burnside problem (1902)

suppose G is a finitely generated, torsion group

is G finite?

answer no; but examples are hard to get

approach 2: start with $Aut(T)$

for some binary tree

Fractal groups

approach 1: $G = \langle \overset{\text{finite}}{D} \mid \underset{\text{don't care}}{R} \rangle$

while making G, ensure that each

for each $w \in \langle D \rangle$, $w^n \in R$ for some n

problem: no idea if G is infinite

(if $|D|=2$, $R = \{w^5 \mid w \in \langle D \rangle\}$, don't know if G is finite

Golod Shaferevich groups:

thm: if $\langle D, R \rangle$ is a finite group,

then $|R| \geq \frac{|D|^2}{4}$

(intuition: R needs to say "all gens commute $\sim |D|^2$ relations

+ finite order ($\sim |D|$ relations"

analogue for algebras:

A is assoc, central K-algebra, graded

generated by $U = \{u_1, \dots u_d\}$

A is a quotient of $K\langle \underbrace{u_1, \dots u_d}_{\text{non-commuting}} \rangle / I$ where I is homogeneous

$$K\langle U \rangle = \bigoplus_n \underbrace{\text{span}_K \{\text{homo. words of length } n\}}_{K\langle U\rangle_n}$$

$$I = \bigoplus_n I \cap K\langle U \rangle_n$$

relations: R is set of homo. relations

e.g. $u_1 + u_2 = u_2 + u_3$ $\qquad$ $u_1 u_2 = u_2 u_1$

$I = \langle R \rangle$, $\quad R_n = R \cap k\langle u \rangle_n$, $r_n = |R_n|$

$A_n = k\langle u \rangle_n / I_n$, $\quad a_n = \dim_k (A_n)$

let $H_R(t) = \sum_n r_n t^n$

$H_A(t) = \sum_n a_n t^n$

theorem:

$$\underline{\left(1 - |U| t + H_R(t)\right) \cdot H_A(t)} \geq 1 + 0t + 0t_2 + \cdots$$

(where $\geq$ means every term is bigger)

pf: I is generated by R, so $I = \operatorname{span}_k(\{v r w\})$

where $v, w \in k\langle u \rangle$

I_n is spanned by $\underbrace{v}_{\text{weights } s}\ \underbrace{r}_{m}\ \underbrace{w}_{\ell}$ $\qquad s+m+\ell=n$

- if $s>0$ $\qquad v = u_i \cdot v'$ for some $v' \in k\langle u\rangle$

$$v \cdot r\, w = u_i \cdot (\underbrace{v'\, r\, w}_{I_{n-1}})$$

these live in $(k\langle u\rangle_1) \cdot I_{n-1}$

- if $s=0$, then $v \in k$

$$r \cdot w \quad \text{lives in} \quad \sum_{m=1}^{n} r_m \cdot (k\langle u\rangle_{n-m})$$

write $k\langle u\rangle_n = I_n \oplus B_n$ for some B_n

if $w \in I_{n-m} \oplus B_{n-m}$

$$r_m \cdot w = r_m \cdot (i_{n-m} + b_{n-m})$$

$$r_m \cdot \boxed{i_{n-m}}$$

$$\in K\langle u\rangle_1 \cdot I_{n-m}$$

So $\boxed{I_n \subseteq K\langle u\rangle_1 \cdot I_{n-1} + \sum_{m=1}^{n} R_m \cdot B_{n-m}}$

$$\dim I_n + \dim(A_n) = d^n = \dim(K\langle u\rangle_d)$$

$$\dim(I_n) = d^n - a_n$$

$$\Big\downarrow \text{take dim}$$

$$\cancel{d^n} - a_n \le \cancel{d \cdot (d^{n-1}} - a_{n-1}) + \sum_{m=1}^{n} r_m \cdot a_{n-m}$$

$$0 \le \underbrace{a_n - d\cdot a_{n-1} + \sum_{m=1}^{n} r_m a_{n-m}}$$

$= t^n$ coefficient of $(1 - dt + H_R(t))\cdot H_A(t)$

t_n in $1\cdot H_A(t)$ $\quad a_n$

$-dt\cdot H_A(t)$ $\quad -da_{n-1}$

$H_R(t)\cdot H_A(t)$ $\quad \sum_{i+j=n} r_i\cdot a_j$

for degree 0, $A_0 = K$, dimension 1 □

Cor: Suppose $1 - d\tau + H_R(\tau) \le 0$ for some $\tau\in\mathbb{R}$

$\tau > 0$

- $H_A(\tau)$ diverges

Otherwise $1 \le (\quad)\cdot \overbrace{H_A(\tau)}^{>0} < 0$

moreover, if $\tau \in (0,1)$ and $1 - d\tau + H_R(\tau) < 0$
then A has exponential growth
(in particular, A is infinite dimensional)

pf: Since $H_A(\tau)$ diverges,

τ outside radius of cvg. for H_A

$$\text{r.o.c.} = \frac{1}{\overline{\lim}\left(\sqrt[n]{a_n}\right)} < \tau$$

$$\text{so } \boxed{\overline{\lim}}\, \overline{\sqrt[n]{a_n}} > \frac{1}{\tau} > 1$$

also $\underbrace{a_n \cdot a_m \gtreqless a_{n+m}}_{?}$ don't really care

a_n is not eventually 0.

how to build a G-S alg:

$k = \mathbb{F}_p$ / enumerate $k\langle u \rangle$

we want to add add w^n for each $w \in k\langle u \rangle$
to R

w_0 w_1 w_2

goal: want $\underbrace{1 - \tau d + H_R(\overset{\uparrow}{\tau})} < 0$

Choose n_1 s.t. $\deg(w_0) \cdot n_1 \cdot \tau^{n_1}$ really small

add relator $w_0^{n_1}$

$n_2 \ \cdots n$

how does $|R| > \frac{|D|^2}{4}$ appear?

idea: let U be a minimal set of generators

$R_1 = \emptyset$ or U is not minimal

$H_R(t) > |R| \cdot t^2$

if A fin. dim, $H_A(t)$ converges, so

$$(1 - |D|t + |R|t^2) > 0$$

$t = \frac{|D|^2}{2}$ should give similar