Introduction to Lie Algebras and Representation Theory

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Goal

Humphreys really starts with section 8 (maybe more experienced readers will disagree).

I found sections 1-6 hard. (8 onward too, but whatever).

My goal is to give an outline and some perspective on the first 7 sections that can make a first read a little easier.

Section 7 is really cool. And it comes up all over in mathematics!

Definition

A Lie algebra is a vector space L over a field F with an operation

$$L \times L \to L$$
$$(x, y) \mapsto [xy]$$

satisfying

- [ ] is bilinear
- $[xx] = 0$
- $[x[yz]] + [y[zx]] + [z[xy]] = 0$ This condition is called the Jacobi identity.

First Properties

[ ] is anticommutative:

$$\begin{aligned}[x+y, x+y] &= 0 \\ [xx]+[xy]+[yx]+[yy] &= 0 \\ [xy]+[yx] &= 0 \\ [xy] &= -[yx]\end{aligned}$$

A Lie algebra homomorphism is a homomorphism of vector spaces that respects the bracket: $\varphi([xy]) = [\varphi(x)\varphi(y)]$.

An ideal is a vector subspace I satisfying $[x, i] \in I \ \forall x \in L, i \in I$.

Quotient Lie algebras, normalizers, centralizers, isomorphism theorems

Examples

Let V be a vector space of dimension n over a field F

$\mathfrak{gl}(V) = \text{End}(V)$. Equivalently,
$\mathfrak{gl}(n, F) = \{n \times n \text{ matrices with entries in } F\}$. In both cases, the bracket is given by $[xy] = xy - yx$.

$\mathfrak{sl}(V) = \mathfrak{sl}(n, F) = \{x \in \mathfrak{gl}(V) | \text{ Trace}(x) = 0\}$.

$\mathfrak{t}(n, F)$ is upper triangular matrices.
$\mathfrak{n}(n, F)$ is the strictly upper triangular matrices.
$\mathfrak{d}(n, F)$ is the diagonal matrices.

Any associative algebra can be made to be a Lie algebra by defining $[xy] = xy - yx$.

Other important ones: symplectic alegbra, orthogonal algebras.

$\mathfrak{sl}(2, F)$

The standard basis of $\mathfrak{sl}(2, F)$ is

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[hx] = 2x, [hy] = -2y, [xy] = h$$

Remark: If L is a Lie subalgebra of some $\mathfrak{gl}(V)$ then we call L a linear Lie algebra.

A Word on Lie Groups

A Lie group G is a topological group with a smooth manifold structure.

$GL(n,\mathbb{R}), SL(n,\mathbb{R}), O(n), U(n)$, etc.

Given a Lie group G, the Lie algebra $\mathfrak{g}$ is defined as the tangent space at id. There exists an map exp: $\mathfrak{g} \to G$ that satisfies a lot of nice properties and is pretty important.

This doesn't really come up at GARTS

Representations

A representation of a Lie algebra L is a pair (φ, V), where φ is a Lie algebra homomorphism

$$\varphi : L \to \text{End}(V)$$

If (φ, V) is a representation of L, then V can be viewed as an L-module by

$$x.v = \varphi(x)(v)$$

for $x \in L, v \in V$.

Irreducible Representations

A representation (φ, V) is called irreducible if V has no nontrivial L-invariant subspaces.

A representation is called completely reducible if V can be written as a direct sum of L-invariant subspaces.

Remark: sometimes V will have a L-invariant subspace (say, W), but there may not be another L-invariant subspace $\tilde{W}$ with $V = W \oplus \tilde{W}$.

Theorem (Schur's Lemma)

Let (φ, V) be an irreducible representation of L. Let $\alpha \in End(V)$. If $[\alpha x] = 0 \ \forall x \in \varphi(L)$, then α is a scalar.

Adjoint Representation

For $x \in L$, define a function $ad_x : L \to L$ by $ad_x(y) = [xy]$. This function is a derivation.

Then define a representation $ad : L \to \text{End}(L)$ by $x \mapsto ad_x$. This is called the adjoint representation.

Recall a map ϕ is called nilpotent if $\phi^n = 0$ for some n.

An element $x \in L$ is called ad-nilpotent if ad_x is nilpotent.

Let $L^0 = L$, $L^{i+1} = [L, L^i]$. Then we say L is nilpotent if $L^n = 0$ for some n. Let's not worry about this.

Theorem (Engel's Theorem)

L is nilpotent if and only if x is ad-nilpotent $\forall x \in L$.

Semisimple Lie Algebras

Let $L^{(0)} = L$ and let $L^{(i+1)} = [L^{(i)} L^{(i)}]$. We call L solvable if $L^{(n)} = 0$ for some n.

Proposition: if I and J are ideals of L, and I and J are solvable, then $I + J = \{x + y | x \in I, y \in J\}$ is solvable.

Corollary: Let S be a maximal solvable ideal of L. Let I be a different solvable ideal of L. Then $S \subset I + S$, so $I + S = S$ by maximality. Thus there is a unique maximal solvable ideal of L. We call it the radical: Rad L.

Definition: L is called semisimple if Rad $L = 0$.

There are 4 infinite families of semisimple Lie algebras, plus 5 other exceptional Lie algebras.

Theorem (Cartan's Criterion)

Let L be a linear Lie algebra and assume $Tr(xy) = 0 \; \forall x \in [LL], y \in L$. Then L is solvable.

Corollary: If L is not necessarily linear, but $\mathrm{Tr}(ad_x \; ad_y) = 0 \; \forall x \in [LL], y \in L$ then L is solvable.

Proof of corollary: $ad \; L$ is a linear Lie algebra. Then $\mathrm{Tr}(ad_x \; ad_y) = 0$ means that $ad \; L$ is solvable.
The kernel of ad is $Z(L)$. $Z(L)$ is always solvable. Then

$$L/Z(L) \simeq ad \; L$$

This implies L is solvable by a property of solvable Lie algebras.

Killing Form

For $x, y \in L$, let

$$\kappa(x, y) = \text{Tr}(ad_x \; ad_y)$$

This is a symmetric bilinear form called the Killing form.

κ is called nondegenerate if $\forall x \in L, \exists y \in L$ so that $\kappa(x, y) \neq 0$.

κ is associative: $\kappa([xy], z) = \kappa(x, [yz])$.

Killing Semisimplicity

Theorem

L is semisimple if and only if its killing form is nondegenerate.

Proof idea: Let $S = \{x \in L | \kappa(x, y) = 0 \ \forall y \in L\}$. S is an ideal of L. Notice that κ is nondegenerate iff $S = 0$.

First suppose L is semisimple. There are no nonzero solvable ideals of L. Let $x \in [SS], y \in L$. Then $\kappa(x, y) =$Tr$(ad_x \ ad_y) = 0$. Apply the corollary of Cartan's Criterion: S is solvable. Hence it's 0.

Now κ be nondegenerate, so $S = 0$. Let J be a solvable ideal of L. That means $[J^{(i)} J^{(i)}] = 0$ for some nonzero $J^{(i)}$. Let $I = J^{(i)}$.

Let $x \in I, y \in L$. Then $ad_x \ ad_y : L \to I$ and $(ad_x \ ad_y)^2 : L \to [I, I] = 0$ This shows that $ad_x \ ad_y$ is nilpotent, and hence has trace 0. Therefore $\kappa(x, y) = 0$, so $y \in S$ and this shows that $I = 0$.

Theorem

Let L be semisimple. Then L is a direct sum of simple ideals. This decomposition is unique.

Proof idea: Let's assume L is not simple (since simple implies semisimple). Let I be an ideal of L. Let $I^{\perp} = \{x \in L | \kappa(x, y) = 0 \ \forall y \in I\}$.

$I \cap I^{\perp}$ is an ideal of I. And by the definition of $I^{\perp}$, we have for $x \in I, y \in I \cap I^{\perp}$, that $\kappa(x, y) = \text{Tr}(ad_x \ ad_y) = 0$. By the corollary to Cartan's Criterion, $I \cap I^{\perp}$ is solvable, hence 0. Therefore $L = I \oplus I^{\perp}$. Repeat until each piece is simple.

Now write $L = L_1 \oplus L_2 \oplus ... \oplus L_n$. Let I be any simple ideal of L. Becuase L is semisimple, it contains no abelian ideals, so $[IL] \neq 0$. Since I is simple, we have $[IL] = I$. Notice

$$[IL] = [IL_1] \oplus ... \oplus [IL_n]$$

Casimir Element

Assume L is semisimple. Fix a basis $(x_1, ..., x_n)$ of L. Then κ induces a dual basis $(y_1, ..., y_n)$ on L satisfying

$$\kappa(x_i y_j) = \delta_{ij}$$

For $\mathfrak{sl}(2, F)$ the dual basis to (x, h, y) is $(y, h/2, x)$.
Then $\sum ad_{x_i}\ ad_{y_i} = c$ is called the casimir element of the adjoint representation.

This can be done more generally: for a representation φ, replace κ with $\mathrm{Tr}(\varphi(x)\varphi(y)$

In general, $c \notin \mathrm{Image}(\varphi)$. Instead it lives in the enveloping algebra of φ.

Weyl's Theorem

Theorem

Let L be a semisimple Lie algebra. Let $\phi : L \to \mathfrak{gl}(V)$ be a finite dimensional representation of L. Then ϕ is completely reducible.

The proof is hard. It does a lot of abstract nonsense to reduce the problem in a couple different ways. First it handles the case where there is an L-invariant space W of codimension 1. Then it uses the casimir element of the representation to construct a one-dimensional L-invariant subspace of V that complements W. Then it does more abstract nonsense to get the result when there is not an L-invariant space of codimension 1.

Jordan Decomposition

Recall that $y \in End(V)$ is called nilpotent if $y^n = 0$ for some n.
We call $z \in End(V)$ semisimple if it is diagonalizable.

Let $x \in End(V)$. There is a unique decomposition $x = x_s + x_n$ where x_s is semisimple and x_n is nilpotent.

$ad_x = ad_{x_s} + ad_{x_n}$ is the Jordan Decomposition of ad_x.

Abstract Jordan Decomposition

There is another decomposition we can induce: let L be semisimple. The kernel of ad is $Z(L)$, but since $Z(L) = 0$, we have $adL \simeq L$. It turns out further that $adL \simeq Der(L)$, the space of derivations of L. That is, all derivations of L are given by adjoints.

If $X \in DerL$ and X_n, X_s are the nilpotent and semisimple parts of X in $End(L)$, it also turns out that $X_n, X_s \in Der(L)$, hence $X_n, X_s \in adL$. Thus there are unique $x_n, x_s \in L$ such that $ad_{x_n} = X_n$ and $ad_{x_s} = X_s$. We can then write $x = x_n + x_s$.

We do not know if L is a linear Lie algebra, so x_n and x_s need not be endomorphisms of a vector space V. If L is linear, then a corollary to Weyl's Theorem tells us that this new decomposition agrees with the old one.

Corollary to Weyl

Theorem

Let L be semisimple and (φ, V) a representation of L. If $x = x_n + x_s$ is the abstract Jordan decomposition of x, then $\varphi(x) = \varphi(x_n) + \varphi(x_s)$ is the Jordan decomposition of $\varphi(x)$.

Representations of $\mathfrak{sl}(2, F)$

Let (φ, V) be an irreducible representation of $\mathfrak{sl}(2, F)$.

$\varphi(h)$ is diagonalizable: h is diagonal in the usual basis, hence it is semisimple. The corollary to Weyl's theorem says that $\varphi(h)$ is also semisimple, hence diagonalizable.

Write $V = \oplus V_\lambda$ where $\lambda \in F$ is an eigenvalue of $\varphi(h)$ and V_λ is the eigenspace.

Call λ weights of h and V_λ weight spaces.

For convenience, when $\lambda \in F$ is not an eigenvalue of $\varphi(h)$, let $V_\lambda = 0$.

Representations of $\mathfrak{sl}(2, F)$

Proposition

Let $v \in V_\lambda$. Then $\varphi(x)(v) \in V_{\lambda+2}$ and $\varphi(y)(v) \in V_{\lambda-2}$

For ease of notation, denote $\varphi(x)(v)$ by $x.v$

$$h.x.v = x.h.v + [hx].v = x.\lambda v + 2x.v = (\lambda + 2)x.v$$

$$h.y.v = y.h.v + [hy].v = y.\lambda v - 2y.v = (\lambda - 2)y.v$$

Maximal Vectors

Remark: there must exist a weight λ such that $\lambda + 2 = 0$. Call $v \in V_\lambda$ a maximal vector if this is the case.

Let v_0 be a maximal vector in the space V_{λ_0}. Define $v_i = \frac{1}{i!} y^i . v_0$. Then

$$y.v_i = (i+1) v_{i+1}$$

By the last slide, $v_i \in V_{\lambda_0 - 2i}$. Whence,

$$h.v_i = (\lambda_0 - 2i) v_i$$

Maximal Vectors

Now we compute that $x.v_i = (\lambda_0 - i + 1)v_{i-1}$ by induction.

$$
\begin{aligned}
x.v_i &= \frac{1}{i} x.y.v_{i-1} \\
ix.v_i &= [xy].v_{i-1} + y.x.v_{i-1} \\
&= h.v_{i-1} + y.x.v_{i-1} \\
&= (\lambda_0 - 2i + 2)v_{i-1} + (\lambda_0 - i + 2)y.v_{i-2} \\
&= (\lambda_0 - 2i + 2)v_{i-1} + (\lambda_0 - i + 2)(i-1)v_{i-1} \\
&= i(\lambda_0 - i + 1)v_{i-1} \\
x.v_i &= (\lambda_0 - i + 1)v_{i-1}
\end{aligned}
$$

Maximal Vectors

Summary: given a maximal vector v_0, there is a set of v_i satisfying

- $y.v_i = (i+1)v_{i+1}$
- $h.v_i = (\lambda_0 - 2i)v_i$
- $x.v_i = (\lambda_0 - i + 1)v_{i-1}$

The 2nd one implies the v_i are linearly independent. But then only finitely many v_i can be nonzero. Let $v_m \neq 0$ but $v_{m+1} = 0$.

Then $\text{span}(v_0, ..., v_m)$ is a $\mathfrak{sl}(2, F)$-invariant subspace of V. (We can see it's $\mathfrak{sl}(2, F)$-invariant because we can see how $\mathfrak{sl}(2, F)$ acts on it above.) Since (φ, V) is an irreducible representation, $\text{span}(v_0, ..., v_m) = V$

Maximal Vectors

Consider $x.v_{m+1} = (\lambda_0 - m)v_m$. Since $v_{m+1} = 0$ and $v_m \neq 0$, we conclude $\lambda_0 = m$.

$\lambda_0 = m = \dim(V) - 1$ is an integer. We call it the highest weight of V. Each weight space of an irreducible representation is 1-dimensional. V admits weights $-m, -m+2, ..., m-2, m$.

The maximal vector we chose is unique up to scalars.

By Weyl's Theorem, general representations of $\mathfrak{sl}(2, F)$ are direct sums of irreducible representations described in this way.

Applications

Actions of $\mathfrak{sl}(2, F)$ appear a lot.

Raising and lowering operators

Quantum mechanics. Spinors?

The root space decomposition of a semisimple Lie algebra admits triples which generate subalgebras isomorphic to $\mathfrak{sl}(2, F)$. Knowing how the adjoint representation of these subalgebras act is helpful.

References

Humphreys, James E.
Introduction to Lie Algebras and Representation Theory