the simple roots form a basis and are closed under σ.

Therefore, $w\alpha$ and α_i are in the same σ orbit. So σ is the same order on α_i as it does on $w\alpha$. And since w commutes with σ, it also has this same order on α.

Now since σ is orthogonal and α_i is simple, $(\alpha_i^\sigma, \alpha_i^\sigma) = \frac{2}{p_{\alpha_i}}$. And since w preserves length, we have

$$(\alpha^\sigma, \alpha^\sigma) = \frac{2}{p_{\alpha_i}} = \frac{2}{p_\alpha}$$

Thus,

$$h_\alpha^\sigma = \frac{2\alpha^\sigma}{(\alpha^\sigma, \alpha^\sigma)} = \frac{2\alpha^\sigma}{2/p_\alpha} = p_\alpha \alpha^\sigma = \alpha + \sigma(\alpha) + \ldots \sigma^{p_\alpha - 1}(\alpha)$$

□

7 Chevalley Bases

In section 3, we showed how to construct any simply-laced simple Lie algebra from a positive definite even lattice L and central extension $\hat{L}$ which comes from the form on L. Set $\Phi = L_2$ and let Δ be a base for Φ in the usual sense.

Let P be a set of representative sections for each $\alpha \in \Phi = L_2$ in $\hat{L}$. Then the following is a Chevalley basis for $\mathfrak{g}$:

$$\Delta \cup \{x_a\}_{a \in P}$$

Now recall from section 4 that any non-simply-laced simple lie algebra can realized as the fixed points of an automorphism Ψ on a simply-laced simple Lie algebra $\mathfrak{g}$, where Ψ restricts to an orthogonal folding σ on Δ.

Let P be a set of representative sections in $\hat{L}$ for one root in each orbit of σ on Φ.

The following is a Chevalley basis for $\mathfrak{g}^\Psi$:

$$\{h_{O_i}\}_{i=1}^n \cup \{x_a^\sigma\}_{a \in P}$$

It is worthwhile to show what some of these structure constants are. We know how h_{O_i} acts on root vectors (and these are integral since the Cartan matrix has integral entries), so we only focus on the brackets of root vectors.

We will ignore choices of section for the root vectors as these just introduce minus signs. We also only

For the simply laced case, note we have:

$$[x_a, x_{a^{-1}}] = \bar{a}$$

which is an integral combination of the $\{\alpha\} \in \Delta\}$.

And clearly, if $ab \in \hat{L_2}$, we have

$$[x_a, x_b] = x_{ab}$$

and 0 otherwise

For the non-simply laced case, we have

$$[x_a^\sigma, x_{a^{-1}}^\sigma] = h_{\bar{a}}^\sigma$$

and we know all coroots are integral combinations of the simple coroots.

Next, consider a non-zero

$$[x_a^\sigma, x_b^\sigma]$$

and assume without loss of generality that $\bar{ab} \in \Phi$.

If $\bar{a}$ is fixed by σ, we have

$$[x_a^\sigma, x_b^\sigma] = [x_a, x_b^\sigma] = x_{ab}^\sigma$$

since $(\bar{a}, \sigma(\bar{b})) = (\bar{a}, \bar{b})$.

Of course, we have the same result if $\bar{b}$ is fixed by σ. This just leaves the case of when both $\bar{a}$ and $\bar{b}$ have orbit of size p:

$$[x_a + \Psi(x_a) + \ldots \Psi^{p-1}(x_a), x_b + \Psi(x_b) + \ldots \Psi^{p-1}(x_b)]$$

this must be a multiple of x^σ_{ab}. Now x^σ_{ab} has $p_{\bar{ab}}$ terms. So if this has m non-zero cross terms, then we have

$$[x_a + \Psi(x_a) + \ldots \Psi^{p-1}(x_a), x_b + \Psi(x_b) + \ldots \Psi^{p-1}(x_b)] = \frac{m}{p_{\bar{ab}}} x^\sigma_{ab}$$

How can m be more easily computed? Observe we have

$$m = (h^\sigma_{\bar{a}}, h^\sigma_{\bar{b}}) = p(\bar{a}^\sigma, h^\sigma_{\bar{b}})$$

Putting all this together, if $[x^\sigma_a, x^\sigma_b] \neq 0$, we have

$$[x^\sigma_a, x^\sigma_b] = x^\sigma_{ab}$$

unless $\bar{a}$ and $\bar{b}$ are not fixed by σ (i.e. $\bar{a}$ and $\bar{b}$ are short) in which case we have

$$[x^\sigma_a, x^\sigma_b] = \frac{p\bar{a}^\sigma(h^\sigma_{\bar{b}})}{p_{\bar{ab}}} x^\sigma_{ab}$$