GARTS talk: Algebras and their modules in monoidal categories.

Monday, 1 November 2021 3:00 PM

Outline of today's talk:

1. Monoidal categories 2. Associative unital algebras 3. A-modules.

Philosophy 1. Write element-theoretic stuff in terms of morphisms.

Philosophy 2. An A-module is an object that A acts on like how A acts on itself.

1. Monoidal cats

Recall the definition of a monoid.

Def^{n}: A <u>monoid</u> $(M, \cdot, 1)$ consists of the data:

- a set M
- a function $\cdot : M \times M \to M$ call multiplication/product/operation.
- an element $1 \in M$ called the unit element

satisfying the conditions:

- (associativity)

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \text{for all } x, y, z \in M.$$

i.e. the functions

$$- \cdot (- \cdot -) : M \times M \times M \to M$$

and

$$(- \cdot -) \cdot - : M \times M \times M \to M$$

are equal.

- (unit property)

$$1 \cdot x = x = x \cdot 1 \quad \text{for all } x \in M.$$

i.e. the functions

$$1 \cdot - : M \to M, \quad id_M : M \to M \quad \text{and} \quad - \cdot 1 : M \to M$$

are equal.

Recall these three notions:

1. <u>Categories</u>: $\mathcal{C} = (ob(\mathcal{C}), hom(\mathcal{C}), \circ)$

DATA

$$\begin{tikzcd} X \arrow[rr, "g \circ f"] \arrow[dr, "f"'] & & Y \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$

CONDITIONS

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] \arrow[rrr, bend left=60, "h\circ(g\circ f)"] \arrow[rrr, bend right=60, "(h\circ g)\circ f"'] & X \arrow[r, "g"] \arrow[rr, bend right, "h\circ g"'] & Y \arrow[r, "h"] & Z \end{tikzcd}
\qquad
\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "f"] & Y \arrow[r, "id_Y"] \arrow[rr, bend right, "g"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

2. Functors $F : \mathcal{C} \to \mathcal{D}$

$\mathcal{C}$ $\qquad\qquad\qquad\qquad$ $\mathcal{D}$

DATA

X $\qquad\qquad$ $F(X)$

$X \xrightarrow{f} Y$ $\qquad\qquad$ $F(X) \xrightarrow{F(f)} F(Y)$

$\xrightarrow{F}$

$$\begin{tikzcd} X \arrow[rr, "g\circ f"] \arrow[dr, "f"'] & & Y \\ & Z \arrow[ur, "g"'] & \end{tikzcd}
\qquad
\begin{tikzcd} F(X) \arrow[rr, "F(g\circ f)"] \arrow[dr, "F(f)"'] & & F(Y) \\ & F(Z) \arrow[ur, "F(g)"'] & \end{tikzcd}$$

CONDITIONS

$$\begin{tikzcd} X \arrow[loop above, "id_X"] \end{tikzcd}
\qquad
\begin{tikzcd} F(X) \arrow[loop above, "id_{F(X)}"] \end{tikzcd}$$

3. Natural transformations: $\eta : F \Rightarrow G$.

$\mathcal{C}$ $\qquad\qquad\qquad\qquad$ $\mathcal{D}$

DATA

X

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\eta\Downarrow"] & \mathcal{D} \end{tikzcd}$$

$F(X) \xrightarrow{\eta_X} F(Y)$

$F(X) \xrightarrow{F(f)} F(Y)$

$$X \xrightarrow{f} Y \qquad \xrightarrow[G]{} $$

CONDITIONS.

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\
G(X) \arrow[r, "G(f)"'] & G(Y)
\end{tikzcd}$$

Def$^{\underline{n}}$: A <u>natural</u> <u>isomorphism</u> is a natural transformation $\eta : F \to G$ where η_X is an isomorphism in $\mathcal{D}$ for each object in $\mathcal{C}$.

IDEA: F and G are the "same" functor.

We can categorify the definition of a monoid by upgrading:

sets	categories
elements	objects
functions	functors
CONDITION → equality of functions	natural isomorphisms ← DATA.
	coherence ← CONDITION.

Def$^{\underline{n}}$: A <u>monoidal</u> <u>category</u> $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$ consists of the data:

- a category $\mathcal{C}$
- a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product
- an object $\mathbb{1} \in ob(\mathcal{C})$ called the unit object
- a natural isomorphism
$$\alpha : - \otimes (- \otimes -) \Rightarrow (- \otimes -) \otimes -$$
called the associator
- two natural isomorphisms
$$l : (\mathbb{1} \otimes -) \Rightarrow id_{\mathcal{C}} \quad \text{and} \quad r : (- \otimes \mathbb{1}) \Rightarrow id_{\mathcal{C}}$$
called the left and right unitors.

satisfying the conditions:

- (pentagon)

$$\xrightarrow{\alpha_{W,X,Y\otimes Z}} (W \otimes X) \otimes (Y \otimes Z) \xrightarrow{\alpha_{W\otimes X,Y,Z}}$$

$$\begin{tikzcd}[column sep=small]
& (W\otimes X)\otimes(Y\otimes Z) \arrow[dr, "\alpha_{W\otimes X,Y,Z}"] & \\
W\otimes(X\otimes(Y\otimes Z)) \arrow[ur, "\alpha_{W,X,Y\otimes Z}"] \arrow[d, "\mathrm{id}_W\otimes\alpha_{X,Y,Z}"'] & & ((W\otimes X)\otimes Y)\otimes Z \\
W\otimes((X\otimes Y)\otimes Z) \arrow[rr, "\alpha_{W,X\otimes Y,Z}"'] & & (W\otimes(X\otimes Y))\otimes Z \arrow[u, "\alpha_{W,X,Y}\otimes\mathrm{id}_Z"']
\end{tikzcd}$$

- (triangle)

$$\begin{tikzcd}
X\otimes(\mathbb{1}\otimes Y) \arrow[rr, "\alpha_{X,\mathbb{1},Y}"] \arrow[dr, "\mathrm{id}_X\otimes l_Y"'] & & (X\otimes\mathbb{1})\otimes Y \arrow[dl, "r_X\otimes\mathrm{id}_Y"] \\
& X\otimes Y &
\end{tikzcd}$$

Fact: The pentagon and triangle identities are sufficient for coherence.

IDEA: any way to relate two objects using ONLY the structure isomorphisms $\alpha, l, r, \alpha^{-1}, l^{-1}, r^{-1}$ are equal. Hence,

$$X_1\otimes\cdots\otimes X_n$$

is canonically isomorphic to

$$((X_1\otimes X_2)\otimes(X_3\otimes(\cdots\mathbb{1}))\otimes\cdots)\otimes(X_{n-1}\otimes(\mathbb{1}\otimes X_n))$$

and

$$X_1\otimes(X_2\otimes(X_3\otimes(\cdots X_{n-2}\otimes(X_{n-1}\otimes X_n)\cdots)))$$

etc.

Examples:

- The best example to keep in your mind for this talk is

$$(\mathbb{k}\text{-Vect}, \otimes, \mathbb{1}, \alpha, l, r).$$

$x\otimes 1 \mapsto x$.

$1\otimes x \mapsto x$

Category of $\mathbb{k}$-vector spaces, $\mathbb{k}$-linear maps and their compositions.

$\mathbb{k}$

$x\otimes(y\otimes z)\mapsto(x\otimes y)\otimes z$

tensor product $\otimes_{\mathbb{k}}$

This also has compatible abelian structure (finite biproducts, zero object, kernels, cokernels etc.)

- another familiar example is
$$(\mathrm{Set}, \times, \{*\}, (x,(y,z)) \mapsto ((x,y),z), (*,x) \mapsto x, (x,*) \mapsto x)$$

- a Hopf algebra H has the purpose in life to supply the data to construct the monoidal category
$$(H\text{-}\mathrm{Mod}_{fd}, \otimes, \mathbb{1}, \alpha, l, r)$$
with additional $\mathbb{k}$-linear tensor structure (including duals).
- a vertex operator algebra $(V, Y, |0\rangle, \omega)$ gives
$$(V\text{-}\mathrm{Mod}, \boxtimes, \mathbb{1}, \underbrace{\alpha, l, r}_{})$$

fusion product   vacuum module V defined analytically.

2. ASSOCIATIVE UNITAL ALGEBRAS.

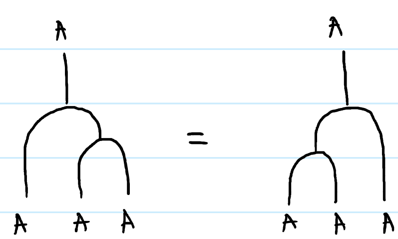
Let $\mathcal{C}$ be a monoidal category.

Def^n: An (<u>associative</u> <u>unital</u>) <u>algebra</u> (<u>object</u>) (A, ∇, η) in $\mathcal{C}$ consists of the data:

- an object A in $\mathcal{C}$
- a morphism  $= \nabla : A \otimes A \to A$ in $\mathcal{C}$, called <u>multiplication</u>
- a morphism  $= \eta : \mathbb{1} \to A$ in $\mathcal{C}$, called the <u>unit</u>

satisfying the conditions

- (associativity)

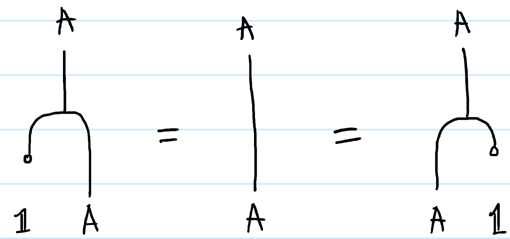


i.e.

$$\begin{tikzcd} & A & \\ A\otimes A \arrow[ur, "\nabla"] & & A\otimes A \arrow[ul, "\nabla"'] \end{tikzcd}$$

$$
\begin{tikzcd}
A\otimes A & & A\otimes A \\
A\otimes(A\otimes A) \arrow[u, "\mathrm{id}_A\otimes\nabla"] \arrow[rr, "\alpha_{A,A,A}"'] & & (A\otimes A)\otimes A \arrow[u, "\nabla\otimes\mathrm{id}_A"']
\end{tikzcd}
$$

∘ (unit identity)



i.e.

$$
\begin{tikzcd}
 & A & \\
A\otimes A \arrow[ur, "\nabla"] & & A\otimes A \arrow[ul, "\nabla"'] \\
\mathbb{1}\otimes A \arrow[u, "\eta\otimes A"] \arrow[dr, "l_A"'] & & A\otimes\mathbb{1} \arrow[u, "\mathrm{id}_A\otimes\eta"'] \arrow[dl, "r_A"] \\
 & A \arrow[uuu, "\mathrm{id}_A"] &
\end{tikzcd}
$$

Claim: This recovers the usual notion of an algebra when $\mathcal{C} = \Bbbk\text{-Vect}$.

Pf sketch: Given an algebra object (A, ∇, η) in $\Bbbk$-Vect, define

- the multiplication $\cdot : A \times A \to A$ to be the $\Bbbk$-bilinear map corresponding to $\nabla : A\otimes A \to A$,
- the unit element $1 = \eta(1_{\Bbbk}) \in A$.

Then

$$
\begin{tikzcd}[column sep=small]
 & & x\cdot(y\cdot z) = (x\cdot y)\cdot z & & \\
x\otimes(y\cdot z) \arrow[urr, mapsto] & & A & & (x\cdot y)\otimes z \arrow[ull, mapsto] \\
 & A\otimes A \arrow[ur, "\nabla"] & & A\otimes A \arrow[ul, "\nabla"'] & \\
 & A\otimes(A\otimes A) \arrow[u, "\mathrm{id}_A\otimes\nabla"] \arrow[rr, "\alpha_{A,A,A}"'] & & (A\otimes A)\otimes A \arrow[u, "\nabla\otimes\mathrm{id}_A"'] & \\
x\otimes(y\otimes z) \arrow[uuu, mapsto] \arrow[rrrr, mapsto] & & & & (x\otimes y)\otimes z \arrow[uuu, mapsto]
\end{tikzcd}
$$

and

$$\begin{tikzcd}
A\otimes A \arrow[r, "\nabla"] & A & A\otimes A \arrow[l, "\nabla"'] \\
\mathbb{1}\otimes A \arrow[u, "\eta\otimes A"] \arrow[r, "l_A"'] & A \arrow[u, "\mathrm{id}_A"] & A\otimes \mathbb{1} \arrow[u, "\mathrm{id}_A\otimes\eta"'] \arrow[l, "r_A"]
\end{tikzcd}$$

$$\begin{tikzcd}
1\otimes x \arrow[r, mapsto] & 1\cdot x = x = x\cdot 1 & x\otimes 1 \arrow[l, mapsto] \\
1_{\mathbb{k}}\otimes x \arrow[u, mapsto] \arrow[r, mapsto] & x \arrow[u, mapsto] & x\otimes 1_{\mathbb{k}} \arrow[u, mapsto] \arrow[l, mapsto]
\end{tikzcd}$$

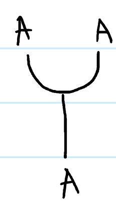
So $(A, \cdot, 1)$ is an associative unital algebra.

There is an inverse construction. □

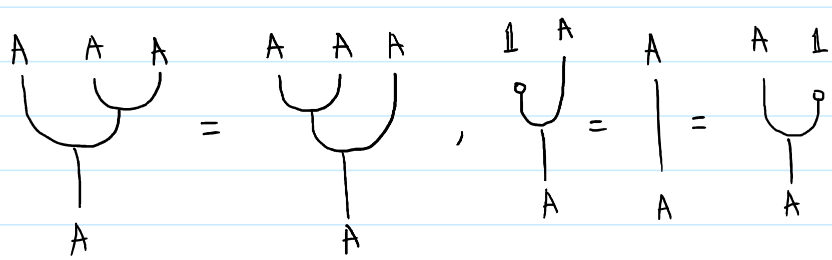
Philosophy 1. Write element-theoretic stuff in terms of morphisms.

— Our philosophy of writing set-theoretic notions/properties in terms of morphisms allows us to abstract to the general categorical settings.

— Fun with pictures. Q: What should a coalgebra in $\mathcal{C}$ be?
A: Reflect all pictures in the horizontal to get

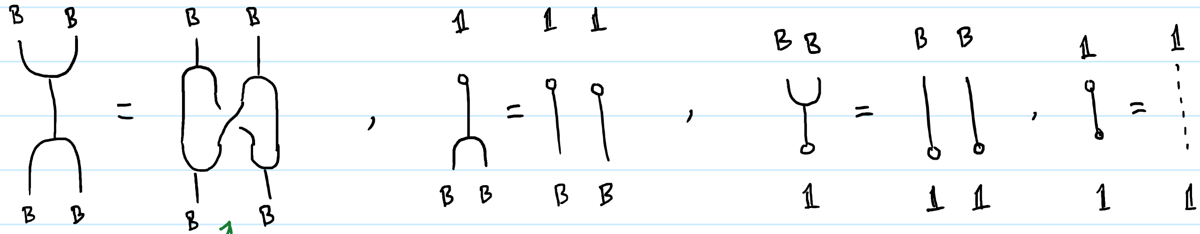
DATA: $C \in ob(\mathcal{C})$, $\Delta =$  , $\varepsilon =$ 

CONDITIONS:

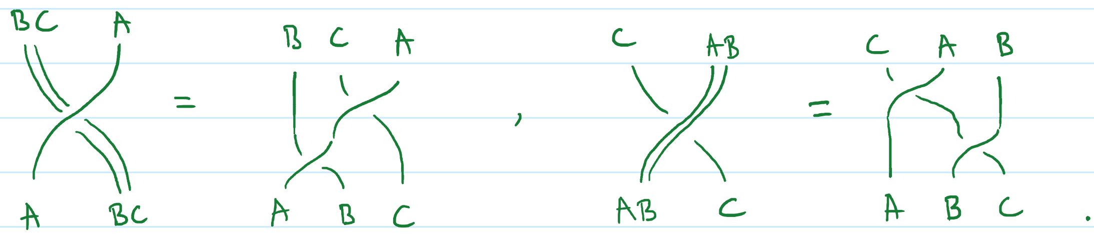


Can also define a bialgebra $(B, \nabla, \eta, \Delta, \varepsilon)$ to be a object B with an algebra structure (B, ∇, η) and a coalgebra structure

(B, Δ, ε) that are compatible in the following sense

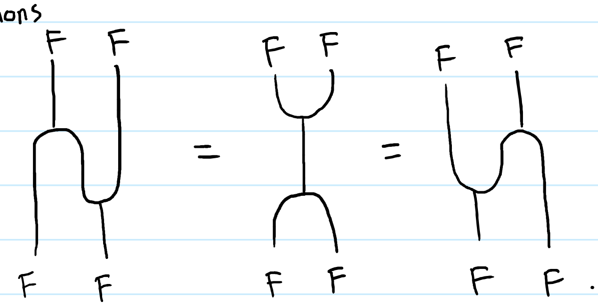


braiding on $\mathcal{C}$ satisfies

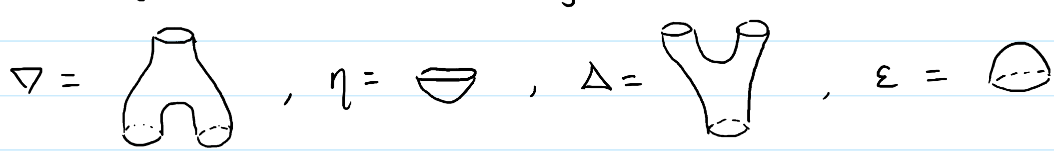


Applications: Hopf algebras, quantum groups.

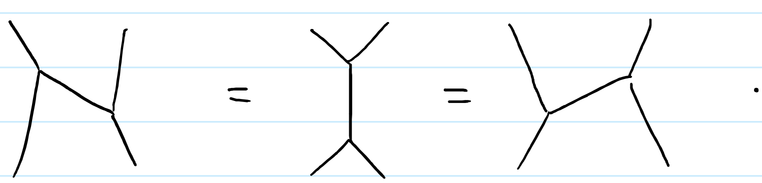
Can also define <u>Frobenius algebras</u> $(F, \nabla, \eta, \Delta, \varepsilon)$ using different compatibility conditions



Spooky observations : Drawing



gives all the Frobenius algebra conditions and looks like s-t-channel duality



If we construct an algebra in $\mathcal{C}$, one is then tempted to look for representations in $\mathcal{C}$.

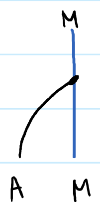
3. A-MODULES.

Philosophy 2. An A-module is an object that A acts on like how A acts on itself.

Keep $\mathcal{C}$ as a monoidal category and $(A, \nabla, \eta, \Delta, \varepsilon)$ an algebra in $\mathcal{C}$.

Def^{n}: A <u>left</u> <u>A-module</u> (M, ρ_M) consists of the data:

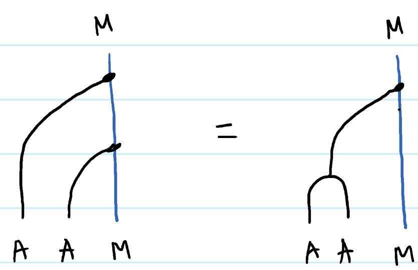
- an object M in $\mathcal{C}$,
- a morphism



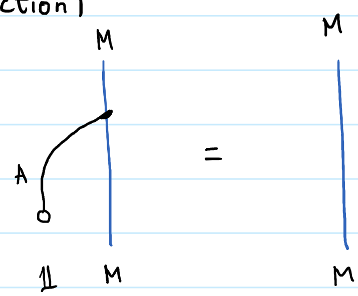
$= \rho_M : A \otimes M \longrightarrow M$

satisfying the conditions

- (associative left action)



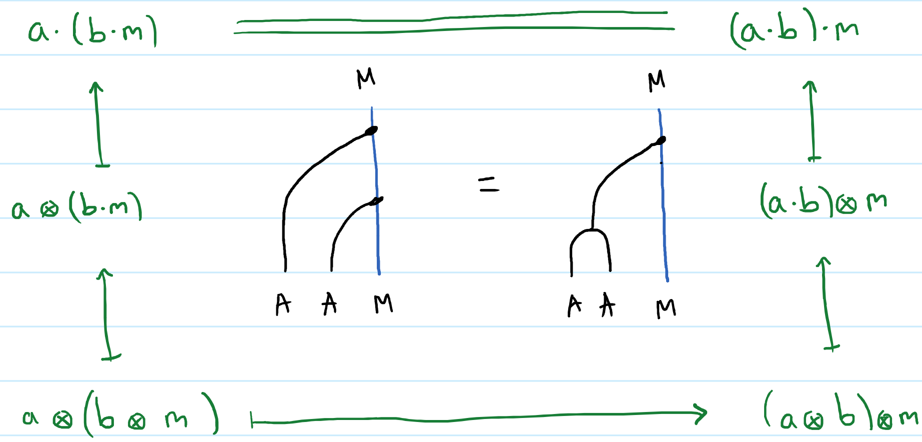
- (unit left action)

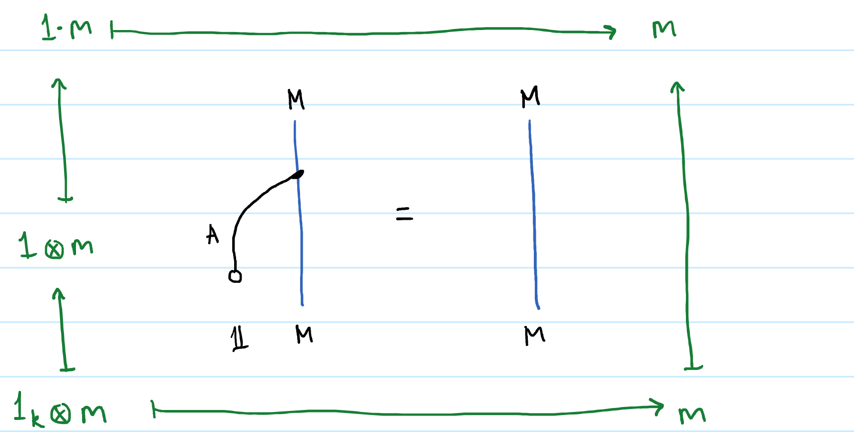


Visually, this is how we "should" define a left A-module because follows Philosophy 2. It also agrees with Philosophy 1. Write element-theoretic stuff in terms of morphisms.

Claim: When $\mathcal{C} = \Bbbk\text{-Vect}$ we recover the usual notion of an A-representation.

Pf sketch: Chase the diagrams.



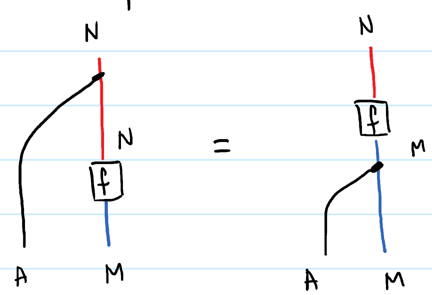


This recovers the notion of an A-module $\rho_M : A \otimes M \to M$.
Then we get a representation (associative unital algebra homomorphism)

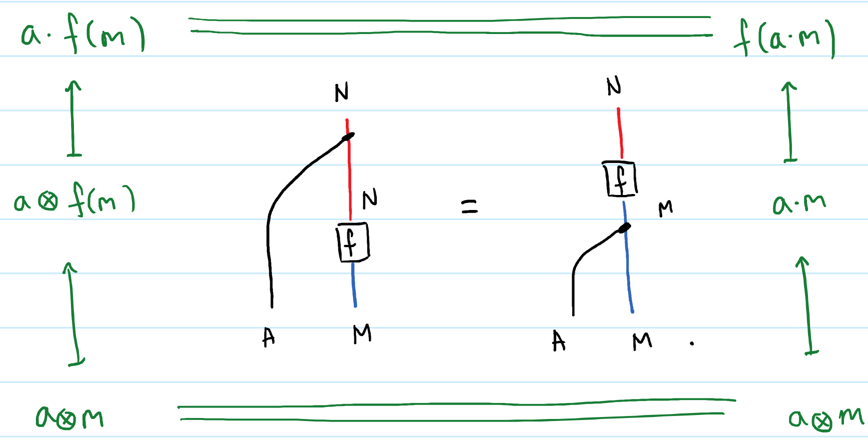
$$\pi_M : A \longrightarrow \mathrm{End}(M).$$

$$\pi_M(a) \circ \pi_M(b) = \pi_M(a \cdot b), \qquad \pi_M(1) = \mathrm{id}_M. \qquad \square$$

Def^{n}: An A-module homomorphism $f : (M, \rho_M) \to (N, \rho_N)$ is a morphism f in $\mathcal{C}$ that preserves the A-module structure i.e.



This is what you expect in $\Bbbk\text{-Vect}$ since



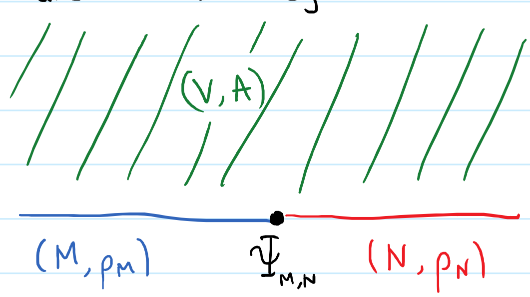
Def^{n} / Prop: The collection of A-modules, A-module homomorphisms and composition inherited from $\mathcal{C}$ defines a category

A - Mod.

Applications: Boundary conformal field theory using the Fuchs - Runkel - Schweigert construction:

- Start with a strongly rational vertex operator algebra V.
- Then V-Mod is a modular tensor category as proved by Huang.
- Then we can construct algebras using the monoidal structure in V-Mod.
- Require symmetric-special-Frobenius structure on the algebra A.

Then (V, A) describes a full CFT in the bulk theory. The boundary theories are described by A-modules



The physical theories can motivate equivalences between monoidal categories with extremely different constructions e.g.

$$\widehat{su(2)}_{d-2}\text{-Mod} \boxtimes \widehat{u(1)}_{2d}\text{-Mod}$$ ← VOAs

is tensor equivalent to a tensor category constructed from

$$(\mathbb{C}[x,y],\ W = x^d - y^d)$$ ← Matrix factorisations.