From number theory to automorphic forms, and then to automorphic distributions

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December 13, 2021

Outline of the talk

Sums of squares

- $r_k(n)$=the number of ways of writing n as a sum of k squares.
- $r_2(8) = 4$ because $8 = (\pm 2)^2 + (\pm 2)^2$.
- Many well known mathematicians of the nineteenth century including Gauss, Jacobi, Eisenstein, Liouville worked on finding formula for $r_k(n)$ for small values of k. Gauss found one for $k = 3$

Sums of squares

- Jacobi proved that

$$r_2(n) = 4(d_1(n) - d_3(n))$$

where $d_i(n)$ is the number of divisors of n congruent to $i \pmod 4$, and

$$r_4(n) = \sum_{\substack{d|n \\ d \not\equiv 0 \pmod 4}} d.$$

Sums of squares

- To prove this, Jacobi used the generating function (now called Jacobi's theta function)

$$\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots$$

- θ can be defined as a holomorphic function on the upper half plane $\mathbb{H}$ by setting $q = e^{2\pi i z}$.
- Then

$$\theta^2(z) = \sum_{m,n} q^{m^2+n^2} = \sum_{n=0}^{\infty} r_2(n) q^n$$

and

$$\theta^4(z) = \sum_{n=0}^{\infty} r_4(n) q^n$$

Sums of squares

- $\theta(z)$ is an example of a *modular form*.
- Definition: Modular forms of weight k for $\mathrm{SL}(2,\mathbb{Z})$ are holomorphic functions on the upper half plane such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

  for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z})$ which is holomorphic at $i\infty$.
- If f vanishes at $i\infty$ then we call f a *cusp form*.

Sums of squares

- Since $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate $\mathrm{SL}(2, \mathbb{Z})$, you only need to check the modularity condition for just the two matrices:
- $f(z+1) = f(z)$, $f(-1/z) = z^k f(z)$.
- This observation allows us to construct a modular form:

$$G_k(z) = \sum_{(m,n)\in\mathbb{Z}^2-(0,0)} \frac{1}{(mz+n)^k}$$

  defined for $k > 2$.
- This series is called the *Eisenstein series*.

Sums of squares

- The Eisenstein series $G_k(z)$ has Fourier expansion:

$$G_k(z) = 2\zeta(k) + \frac{2(-2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \sigma_{k-1}(n) q^n.$$

- You see the divisor sum function comes up.

Sums of squares

- How can we prove Jacobi's two-square and four-square theorem?
- The function $\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$, $q = e^{2\pi i z}$ satisfies the two functional equations

$$\theta(z) = \theta(z+1), \quad \theta\left(\frac{-1}{4z}\right) = \sqrt{\frac{2z}{i}}\theta(z).$$

- $\theta(z)$ is a modular form of weight $1/2$ for the congruence subgroup

$$\Gamma_0(4) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{Z}) : c \equiv 0 \pmod 4 \right\}$$

Sums of squares

- $\theta^2(z)$ is a modular form of weight 1 and $\theta^4(z)$ is a modular form of weight 2.
- The space of modular forms of weight k is finite-dimensional and they are generated by Eisenstein series. For example,

$$\theta^4(z) = 8(\mathbb{G}_2(z) - 2\mathbb{G}_2(2z)) + 16(\mathbb{G}_2(2z) - 2\mathbb{G}_2(4z))$$

  where $\mathbb{G}_2(z) = -G_2(z)/(4\pi^2)$.
- By comparing the coefficient of q^n we prove Jacobi's theorems.

Sums of squares

- In 1916, Ramanujan found formulas for r_{20}, r_{22}, r_{24}. For r_{24}, the formula is

$$r_{24}(n) = \left(\frac{16}{691}\sigma_{11}(n) - \frac{32}{691}\sigma_{11}(n/2)\right) + \left(\frac{33152}{691}(-1)^{n-1}\tau(n) - \frac{66536}{691}\tau(n/2)\right)$$

where

$$\Delta(z) = \sum_{n\geq 1}\tau(n)q^n = q\prod_{k\geq 1}(1-q^n)^{24}.$$

Sums of squares

- Ramanujan studied the modular discriminant $\Delta(z) = \sum_{n\geq 1} \tau(n)q^n$ and made the following observations/conjectures:
  1. If m and n are coprime then $\tau(mn) = \tau(m)\tau(n)$
  2. If p is prime then $\tau(p^{\alpha+2}) = \tau(p)\tau(p^{\alpha+1}) - p^{11}\tau(p^{\alpha})$
  3. If p is prime then $\left|\tau(p)\right| \leq 2p^{11/2}$

Sums of squares

- (1) and (2) were proved by Mordell in 1917 and Deligne proved (3) in 1974.
- (1) and (2) are resolved by Hecke operators.
- (2) leads to the Euler product of the L-function

$$L(s, \Delta) = \sum_{n \geq 1} \frac{\tau(n)}{n^s} = \prod_{n \geq 1} \frac{1}{1 - \tau(p)p^{-s} + p^{11-2s}}.$$

Sums of squares

- Jacobi's θ is also used to derive a functional equation of Riemann's ζ-function.
- The modular discriminant $\Delta(z)$ is related to elliptic curves.
- The theory of automorphic forms is an inevitable component of number theory.

Outline of the talk

Representation-theoretic approach

- The upper half plane is isomorphic to $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R})$.
- So the modular forms are functions on

$$\mathrm{SL}(2,\mathbb{Z})\backslash\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2,\mathbb{R}).$$

- Gelfand-Graev lift: lifting from $\mathbb{H}$ to $G = \mathrm{SL}(2,\mathbb{R})$

$$F\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = (ci+d)^{-k} f\left(\frac{ai+b}{ci+d}\right).$$

- $F(\gamma g) = F(g)$ for all $\gamma \in \Gamma = \mathrm{SL}(2,\mathbb{Z})$.

Representation-theoretic approach

- This gives a perspective change from $\mathbb{H}$ to a Lie group $G = \mathrm{SL}(2, \mathbb{R})$.
- The group $G = \mathrm{SL}(2, \mathbb{R})$ acts on the space of functions $L^2(\Gamma \backslash G)$ via right regular representation:

$$\left[\varphi(g) f\right](x) = f(xg).$$

- This is an example of an *automorphic representation*.

Outline of the talk

Automorphic distributions

- G=some real Lie group, $\Gamma \subset G$ a discrete subgroup.
- Automorphic forms are smooth functions on G that are Γ-invariant.
- Idea: Consider instead the space of *distributions*, which is the dual space of the space of smooth functions. Its Γ-invariant subspace is the space of *automorphic distributions* for Γ.

Distributions on $\mathbb{R}$

- There are two ways to think of distributions.
- Example: the Dirac δ-function

$$\delta_0(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{otherwise} \end{cases}$$

with the property

$$\int_{-\infty}^{\infty} \delta_0(x)\, dx = 1.$$

- Of course, this is not really a function.

Distributions on $\mathbb{R}$

- One way to define distributions is that they are continuous linear functionals on a space of functions of compact support.

$$< \delta_0, \phi >= \phi(0).$$

- Another way is that they are sums of 'higher order derivatives' of continuous functions, even if the functions are not really differentiable.

$$\delta_0^{(-2)}(x) = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{otherwise} \end{cases}$$

Distributions on $\mathbb{R}$

- We can think of the derivative as the formal rule for integration by parts. For $\phi(x) \in C_c^\infty(\mathbb{R})$, we have

$$\int_{-\infty}^{\infty} \delta_0(x)\phi(x)\,dx = \int_{-\infty}^{\infty} \delta_0^{(-2)}\phi''(x)\,dx = \int_0^\infty x\phi''(x)\,dx$$
$$= \left[x\phi'(x)\right]_0^\infty - \int_0^\infty \phi'(x)\,dx$$
$$= \phi(0)$$

- The two definitions are equivalent. I will mainly use the second definition- distributions act on smooth functions by integrating against them.

Automorphic distributions on $\mathbb{R}$

- $G = \mathrm{SL}(2,\mathbb{R})$, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, B_- is the subgroup of invertible lower triangular matrices, N is the subgroup of unipotent upper triangular matrices.
- Let $\lambda \in \mathbb{C}$, $\delta \in \{\pm 1\}$ be two parameters.
- The principal series representation of $G = \mathrm{SL}(2,\mathbb{R})$ is the representation $(\pi_{\lambda,\delta}, V_{\lambda,\delta})$, where

$$V_{\lambda,\delta} = \left\{F : G \to \mathbb{C} \mid F\left(g\left(\begin{smallmatrix} a & 0 \\ * & a^{-1}\end{smallmatrix}\right)\right) = |a|^{1-2\lambda}\,\mathrm{sgn}(a)^{\delta} F(g)\right\}$$

and $\left[\pi_{\lambda,\delta}(h)f\right](g) = f(h^{-1}g)$.

Automorphic distributions on $\mathbb{R}$

- Most of the matrices in $G = \mathrm{SL}(2,\mathbb{R})$ can be written as a product of upper triangular times lower triangular matrices.
- If $f \in V_{\lambda,\delta}$, then

$$f\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right) = f\left(\left(\begin{smallmatrix} 1 & b/d \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1/d & 0 \\ c & d \end{smallmatrix}\right)\right)$$
$$= |d|^{2\lambda-1}\,\mathrm{sgn}\,(d)^{\delta} f\left(\left(\begin{smallmatrix} 1 & b/d \\ 0 & 1 \end{smallmatrix}\right)\right)$$

- Thus if $d \neq 0$, f is completely determined by its restriction to $N \equiv \mathbb{R}$.

Automorphic distributions on $\mathbb{R}$

- Restricting functions $f \in V_{\lambda,\delta}$ to $\mathbb{R}$ by $h(x) = f\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, we get the *line model* of a principal series representation.
- For simplicity we fix $\delta = 0$. The G-action in the line model is

$$\pi\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^{-1}\right) h(x) = |cx + d|^{1-2\nu}\, h\left(\frac{ax+b}{cx+d}\right).$$

- This looks like the modular transformation!

Automorphic distributions on $\mathbb{R}$

- Consider the space of distributions $V_{\lambda,\delta}^{-\infty}$. This is a space of distributions on $\mathbb{R}$, and G acts on a distribution $\tau \in V_{\lambda,\delta}^{-\infty}$ by

$$\pi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right)\tau(x) = |cx+d|^{1-2\nu}\,\tau\left(\frac{ax+b}{cx+d}\right).$$

- An automorphic distribution is a distribution such that

$$\tau(x) = |cx+d|^{1-2\nu}\,\tau\left(\frac{ax+b}{cx+d}\right) \text{ for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Forming distributions from modular forms

- Let $F(z)$ be a cusp form of weight k for $SL(2,\mathbb{Z})$.
- $F(z)$ has distribution boundary values:

$$\tau(x) = \tau_F(x) = \lim_{y \to 0+} F(x+iy).$$

- Starting with a q-expansion

$$F(z) = \sum_{n \geq 1} a_n n^{\frac{k-1}{2}} e^{2\pi i n z}$$

we can write

$$\tau(x) = \sum_{n \geq 1} a_n n^{\frac{k-1}{2}} e^{2\pi i n x}.$$

Forming distributions from modular forms

- For sufficiently large j, $\tau(x)$ has a continuous j-th antiderivative:

$$\tau^{(-j)}(x) = \sum_{n\geq 1} (2\pi i n)^{-j} a_n n^{\frac{k-1}{2}} e(nx).$$

- As a consequence of the limit formula, τ inherits automorphy from F:

$$\tau\left(\frac{ax+b}{cx+d}\right) = (cx+d)^k \tau(x).$$

- For these reasons, we call τ the *automorphic distribution* attached to F.

Automorphic distributions on $\mathbb{R}$

- Riemann studied the function

$$\varphi(x) = \sum_{n \geq 1} \frac{\sin(2\pi n^2 x)}{n^2}$$

  which is continuous but fails to be differentiable at "most" points.
- Hardy proved that the function is nondifferentiable at irrationals. In fact, $\varphi(x)$ is non-differentiable except at points $x = p/2q$ with p and q odd. At those, the derivative equals to $-\pi$.

Automorphic distributions on $\mathbb{R}$

- The function is actually the automorphic distribution attached to the Jacobi theta function

$$\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z}$$

- The automorphic distributions have a really interesting analytic behavior.

The automorphic distribution attached to Jacobi θ-function

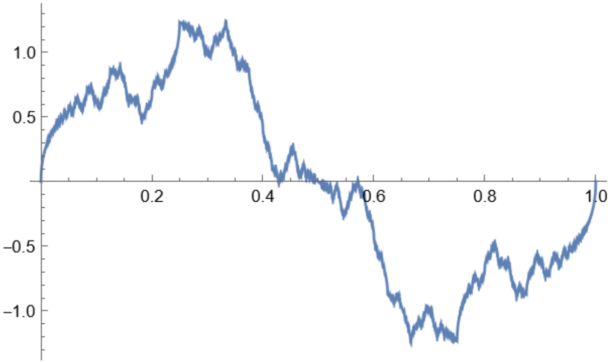

Figure: $\varphi(x)$, the antiderivative of the automorphic distribution attached to $\theta(z)$

The automorphic distribution attached to Jacobi θ-function

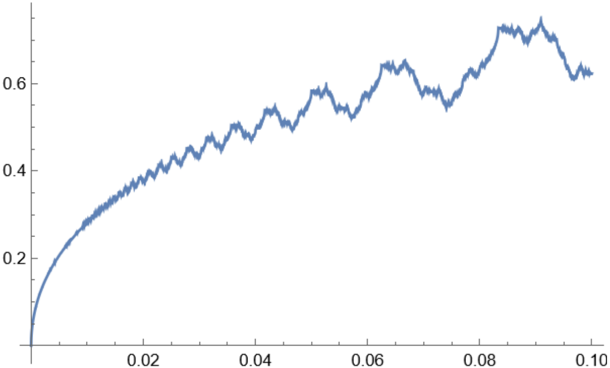

Figure: $\varphi(x)$ near the origin

The automorphic distribution attached to the modular discriminant

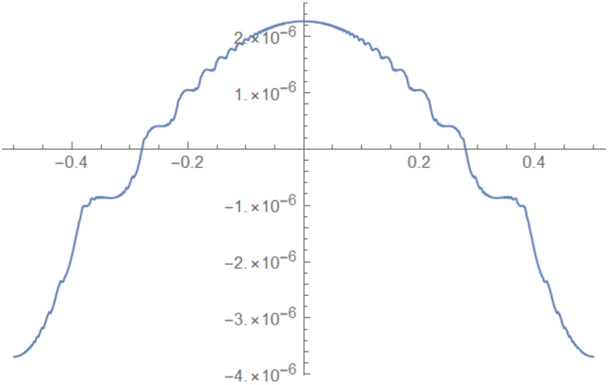

Figure: Imaginary part of $\tau_{\Delta}^{(-7)}(x)$

Outline of the talk

The mechanism of the Hardy-Littlewood method

- For an L-function $L(s) = a_n n^{-s}$ of interest, construct a real-valued function $Z(t)$ of the form

$$Z(t) = \chi_L(1/2 + it)^{-1/2} L(1/2 + it).$$

- If $L(1/2 + it)$ does not vanish on $t \in [T, 2T]$ then

$$\left| \int_T^{2T} Z(t)\, dt \right| = \int_T^{2T} \left| Z(t) \right| \, dt.$$

- If the RHS is $\gg T$ and the LHS is $o(T)$ as $T \to \infty$ then the L-function has infinitely many zeros on the critical line.

The mechanism of the Hardy-Littlewood method

- For L-functions of holomorphic modular forms, $Z(t)$ is constructed from the Mellin transform of the corresponding modular forms:

$$Mf(s) = \int_0^{i\infty} f(y) y^{s-1}\, dy$$

  which is a product of the L-function and some Γ-factors.
- The most difficult part is finding the bound of the LHS. Typically, the bound requires cancellation of the exponential sum

$$\sum_{m \le T} a_m e(mx) = o(T), \quad \text{as} \quad T \to \infty.$$

A distributional framework

- We write $Z(t)$ as Mellin transform of automorphic distributions instead of automorphic forms.
- In certain arithmetic instances, this gives a quick proof that the integral
$$\int_T^{2T} Z(t)\,dt$$
decays rapidly as $T \to \infty$.
- This uses the notion of *a distribution vanishing to infinite order*.

A distributional framework

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A distributional framework

- A distribution can be written (locally) as a higher order derivative of a continuous function.
- Roughly speaking, if the continuous function vanishes to a high order at a point p, then we say that the distribution vanishes to a high order at p.
- The automorphic distribution attached to a cusp form vanishes to *infinite order* at every rational point.

A distributional framework

- The local presentation of distributions vanishing to order $m \geq 0$ at 0 can be used to bound integrals against certain test functions.

Lemma

Suppose that σ is a distribution that vanishes to order $m \geq 0$ at 0 and let ϕ be a bump function near 0. Then

$$\int_{-\infty}^{\infty} \sigma(x)\left(T\phi(Tx)\right)\, dx = O(T^{-m}) \tag{1}$$

as $T \to \infty$.

A distributional framework

- By constructing the real-valued function $Z(t)$ using automorphic distributions and using the Lemma, we can give a quick prove that the integral

$$\int_{T}^{2T} Z(t)\, dt$$

  decays rapidly as $T \to \infty$.
- This completely avoids the use of exponential sums.
- This makes the method particularly effective in taking care of complications due to half-integral weights or additive twists.

A distributional framework

Theorem (K. 2020)

Let $k \in \frac{1}{2}\mathbb{Z}$, $k \geq 1$ and let $F(z)$ be a cusp form of weight k with respect to the congruence subgroup $\Gamma_0(N)$. Whose Fourier coefficients satisfy $a_i \in \mathbb{R}$ and $a_1 \neq 0$.

a. *If N is a perfect square and $F(z)$ is an eigenfunction of the involution W_N, then the L-function of $F(z)$ has infinitely many zeros on the critical line.*

b. *If $\frac{p}{q} \in \mathbb{Q}$ is a cusp $\Gamma_0(N)$-equivalent to $i\infty$ such that $p^2 \equiv 1 \pmod{q}$ then the additively twisted L-function*

$$L_{p/q}(s) = \sum_{n \geq 1} a_n e^{2\pi i n \frac{p}{q}} n^{-s}$$

has infinitely many zeros on the critical line.

Thank you!