# Representation Theory of $GL_2(\mathbb{F}_q)$

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# Outline

- Review some facts about representations of finite groups
- Find the irreducible representations of many subgroups of $GL_2$.
- Induce these representations to $GL_2$ and then decompose those into irreducibles.
- Find the remaining irreducible representations.

# Representations of Finite Groups

Let $G$ be a finite group.

The number of irreducible representations equals the number of conjugacy classes of $G$. Their dimensions satisfy

$$\sum (\dim \rho_i)^2 = |G|$$

The number of characters (1-dimensional representations) of $G$ is equal to $(G : G')$, where $G'$ is the commutator subgroup of $G$.

Representations of $G$ can be decomposed into a direct sum of irreducibles. In particular, irreducible representations of finite groups have complements.

# Bilinear Form on Representations

Let $\rho$ be a representation of $G$. Then $V_\rho$ can be given the structure of a $\mathbb{C}[G]$-module.

Define a symmetric bilinear (w.r.t. direct sum) form on the representations of $G$ by

$$(\rho, \rho') = \dim \operatorname{Hom}_{\mathbb{C}[G]}(V_\rho, V_{\rho'})$$

By Schur's lemma, if $\rho$ and $\rho'$ are irreducible, then
$(\rho, \rho') = \begin{cases} 1 & \rho = \rho' \\ 0 & \rho \neq \rho' \end{cases}$

Thus, in general, this form is the number of common irreducible subrepresentations, counted with multiplicity, between $\rho$ and $\rho'$.

# Induced Representaion

Let $H < G$ and let $\tau$ be a representation of $H$.

Let $V = \{f : G \to V_\tau \mid f(hg) = \tau(h)f(g) \; \forall \; h \in H, g \in G\}$

Such a function is determined by its values on a set of representatives of $H \backslash G$.

Let $s, g \in G$ and define an action of $G$ on $V$ by $(sf)(g) = f(gs)$. This is the induced representation $\mathrm{Ind}_H^G V_\tau$. Also denoted $\mathrm{Ind}_H^G \tau$.

Transitivity: $\mathrm{Ind}_J^G \mathrm{Ind}_H^J V_\tau = \mathrm{Ind}_H^G V_\tau$

Frobenius Reciprocity: if $\rho$ is a representation of $G$, then $\mathrm{Hom}_{\mathbb{C}[G]}(\mathrm{Ind}_H^G V_\tau, V_\rho) \simeq \mathrm{Hom}_{\mathbb{C}[H]}(V_\tau, \mathrm{Res}_H^G V_\rho)$. In particular, $(\mathrm{Ind}_H^G \tau, \rho) = (\tau, \mathrm{Res}_H^G \rho)$

# Dimension

We can embed $V_\tau$ into $V = \mathrm{Ind}_H^G \tau$. Let $w \in V_\tau$. Define $f_w \in V$ by

$$f_w(g) = \begin{cases} \tau(g)w & g \in H \\ 0 & g \notin H \end{cases}$$

Let $R$ be a set of representatives of $H \backslash G$. Write $G = \cup_{r \in R} rH$ as a decomposition of $G$ into left cosets of $H$.

For $f \in V$, define

$$f_r(g) = \begin{cases} f(g) & g \in Hr^{-1} \\ 0 & g \neq Hr^{-1} \end{cases}$$

Then $f = \sum_{r \in R} f_r$. It follows that $V = \oplus_{r \in R} rV_\tau$

$\dim \mathrm{Ind}_H^G \tau = (G : H) \dim V_\tau$

## $GL_2(k)$

Let $k = \mathbb{F}_q$. We have the following subgroups of $GL_2(k) = G$

- $U = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \right\} \simeq k^+ \quad A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right\} \simeq k^*$
- $B = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \right\}$ Notice $\alpha, \delta \in k^*$ and $\beta \in k$.
- $P = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \right\}$
- $D = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \right\}$
- $Z = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right\}$
- $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

# Conjugacy Classes

There are:

- q-1 classes with repeated eigenvalue $\lambda \in k$. $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$
- q-1 classes with repeated eigenvalue $\lambda \in k$. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$
- $(q-1)(q-2)/2$ classes with distinct eigenvalues $\lambda_1, \lambda_2 \in k$. $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
- $(q^2-q)/2$ classes with distinct eigenvalues $\lambda, \bar{\lambda} \in L \setminus k$, where $L$ is the degree 2 field extension of $k$. $\begin{pmatrix} 0 & -N(\lambda) \\ 1 & Tr(\lambda) \end{pmatrix}$

# Representations of U

$$U = \left\{ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \right\} \simeq k^+ \qquad A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \right\} \simeq k^*$$

$A$ acts on $U$ by conjugation, corresponding to the action of $k^*$ on $k^+$ by multiplication:

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha\beta \\ 0 & 1 \end{pmatrix}$$

Let $\psi$ be a (nontrivial) character of $k^+$. Then $\psi$ is also a character of $U$.

For $a \in A$ and $u \in U$, let $\psi_a(u) = \psi(aua^{-1})$.

These are inequivalent representations (and are equal to the nontrivial representations of $k^+$). Thus, together with the trivial representation, we have $q$ irreducible representations of $U$. Since $|U| = q$, this is all of them.

# Representations of P

$$P = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \right\} = U \rtimes A$$

If $\mu$ is a character of $A$, then $\widetilde{\mu}(ua) := \mu(a)$ is a character of $P$. There are $q-1$ such characters of $P$, which is all of them: $P' = U$ and $(P : U) = q - 1$.

There is 1 more irreducible representation: $\pi = \mathrm{Ind}_U^P \psi$. Note $\dim \pi = q - 1$

$\mathrm{Res}_U^P \mathrm{Ind}_U^P \psi = \oplus \psi_a$. (Recall $\psi = \psi_1$) This is proven by induced representation bashing. But then, by Frobenius reciprocity, we can show $\mathrm{Ind}_U^P \psi$ is irreducible:

$$(\mathrm{Ind}_U^P \psi, \mathrm{Ind}_U^P \psi) = (\psi, \mathrm{Res}_U^P \mathrm{Ind}_U^P \psi) = (\psi, \oplus \psi_a) = 1$$

$\sum \dim \rho^2 = (q-1) + (q-1)^2 = |P|$

# Induced Representation Bashing

Let $a \in A$. Then define $f_a \in \mathrm{Ind}_U^P \psi$ by $f_a(a') = \begin{cases} 1 & a = a' \\ 0 & a \neq a' \end{cases}$

Claim: $f_a$ is an eigenvector of $U$ with eigenvalue $\psi_a$. This means $f_a(pu) = \psi_a(u) f_a(p)$ for all $p \in P$, $u \in U$.

Write $p = u'a'$. Then the condition above becomes

$$f_a(pu) = f_a(u'a'u) = \psi(u') f_a(a'u) = \psi(u')\psi_a(u) f_a(a')$$

Hence, it suffices to show $f_a(a'u) = \psi_a(u) f_a(a')$. We calculate:

$$f_a(a'u) = f_a(a'ua'^{-1}a') = \psi(a'ua'^{-1}) f_a(a') = \psi_{a'}(u) f_a(a')$$

If $a = a'$ we get the equation we wanted. If $a \neq a'$ we get 0.

# More Bashing

We have found a 1-dimensional subspace of $\text{Res}_U^P \text{Ind}_U^P \psi$. If we do this for each $a \in A$, then we get $q - 1$ linearly independent subspaces, and hence $\text{Res}_U^P \text{Ind}_U^P \psi = \oplus \psi_a$

Notice that if $\text{Res}_U^P \text{Ind}_U^P \psi = \oplus \psi_a$, then $\pi$ is independent of the choice of $\psi$.

# Characters of B

$$B = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \right\} = U \rtimes D = Z \rtimes P. \ |B| = q(q-1)^2.$$

Also $B' = U$. Recall $w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

If $\mu_1$ and $\mu_2$ are characters of $k^*$, then $\mu(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}) = \mu_1(\alpha)\mu_2(\delta)$ is a character of $B$. This gives all $(q-1)^2$ characters of $B$.

Remark: If $\mu$ is a character of $B$, then there is another character $\mu_w$ defined by $\mu_w(b) = \mu(wbw^{-1})$.

$$\mu_w(\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}) = \mu_1(\delta)\mu_2(\alpha)$$

Note $\mu = \mu_w$ if and only if $\mu_1 = \mu_2$.

# Other representations of B

Now we use that $B = Z \rtimes P$.

$Z$ has $q-1$ characters $\chi$, given by the characters of $k^*$. A character $\widetilde{\chi}$ can be defined on $B$ by $\widetilde{\chi}(zp) = \chi(z)$

Recall $P$ has the $q-1$ dimensional representation $\pi$. Then $\widetilde{\pi}(zp) = \pi(p)$ is a $q-1$ dimensional representation of $B$.

$(\widetilde{\chi} \otimes \widetilde{\pi})(zp) = \widetilde{\chi}(z)\widetilde{\pi}(p)$ is a $q-1$ dimensional representation of $B$, and is irreducible. Letting $\chi$ vary gives $q-1$ such representations.

$$\sum \dim(\rho)^2 = (q-1)^2 + (q-1)(q-1)^2 = q(q-1)^2 = |B|$$

# Jacquet Module

Now we will induce characters from $B$ to $G$ and identify the irreducible subrepresentations.

There is a great tool called the Jacquet Module for identifying the irreducibles.

Notation: for a character $\mu$ of $B$, let $\hat{\mu} = \mathrm{Ind}_B^G \mu$. $\dim \hat{\mu} = q + 1$.

If $\rho$ is a representation of $G$, let
$J(\rho) = \{\nu \in V_\rho \mid \forall u \in U,\ \rho(u)\nu = \nu\}$

$B$ acts on $J(\rho)$. Indeed, $\rho(u)\rho(b)\nu = \rho(b)\rho(b^{-1}ub)\nu = \rho(b)\nu$.
The last step follows because $U \trianglelefteq B$, hence $b^{-1}ub \in U$.

In general, $J(\rho)$ is not a representation of $G$.

# Dimension of Jacquet Module

Lemma: If $\mu$ is a character of $B$, then $\dim J(\hat{\mu}) = 2$

Proof: recall

$$V_{\hat{\mu}} = \{f : G \to \mathbb{C} \mid f(bg) = \mu(b)f(g)\}$$

Also recall the action is given by $(hf)(g) = f(gh)$ for $g, h \in G$. In particular, the Jacquet module condition is $(uf)(g) = f(gu) = f(g)$ Thus,

$$J(\mu) = \{f : G \to \mathbb{C} \mid f(bg) = \mu(b)f(g), f(gu) = f(g)\}$$

In particular, $f(b) = \mu(b)f(1)$, $f(bwu) = \mu(b)f(w)$

Recall the Bruhat decomposition: $G = B \cup BwU$. Now we see $f$ is determined by $f(1)$ and $f(w)$, hence $J(\mu)$ is 2-dimensional.

# Why the Jacquet Module is Neat

Lemma: Let $\rho$ be a representation of $G$. $J(\rho) \neq 0$ if and only if $(\rho, \hat{\mu}) \neq 0$ for some character $\mu$ of $B$.

Proof idea: $D = B/U$ is abelian and acts on $J(\rho)$, so $J(\rho)$ splits into a sum of characters. Hence, there is a character $\mu$ of $B$ and $\nu \in J(\rho)$ so that, $\forall b \in B$, we have $\rho(b)\nu = \mu(b)\nu$. Thus, $(\mathrm{Res}_B^G \rho, \mu) \neq 0$, hence by Frobenius reciprocity $(\rho, \hat{\mu}) \neq 0$.

The converse is similar. Assume $(\rho, \hat{\mu}) \neq 0$, so there exists $\nu \in V_\rho$ so that $\rho(b)\nu = \mu(b)\nu$. Since $\mu$ is trivial on $U$, we moreover have $\nu \in J(\rho)$, which is thus nonzero.

Corollary: If $\rho$ is an irreducible representation of $G$, then $J(\rho) \neq 0$ if and only if $\rho \leq \hat{\mu}$ for some character $\mu$ of $B$.

Corollary: $\hat{\mu}$ has at most 2 irreducible components.

# Remarks

Recall that a character $\mu$ of $B$ is given by 2 characters $\mu_1, \mu_2$ of $k^*$. If $\mu_1 = \mu_2$, then $\hat{\mu}$ has a 1-dimensional component. The proof idea is that the representation only depends on $\det(b)$.

Notice that $\mu_1 = \mu_2$ is equivalent to $\mu = \mu_w$.

$\hat{\mu}$ has at most one 1-dimensional component, since $\dim(\hat{\mu}) = q + 1$

$\text{Res}_P \hat{\mu} = \text{Res}_P J(\hat{\mu}) \oplus V_\pi$. The proof idea is that since $B$ acts on $J(\hat{\mu})$, so does $P$. Then $J(\hat{\mu})$ has a complement, which is $q - 1$ dimensional. One can show the complement has no 1-dimensional components, hence it must be $\pi$.

# More remarks

Lemma: If $\hat{\mu}$ is reducible, then $\hat{\mu}$ has a 1-dimensional component and $\mu = \mu_w$.

Proof idea: write $V_{\hat{\mu}} = V_\tau \oplus V_\omega$. Then since $V_\pi \subset \text{Res}_P \hat{\mu}$, without loss of generality, $V_\pi \subset \text{Res}_P V_\tau$. But since $V_\tau \subset V_{\hat{\mu}}$, we know $0 \neq J(\tau) \subset J(\hat{\mu}) \cap V_\tau$. In particular, $V_\pi \neq \text{Res}_P V_\tau$. Since $\dim V_\pi = q - 1$ and $\dim V_{\hat{\mu}} = q + 1$, we have that $\dim V_\tau = q$ and $\dim V_\omega = 1$.

Lemma: If $\mu$ and $\mu'$ are distinct characters of $B$, then $(\hat{\mu}, \hat{\mu'}) \neq 0$ if and only if $\mu_w = \mu'$.

Corollary: $\hat{\mu} = \hat{\mu'}$ if and only if $\mu = \mu'$ or $\mu_w = \mu'$.

# Putting it together

If $\mu$ and $\mu'$ are characters of $B$, then

- $\dim \hat{\mu} = q + 1$
- $\hat{\mu}$ has at most 2 irreducible components.
- if $\mu \neq \mu_w$, then $\hat{\mu}$ is irreducible.
- If $\mu = \mu_w$, then $\hat{\mu}$ is reducible, and is the sum of a character and a $q$-dimensional representation.
- $\hat{\mu} = \hat{\mu'}$ if and only if $\mu = \mu'$ or $\mu_w = \mu'$.

This gives the following irreducible representations of $G$:

- $q - 1$ characters of $G$, as the 1-dimensional components of $\hat{\mu}$ when $\mu = \mu_w$.
- $q - 1$ representations that are $q$-dimensional.
- $(q-1)(q-2)/2$ representations that are $q + 1$ dimensional.

# Cuspidal Representations

Comparing the irreducible representations that we have found with the conjugacy classes of $G$, we are missing $(q^2 - q)/2$ irreducible representations.

An irreducible representation of $G$ that is not a component of $\hat{\mu}$ for any character $\mu$ of $B$ is called cuspidal.

Suppose $\rho$ is cuspidal. Then $J(\rho) = 0$. Further, $\mathrm{Res}_P(\rho)$ cannot have any 1-dimensional components, else they would make $J(\rho) \neq 0$. Hence, $\mathrm{Res}_P(\rho) = n\pi$ for some $n$, and $\dim \mathrm{Res}_P(\rho) = n(q-1)$.

# Dimension

Lemma: If $\rho$ is cuspidal, then $\dim \rho = q - 1$. And if $\mathrm{Res}_P \rho = \pi$, then $\rho$ is irreducible and cuspidal.

Proof: counting dimensions, $|G| = q(q+1)(q-1)^2 = (q-1) + (q-1)q^2 + \frac{1}{2}(q-1)(q-2)(q+1)^2 + \sum_\rho \dim(\rho)^2$. This equation is satisfied only if $\dim \rho = q - 1$ for each cuspidal $\rho$.

Because $\pi$ is irreducible, so is $\rho$. Since the components of $\hat{\mu}$ have dimension 1, $q$, or $q+1$, but $\dim \rho = q + 1$, it follows that $\rho$ is not a component of $\hat{\mu}$, and is hence cuspidal.

# Norms and Traces

To exhibit cuspidal representations, we will need a character of $L$, the degree 2 field extension of $k$.

If $\alpha \in L$, define $N\alpha = \alpha\bar{\alpha}$ and $Tr\alpha = \alpha + \bar{\alpha}$. Both are surjective.

If $\chi$ is a character of $k^*$, then $\widetilde{\chi}(\alpha) = \chi(N\alpha)$ is a character of $L$. Such characters are called decomposable.

If $\nu$ is a non-decomposable character of $L$, then $\sum_{N\alpha=x} \nu(\alpha) = 0$

# Generating G

$$w' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, z = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, s = w'z$$

$G$ is generated as the free group on $B$ and $w'$ with relations

$$w' \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} w'^{-1} = \begin{pmatrix} \delta & 0 \\ 0 & \alpha \end{pmatrix}$$

$$w'^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$s^3 = 1$$

# Explicit Cuspidal Representation

Let $\nu$ be a non-decomposable character of $L$ and let $\psi$ be a character of $k^+$. Let $V = \{f : k^* \to \mathbb{C}\}$

The idea is to define a representation $\rho$ on $B$ and $w'$ such that it respects the necessary relations, and so that the restriction to $P$ is $\pi$.

$$\left(\rho\begin{pmatrix}\alpha & \beta \\ 0 & \delta\end{pmatrix} f\right)(x) = \nu(\delta)\psi(\beta\delta^{-1}x)f(\alpha\delta^{-1}x)$$

Let $j : k^* \to \mathbb{C}$ by $j(u) = \frac{1}{q}\sum_{Nt=u}\psi(t+\bar{t})\nu(t)$. Then define

$$(\rho(w')f)(x) = \sum_{y\in k^*}\nu(y^{-1})j(xy)f(y)$$

# Remarks

Showing this representation as defined is indeed a representation, and that its restriction to $P$ is $\pi$, is a lot of unenlightening computations.

However, there is this cool table.

| elements of $L^\times$ | conjugacy classes | characters of $L^\times, K^\times$ | irr. repr. of $G$ | dim of repr. | no. of elements |
|---|---|---|---|---|---|
| $\alpha \in K^\times$ | $c_1(\alpha)$ | $\mu_1 \in X(K^\times)$ | $\rho'_{(\mu_1,\mu_1)}$ | 1 | $q-1$ |
| | $c_2(\alpha)$ | | $\rho_{(\mu_1,\mu_1)}$ | $q$ | $q-1$ |
| $\alpha, \beta \in K^\times$, $\alpha \neq \beta$ | $c_3(\alpha,\beta)$ | $\mu_1, \mu_2 \in X(K^\times)$, $\mu_1 \neq \mu_2$ | $\rho_{(\mu_1,\mu_1)}$ | $q+1$ | $\frac{1}{2}(q-1)(q-2)$ |
| $\lambda \in L^\times - K^\times$ | $c_4(\lambda)$ | $\nu \in X(L^\times) - X(K^\times)$ | $\rho_\nu$ | $q-1$ | $\frac{1}{2}(q^2-q)$ |

# References

Piatetski-Shapiro, Ilya
Complex Representations of GL(2,K) for Finite Fields K