1 Homework assignment no. 3, due on Thursday February 14

In this assignment, the rules of the game are slightly changed. If you are asked to prove something, the answer could be “It cannot be proved,” followed by a proof that the statement cannot be proved. For example, if you are asked to prove that the equation \( x^2 + 1 = 0 \) has a real solution (i.e., that \( (\exists x \in \mathbb{R})x^2 + 1 = 0 \)), the answer should be “I cannot prove this because it isn’t true. Here is a proof that the statement isn’t true, (i.e., that the equation \( x^2 + 1 = 0 \) does not have a real solution): suppose there exists an \( x \in \mathbb{R} \) such that \( x^2 + 1 = 0 \); pick a such that \( a^2 + 1 = 0 \); then \( a^2 \geq 0 \), because the square of any real number is \( \geq 0 \); so \( a^2 + 1 > 0 \), so \( a^2 + 1 \geq 0 \); but \( a^2 + 1 = 0 \), so \( 0 > 0 \); on the other hand, \( \sim 0 > 0 \); so we got a contradiction. Hence \( \sim (\exists x \in \mathbb{R})x^2 + 1 = 0 \).” For a second example, if you are given the axiom

\[
(\forall x)(A(x) \implies B(x))
\]

and you are asked to prove, from that axiom, the proposition

\[
(\exists x)B(x),
\]

where \( A(x) \) and \( B(x) \) are arbitrary sentences containing the variable \( x \), then your answer should be “This cannot be proved, because \( A(x) \) could be, for example, the sentence “\( x \neq x \)”, and \( B(x) \) could also be “\( x \neq x \)”, and then \( (1) \) would be true (because if \( a \) is an arbitrary object, then \( a \neq a \implies a \neq a \) is true, since \( a \neq a \) is false) but \( (\exists x)x \neq x \) is false, whereas if you could prove \( (2) \) from \( (1) \) then it would follow that \( (2) \) is true whenever \( (1) \) is true, for any choice of the predicates \( A(x) \) and \( B(x) \).”

In the following problems,

i. We use “\( \mid \)” to indicate that \( a \) divides \( b \). The definition of “divides” is as follows:

[DEFINITION] Let \( m, n \) be integers. We say that \( m \) divides \( n \) if there exists an integer \( k \) such that \( n = km \). In formal language,

\[
(\forall m \in \mathbb{Z})(\forall n \in \mathbb{Z})(m \mid n \iff (\exists k \in \mathbb{Z})n = mk).
\]

ii. A rational number is a real number \( x \) such that \( x = \frac{m}{n} \) for some integers \( m, n \) such that \( n \neq 0 \). We write \( \mathbb{Q} \) to denote the set of all rational numbers, and \( \mathbb{R} \) to denote the set of all real numbers. Then

\[
(\forall x \in \mathbb{R})\left( x \in \mathbb{Q} \iff (\exists m \in \mathbb{Z})(\exists n \in \mathbb{Z})(n \neq 0 \land x = \frac{m}{n}) \right).
\]
PROBLEM 1. In this problem, you are given three axioms (or, if you prefer, three hypotheses) and a conclusion, and you are asked to prove the conclusion from the axioms.

The axioms are:

A1. Giraffes are taller than cows (that is, “every giraffe is taller than every cow”).

A2. Cows are taller than sheep.

A3. “Taller than” is a transitive relation (that is, if something is taller than something else, which in turn is taller than a third thing, then the first one is taller than the third one).

And the conclusion is:

C. Giraffes are taller than sheep.

In symbolic language, we introduce predicate symbols $G(x)$, $C(x)$, $S(x)$, $T(x,y)$ to stand, respectively, for “$x$ is a giraffe”, “$x$ is a cow”, “$x$ is a sheep”, and “$x$ is taller than $y$”. Then the axioms are

A1. $(\forall x)(\forall y)((G(x) \land C(y)) \implies T(x,y))$,

A2. $(\forall x)(\forall y)((C(x) \land S(y)) \implies T(x,y))$,

A3. $(\forall x)(\forall y)(\forall z)((T(x,y) \land T(y,z)) \implies T(x,z))$,

the desired conclusion is

$$(\forall x)(\forall y)((G(x) \land S(y)) \implies T(x,y)) .$$

You are asked to prove (3) from Axioms A1, A2, A3, using the logical rules.

PROBLEM 2. Write the following statement in formal language and prove it: If $x, y, z$ are integers such that $x|y$ and $y|z$ then $x|z$.

PROBLEM 3. Write the following statement in formal language and prove it: If $x, y, z$ are integers such that $x|z$ and $y|z$ then $xy|z$.

PROBLEM 4. Write the following statement in formal language and prove it: If $x$ is an integer such that $7|x$ and $11|x$ then $77|x$. (Hint: $1 = 56 - 55$.)

PROBLEM 5. Write the following statement in formal language and prove it: If $x$ is an integer such that $11|x$ and $23|x$ then $253|x$. (Remark: $253 = 23 \times 11$.)
PROBLEM 6. Write the following statement in formal language and prove it: If $x$ is an integer such that $4|x$ and $6|x$ then $24|x$.

PROBLEM 7. Two integers $a, b$ are said to be coprime if there exist integers $u, v$ such that $au + bv = 1$. (For example, 11 and 23 are coprime because $23 \times 1 + (-2) \times 11 = 1$. On the other hand, 12 and 22 are not coprime, because if $u, v$ are any integers then $12u + 22v$ must be even, so $12u + 22v$ cannot be equal to 1.)

i. Prove that if $a, b$ are coprime integers then the following is true: if $n$ is any integer such that $a|n$ and $b|n$ then $ab|n$.

ii. Prove that if $a, b$ are integers that have a common factor greater than 1 (i.e., $(\exists k \in \mathbb{N})(k|a \land k|b \land k > 1)$), then $a$ and $b$ are not coprime.

PROBLEM 8. Rewrite each of the following statements in formal language (using “$\in \mathbb{Q}$” for “is rational”), and then prove it. (You are allowed to use the fact—to be proved later in the course—that $\sqrt{2}$ is irrational.)

6.1. The sum of two rational numbers is a rational number.
6.2. The product of two rational numbers is a rational number.
6.3. The sum of two irrational numbers is an irrational number.
6.4. The product of two irrational numbers is an irrational number.
6.5. The sum of a rational number and an irrational number is a rational number.
6.6. The sum of a rational number and an irrational number is a rational number.
6.7. The sum of a rational number and an irrational one is an irrational number.
6.8. The product of a rational number and an irrational number is a rational number.
6.9. The product of a rational number and an irrational number is an irrational number.