BGG category $\mathcal{O}$ and BGG resolution

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Background setting and notations

- Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra over $\mathbb{C}$, with triangular decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+$, universal enveloping algebra $U(\mathfrak{g})$, root system $\Phi \subset \mathfrak{h}^*$, simple roots (base) $\Delta = \{\alpha_1, ..., \alpha_l\}$, and positive roots Φ_+.
- Let $\mathfrak{b} := \mathfrak{n}_+ + \mathfrak{h}$ be the (standard) Borel subalgebra of $\mathfrak{g}$.
- For any root $\alpha \in \Phi$, let s_α be the reflection through α, i.e. $s_\alpha(\lambda) := \lambda - \frac{2(\lambda,\alpha)}{(\alpha,\alpha)}\alpha$. The Weyl group is defined to be the group generated by reflections through simple roots, i.e. $W = \langle s_{\alpha_i} | i = 1, \ldots, l\rangle$.
- s_{α_i} is called a simple reflection.

Background setting and notations

- Partial order on $\mathfrak{h}^*$: for $\lambda, \mu \in \mathfrak{h}^*$, $\lambda \leq \mu$ iff

$$\lambda - \mu = \sum_{i=1}^{\ell} a_i \alpha_i, \qquad a_i \in \mathbb{Z}_{\geq 0}.$$

- Dot action. Let $\rho := \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha_i$. For any $\lambda \in \mathfrak{h}^*$, $w \in W$, define the dot action:

$$w \cdot \lambda = w(\lambda + \rho) - \rho.$$

Background setting and notations

- Length. For $w \in W$, write $\ell(w) = n$ if $w = s_1 \dots s_n$ with s_i simple reflection and n as small as possible; such an expression is called reduced.
- Bruhat order on W. If $w_1 = s_\alpha w_2$, with $\alpha \in \Phi_+$ and $\ell(w_1) < \ell(w_2)$, we write $w_1 \xrightarrow{s_\alpha} w_2$. Define $w < w'$ if $w = w_1 \to w_2 \to \cdots \to w_n = w'$.

Consider the category $\mathcal{C}$ of finite dimensional $\mathfrak{g}$-modules. We know that

$$\mathcal{C} = \bigoplus_{\lambda \in \Lambda_+} \mathcal{C}_\lambda$$

where $\mathcal{C}_\lambda$ is a full subcatgeory of $\mathcal{C}$ with
$\mathrm{Obj}(\mathcal{C}_\lambda) = \{L(\lambda)^{\oplus n} \mid n \geq 0.\}$
By schur lemma, we know $\mathrm{Hom}(L(\lambda), L(\mu)) = 0$ if $\lambda \neq \mu$, and
$\mathrm{End}(L(\lambda)) = \mathbb{C}\,\mathrm{id}_{L(\lambda)}$

Basics of BGG category $\mathcal{O}$

Definition 1 (BGG category).

The BGG category $\mathcal{O}$ is defined to be the full subcategory of $U(\mathfrak{g})$-Mod whose objects are the modules M satisfying the following three conditions:

- $(\mathcal{O}1)$ M is finitely generated.
- $(\mathcal{O}2)$ M is $\mathfrak{h}$-diagonalizable, i.e. there exists a basis of M consisting of common eigenvectors of $\mathfrak{h}$. That is,

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda,$$

  where $M_\lambda := \{v \in M \mid h.v = \lambda(h)v, \ \forall h \in \mathfrak{h}\}$.
- $(\mathcal{O}3)$ M is locally $\mathfrak{n}_+$-finite, i.e. $U(\mathfrak{n}_+).v$ is finite dimensional, $\forall v \in M$.

Highest weight module

Definition 2 (Maximal vector).

Let $M \in U(\mathfrak{g})$-Mod, $\lambda \in \mathfrak{h}^*$. A nonzero vector $v \in M_\lambda$ is called a maximal vector of weight λ if $\mathfrak{n}_+.v = 0$.

Definition 3 (Highest weight module).

$M \in \mathcal{O}$ is called a highest weight module(h.w.m.) with highest weight $\lambda \in \mathfrak{h}^*$ if it is generated by a maximal vector. That is, there exists a nonzero $v^+ \in M_\lambda$, such that $M = U(\mathfrak{g}).v^+$.

Any highest weight module is in category $\mathcal{O}$.

Construction of the Verma module $M(\lambda)$

Fix $\lambda \in \mathfrak{h}^*$, let $\mathbb{C}_\lambda = \mathbb{C}v_\lambda$ be a one dimensional $\mathfrak{b}$-module such that

$$\mathfrak{n}_+.v_\lambda = 0, \quad h.v_\lambda = \lambda(h)v_\lambda, \ \forall h \in \mathfrak{h}.$$

Define $M(\lambda) = \mathrm{Ind}_{U(\mathfrak{b})}^{U(\mathfrak{g})} \mathbb{C}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$ to be the induced module from $U(\mathfrak{b})$-Mod to $U(\mathfrak{g})$-Mod.

Construction of the Verma module $M(\lambda)$

Verma module $M(\lambda)$ is an h.w.m. of highest weight λ with maximal vector $1 \otimes v_\lambda$. For any h.w.m. $M = U(\mathfrak{g}).v^+$ of highest weight λ, M is a quotient of M_λ: $1 \otimes v_\lambda \mapsto v^+$.
Notice:

$$\begin{aligned}\mathrm{Hom}_{U(\mathfrak{g})}(M(\lambda), M) &= \mathrm{Hom}_{U(\mathfrak{g})}\left(\mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda, M\right)\\ &\cong \mathrm{Hom}_{U(\mathfrak{b})}\left(\mathbb{C}_\lambda, \mathrm{Res}_{\mathfrak{b}}^{\mathfrak{g}} M\right)\\ &= \{v \in M_\lambda \mid v \text{ is a maximal vector or } 0\}.\end{aligned}$$

Theorem 4.

Let $M = U(\mathfrak{g}).v^+$ be an h.w.m., then M has a unique maximal submodule.

Proof.

For any submodule N of M, $N = M$ iff $v^+ \in N$ iff $N_\lambda \neq 0$. So, one can prove $\sum_{N \subsetneq M} N \subsetneq M$ is the unique maximal submodule of M. Notice:

$$\left(\sum_{N \subsetneq M} N\right)_\lambda = \sum_{N \subsetneq M} N_\lambda = \sum_{N \subsetneq M} 0 = 0.$$

□

As a corollary, we know that there is a unique simple h.w.m. of weight λ. Namely, $L(\lambda) := M(\lambda)/N(\lambda)$, where $N(\lambda)$ is the maximal submodule of $M(\lambda)$.
Also, every simple module S is an h.w.m. So we know $S \cong L(\lambda)$ for some $\lambda \in \mathfrak{h}^*$.

Define $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ to be the center of $U(\mathfrak{g})$. We know that $[Z(\mathfrak{g}), U(\mathfrak{h})] = 0$ implies $Z(\mathfrak{g}).M_\lambda \subset M_\lambda$, $\forall \lambda \in \mathfrak{h}^*$.
For an h.w.m. $M = U(\mathfrak{g}).v$ of weight λ, we know $M_\lambda = \mathbb{C}v$. So, $\forall z \in Z(\mathfrak{g})$, $\exists \chi_\lambda(z) \in \mathbb{C}$, such that $z.v = \chi_\lambda(z)v$. This χ_λ is an algebra homomorphism from $Z(\mathfrak{g})$ to $\mathbb{C}$. Moreover, z acts on M as a scalar multiplication $\chi_\lambda(z) \cdot \mathrm{id}_M$:

$$z.(u.v) = u.(z.v) = u.(\chi_\lambda(z)v) = \chi_\lambda(z)u.v, \quad \forall u \in U(\mathfrak{g}).$$

Since every h.w.m. of highest weight λ is a quotient of $M(\lambda)$, χ_λ is independent of the choice of M.

In fact, every algebra homomorphism χ is equal to χ_λ for some λ, i.e. $\mathfrak{h}^* \twoheadrightarrow \mathrm{Hom}_{\mathrm{Alg}}(Z(\mathfrak{g}), \mathbb{C})$.

Let $\chi \in \mathrm{Hom}_{\mathrm{Alg}}(Z(\mathfrak{g}), \mathbb{C})$. Define a full subcategory of $\mathcal{O}$:

$$\mathcal{O}_\chi := \{M \in \mathcal{O} : M = M^\chi\}$$

where

$$M^\chi := \{x \in M : \forall z \in Z(\mathfrak{g}), \exists n \in \mathbb{N}, \text{ s.t. } (z - \chi(z))^n x = 0\}.$$

Theorem 5.

$\mathcal{O} = \bigoplus_\chi \mathcal{O}_\chi$. In other words, $\forall M \in \mathcal{O}$, $M = \bigoplus_\chi M^\chi$. Also, $\mathrm{Hom}_{\mathcal{O}}(M, N) = \bigoplus_\chi \mathrm{Hom}_{\mathcal{O}_\chi}(M^\chi, N^\chi)$.

Lemma 6.

Let $\{x_i, y_i | 1 \le i \le n\} \cup \{h_i | 1 \le i \le \ell\}$ *be a standard basis of* $\mathfrak{g}$. *For all* $k \ge 0$, *and* $1 \le i, j \le \ell$, *we have:*

(a) $\left[x_j, y_i^{k+1}\right] = 0$ *whenever* $j \neq i$.

(b) $\left[h_j, y_i^{k+1}\right] = -(k+1)\alpha_i(h_j)\, y_i^{k+1}$.

(c) $\left[x_i, y_i^{k+1}\right] = -(k+1)y_i^k\,(k \cdot 1 - h_i)$.

Theorem 7.

Given $\lambda \in \mathfrak{h}^*$ *and* $\alpha \in \Delta$, *suppose* $n := \langle \lambda, \alpha^\vee \rangle$ *lies in* $\mathbb{Z}_{\ge 0}$. *If* v^+ *is a maximal vector of weight* λ *in* $M(\lambda)$, *then* $y_\alpha^{n+1}.v^+$ *is a maximal vector of weight* $\mu := \lambda - (n+1)\alpha < \lambda$. *Thus there exists a nonzero homomorphism* $M(\mu) \to M(\lambda)$ *whose image lies in the maximal submodule* $N(\lambda)$.

A nonzero homomorphism $M(\mu) \to M(\lambda)$ implies $\chi_\mu = \chi_\lambda$. So, we get a conclusion: if $\lambda \in \Lambda_+$, then $\chi_{s_\alpha \cdot \lambda} = \chi_\lambda$.
By considering the reduced form of $w \in W$, we can show that if $\lambda \in \Lambda$, $\mu \in W \cdot \lambda$, then $\chi_\lambda = \chi_\mu$.

Theorem 8 (Harish-Chandra).

Suppose $\lambda, \mu \in \mathfrak{h}^*$, then $\chi_\lambda = \chi_\mu$ if and only if $\lambda \in W \cdot \mu$.

As an important corollary, $\mathcal{O}$ is of finite length (or to say, both Noetherian and Artinian), i.e., every $M \in \mathcal{O}$ has a finite length. Namely, M has a filtration:

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

with each M_i/M_{i-1} simple.

Proof.

1. M has a finite filtration with every factor is an h.w.m.
   Since M is finitely generated, we can assume M is generated by one weight vector: $M = U(\mathfrak{g}).v$. Induction on $\dim V$, where $V := U(\mathfrak{n}_+).v$ is a finite dimensional vector space by the definition of $\mathcal{O}$.
2. Every h.w.m. has a finite length.
   By the Harish-Chandra theorem. (cf. Humphreys, page 28)

□

For any $M \in \mathcal{O}$, we can define $[M : L(\lambda)]$ to be the multiplicity of $L(\lambda)$.

A natural question: when is $[M(\lambda) : L(\mu)] \neq 0$?

1. Necessary condition 1: $\boldsymbol{\mu \leq \lambda}$.
   $[M(\lambda) : L(\mu)] \neq 0$ implies $M(\lambda)_\mu \neq 0$, so $\mu \leq \lambda$.
2. Necessary condition 2: $\boldsymbol{\chi_\mu = \chi_\lambda}$.
   $[M(\lambda) : L(\mu)] \neq 0$ implies $L(\mu)$ is isomorphic to a subquotient of $M(\lambda)$.

By the Harish-Chandra theorem, condition 2 $\Leftrightarrow$ $\mu = W \cdot \lambda$.

Strongly linked

Let $\lambda, \mu \in \mathfrak{h}^*$ and write $\mu \uparrow \lambda$ if $\mu = \lambda$ or there is a root $\alpha > 0$ such that $\mu = s_\alpha \cdot \lambda < \lambda$; in other words, $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{>0}$. More generally, if $\mu = \lambda$ or there exist $\alpha_1, \ldots, \alpha_r \in \Phi^+$such that

$$\mu = (s_{\alpha_1} \ldots s_{\alpha_r}) \cdot \lambda \uparrow (s_{\alpha_2} \ldots s_{\alpha_r}) \cdot \lambda \uparrow \ldots \uparrow s_{\alpha_r} \cdot \lambda \uparrow \lambda$$

we say that μ is strongly linked to λ and write $\mu \uparrow \lambda$.
This is a partial order on $\mathfrak{h}^*$.

Theorem 9.

Let $\lambda, \mu \in \mathfrak{h}^*$.
(a) (Verma) *If* $\mu \uparrow \lambda$, *then* $M(\mu) \hookrightarrow M(\lambda)$; *in particular, we know* $[M(\lambda) : L(\mu)] \neq 0$.
(b) (BGG) *If* $[M(\lambda) : L(\mu)] \neq 0$, *then* $\mu \uparrow \lambda$.

Theorem 10.

If $\lambda \in \Lambda^+$, the unique maximal submodule $N(\lambda)$ of $M(\lambda)$ is the sum of the submodules $M\left(s_{\alpha_i} \cdot \lambda\right)$ for $1 \leq i \leq \ell$.

We can express this result by an exact sequence:

$$\bigoplus_{w \in W, \ell(w)=1} M(w \cdot \lambda) \to M(\lambda) \to L(\lambda) \to 0$$

Theorem 11.

If $\lambda \in \Lambda^+$, the unique maximal submodule $N(\lambda)$ of $M(\lambda)$ is the sum of the submodules $M\left(s_{\alpha_i} \cdot \lambda\right)$ for $1 \leq i \leq \ell$.

We can express this result by an exact sequence:

$$??? \rightarrow \bigoplus_{w \in W, \ell(w)=1} M(w \cdot \lambda) \rightarrow M(\lambda) \rightarrow L(\lambda) \rightarrow 0$$

The Weyl group W has a unique maximal element w_0, and $\ell(w_0) = |\Phi_+|$. We define the BGG resolution of $L(\lambda)$ is an exact sequence of the following form.

$$0 \to M(w_0 \cdot \lambda) \xrightarrow{\delta_m} \bigoplus_{\substack{w \in W \\ \ell(w) = |\Phi_+| - 1}} M(w \cdot \lambda) \to \cdots$$

$$\xrightarrow{\delta_2} \bigoplus_{\substack{w \in W \\ \ell(w) = 1}} M(w \cdot \lambda) \xrightarrow{\delta_1} M(\lambda) \xrightarrow{\varepsilon} L(\lambda) \to 0.$$

Theorem 12.

For $\lambda \in \Lambda_+$, there exists a BGG resolution for $L(\lambda)$.

Application: Weyl character formula

We can prove the Weyl character formula from the existence of BGG resolution. Write $C_k := \bigoplus_{\substack{w \in W \\ \ell(w)=k}} M(w \cdot \lambda)$, then

$\operatorname{ch} C_k = \sum_{\substack{w \in W \\ \ell(w)=k}} \operatorname{ch} M(w \cdot \lambda)$.

We know the alternating sum of dimension of each weight space of C_k is 0. The alternating sum of character should also be 0, i.e.

$$\operatorname{ch} L(\lambda) + \sum_{i=0}^{|\Phi_+|} (-1)^{i+1} \operatorname{ch} C_k = 0$$

$$\begin{aligned}
\operatorname{ch} L(\lambda) &= \sum_{i=0}^{|\Phi_+|} (-1)^i \operatorname{ch} C_k \\
&= \sum_{i=0}^{|\Phi_+|} \sum_{\substack{w \in W \\ \ell(w)=i}} (-1)^i \operatorname{ch} M(w \cdot \lambda) \\
&= \sum_{w \in W} (-1)^{\ell(w)} \operatorname{ch} M(w \cdot \lambda).
\end{aligned}$$

This is the Weyl character formula for finite dimensional irreducible module $L(\lambda)$.

Bott's theorem

compute the dimension of Lie algebra cohomology group: $H^k(\mathfrak{n}_-, L(\lambda)) := \mathrm{Ext}^k_{\mathfrak{n}_-}(\mathbb{C}, L(\lambda))$ (It is actually an $\mathfrak{h}$-module.)

Theorem 13 (Bott).

If $\lambda \in \Lambda^+$, then $\dim H^k\left(\mathfrak{n}^-, L(\lambda)\right) = \left|W^{(k)}\right|$, where $W^{(k)}$ denotes the set of elements in W having length k.

Proof.

By using property of dual functor (-)*, we know $\mathrm{Ext}^k_{\mathfrak{n}_-}(\mathbb{C}, L(\lambda)) \cong \mathrm{Ext}^k_{\mathfrak{n}_-}(L(\lambda)^*, \mathbb{C})$, $L(\lambda)^* \cong L(-w_0 \cdot \lambda)$. We know $\lambda^* := -w_0 \cdot \lambda \in \Lambda_+$ So, $L(\lambda^*)$ has a BGG resolution, which is a free resolution as $U(\mathfrak{n}_-)$-module. By applying functor $\mathrm{Hom}_{\mathfrak{n}_-}(\text{-}, \mathbb{C})$ to the BGG resolution, then taking the homology, we get $\mathrm{Ext}^{\cdot}_{\mathfrak{n}_-}(L(\lambda^*, \mathbb{C}))$.
we can natually identify $\mathrm{Hom}_{\mathfrak{n}_-}(M, \mathbb{C})$ with $(M/\mathfrak{n}_- M)^*$. For M is a Verma module $M(\mu)$, $(M/\mathfrak{n}_- M)^* \cong \mathbb{C}^*_\mu = \mathbb{C}_{-\mu}$ is a one dimensional space. Thus, we know $\dim \mathrm{Hom}_{\mathfrak{n}_-}(C_k, \mathbb{C}) = |W^{(k)}|$. Also, because all $-w \cdot \lambda^*$ are distinct, all maps in this chain are zero. □

Other applications of BGG resolution:

1. projective dimension & global dimension
2. compute $\text{Ext}^n_{\mathcal{O}}\left(M\left(w' \cdot \lambda\right), M(w \cdot \lambda)\right)$, $\text{Ext}^n_{\mathcal{O}}\left(M(\mu), M(\lambda)^\vee\right)$.

Thank you!