REVIEW PROBLEMS FOR THE FINAL EXAM

The following list of problems consists mostly of questions that are answered in the notes, or are discussed in the book, or were discussed in class. But there are several problems for which you will have to do some thinking.

Every problem in the final exam is going to be a problem in this list, or very similar to a problem in this list.

The exam will have two parts: you will have to do all the problems of Part I, and in Part II you will be able to choose.

1. Define “finite set” and “infinite set”. (NOTE: You have a correct definition in the notes. Study it carefully, especially if you are one of the students who are having trouble writing definitions.)

2. Define “divisible”, “multiple, “divides”, and “factor”. (NOTE: You have a correct definition in the notes. Study it carefully, especially if you are one of the students who are having trouble writing definitions.)

3. Define “prime number”. (NOTE: You have a correct definition in the notes. Study it carefully, especially if you are one of the students who are having trouble writing definitions.)

4. Define “subset”.

5. Define “empty set”.
6. Define “power set”.

7. Define “inductive set”. (NOTE: You have a correct definition in the notes. Study it carefully, especially if you are one of the students who are having trouble writing definitions.)

8. Define “union” (of two sets).

9. Define “intersection” (of two sets).

10. Define “Cartesian product”.

11. Define “rational number” and “irrational number”. (NOTE: You have a correct definition in the book, page xvi. Study it carefully, especially if you are one of the students who are having trouble writing definitions.)

12. Define “coprime”.

13. Define “greatest common divisor”.

14. Define “absolute value”. (NOTE: You have a correct definition in the notes. Study it carefully, especially if you are one of the students who are having trouble writing definitions.)

15. Define what it means for a function to be “onto” a set\(^1\).

16. Define “one-to-one function”. (This is in the book, page 208.)

17. Define “bijection” (also called “one-to-one correspondence”). (This is in the book, page 214.)

18. Define “sequence”. (This is in the book, page 225.)

19. Define what it means for two sets to have the same cardinality, or to be equivalent. (This is in the book, page 234. The book uses the word “equivalent”. In class I called this “having the same cardinality”. So two sets \(A, B\) are “equivalent” in the sense of the book if and only if they have the same cardinality, that is, if and only if there exists a bijection from \(A\) to \(B\). And a bijection from \(A\) to \(B\) is a one-to-one function \(f : A \to B\) which is onto \(B\).)

\(^1\) ANSWER: If \(A, B\) are sets, and \(f : A \to B\), we say that the function \(f\) is onto \(B\) if \((\forall y \in B)(\exists x \in A)f(x) = y\).
20. Prove that the sets $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality. (NOTE: I showed in class how to construct a bijection $f : \mathbb{N} \to \mathbb{Z}$. This is also done in the book, page 244, Theorem 5.2.2.)

21. Prove Cantor’s theorem: if $S$ is any set, then $S$ and the power $\mathcal{P}(S)$ set are not equivalent. (That is, $S$ and $\mathcal{P}(S)$ do not have the same cardinality.) (NOTE: What you have to prove is that there does not exist a function $f : S \to \mathcal{P}(S)$ which is onto $\mathcal{P}(S)$. I did this proof in class, and it is also done in the book, page 261.) **This question has a 99% chance of appearing in the final exam, and it may even show up in Part I.**

22. Prove or disprove that the function $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = x^2 + 1 \quad \text{for} \quad x \in \mathbb{R}$$
is onto $\mathbb{R}$. (This is done in the book, page 207.)

23. Prove or disprove that the function $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = x^2 \quad \text{for} \quad x \in \mathbb{R}$$
is onto $\mathbb{R}$. (This is done in the book, page 207.)

24. Of all subsets of $\mathbb{R}$, there is one and only one subset $B$ such that the function $f$ given by
$$\begin{align*}
\text{dom}(f) &= \mathbb{R}, \\
f(x) &= x^2 \quad \text{for} \quad x \in \mathbb{R},
\end{align*}$$
is onto $B$. Determine this set.

25. Prove or disprove that the function $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = x^3 + 3x^2 - 24x \quad \text{for} \quad x \in \mathbb{R}$$
is onto $\mathbb{R}$. (This is done in the book, page 207.)

26. Prove or disprove that the function $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = 2x + 1 \quad \text{for} \quad x \in \mathbb{R}$$
is onto $\mathbb{R}$. 
27. Prove or disprove that the function $f : \mathbb{R} \to \mathbb{R}$ given by
   \[ f(x) = 2x + 1 \quad \text{for} \quad x \in \mathbb{R} \]
   is one-to-one. (This is in the book, page 208.)

28. Prove or disprove that the function $f : \mathbb{R} \to \mathbb{R}$ given by
   \[ f(x) = \frac{1}{x^2 + 1} \quad \text{for} \quad x \in \mathbb{R} \]
   is one-to-one. (This is in the book, page 209.)

29. Prove or disprove that the function $f : \mathbb{R} \to \mathbb{R}$ given by
   \[ f(x) = |x| \quad \text{for} \quad x \in \mathbb{R} \]
   is one-to-one.

30. Prove or disprove that the function $f : [0, \infty) \to \mathbb{R}$ given by
   \[ f(x) = |x| \quad \text{for} \quad x \in [0, \infty) \]
   is one-to-one. (This is in the book, page 208.) (NOTE: $[0, \infty)$ is the set $\{ x \in \mathbb{R} : 0 \leq x \}.$)

31. Prove that function $f : \mathbb{R} \to (-1, 1)$ given by
   \[ f(x) = \frac{x}{\sqrt{x^2 + 1}} \quad \text{for} \quad x \in \mathbb{R} \]
   is a bijection. (NOTE: $(-1, 1)$ is the interval $\{ x \in \mathbb{R} : -1 < x < 1 \}.$) (HINT: Use the techniques you learned in your Calculus courses to study the function $f$. In particular, compute the derivative $f'$ of $f$, and use the information you get from $f'$—and, in particular, the sign of $f'(x)$ for different values of $x$—to sketch the graph of $f$. In addition, it will be useful to determine the limits $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$.)

32. If $A$, $B$, $C$ are sets, and $f : A \to B$, $g : B \to C$ are functions, the composite of $f$ and $g$ is the function $g \circ f : A \to C$ whose values are given by
   \[ (g \circ f)(x) = g(f(x)) \quad \text{for} \quad x \in A. \]
   Prove that if $f$ is onto $B$ and $g$ is onto $C$ then $g \circ f$ is onto $C$. (This is done in the book, on page 208.)
33. Prove or disprove that

(a) If $A$, $B$, $C$ are sets, $f : A \rightarrow B$, $g : B \rightarrow C$ are functions, and $g \circ f$ is onto $C$, then $f$ is onto $B$. (For this one, you have to think.)

(b) If $A$, $B$, $C$ are sets, $f : A \rightarrow B$, $g : B \rightarrow C$ are functions, and $g \circ f$ is onto $C$, then $g$ is onto $C$. (For this one, you have to think.)

34. Prove that if $A$, $B$, $C$ are sets, $f : A \rightarrow B$ and $g : B \rightarrow C$ are functions, and $f$ and $g$ are one-to-one then $g \circ f$ is one-to-one. (This is done in the book, on page 209.)

35. Prove or disprove that

(a) If $A$, $B$, $C$ are sets, $f : A \rightarrow B$, $g : B \rightarrow C$ are functions, and $g \circ f$ is one-to-one, then $f$ is one-to-one. (For this one, you have to think.)

(b) If $A$, $B$, $C$ are sets, $f : A \rightarrow B$, $g : B \rightarrow C$ are functions, and $g \circ f$ is one-to-one, then $g$ is one-to-one. (For this one, you have to think.)

36. Prove that if $A$, $B$, $C$ are sets, $f : A \rightarrow B$, and $g : B \rightarrow C$ are functions, and both $f$, $g$ are bijections, then $g \circ f$ is a bijection. (This is explained in the book, on page 214.)

37. Prove or disprove that

(a) If $A$, $B$, $C$ are sets, $f : A \rightarrow B$, $g : B \rightarrow C$ are functions, and $g \circ f$ is a bijection, then $f$ is a bijection. (For this one, you have to think.)

(b) If $A$, $B$, $C$ are sets, $f : A \rightarrow B$, $g : B \rightarrow C$ are functions, and $g \circ f$ is a bijection, then $g$ is a bijection. (For this one, you have to think.)

38. Let $P$ be the set of all people who live in the United States, and let $S$ be the set of all states of the United States. Let $f : P \rightarrow S$ be the function such that

$$f(x) = \text{the state where } x \text{ lives}, \text{ for } x \in S,$$
and let \( g : S \to P \) be the function such that
\[
g(x) = \text{the governor of state } x, \text{ for } x \in S.
\]
Let \( h \) be the composite function \( g \circ f \), so \( h : P \to P \). Answer the following questions, and in each case give a reason for your answer:

(a) How would you describe the function \( h \) in plain English?
(b) Is \( f \) one-to-one?
(c) Is \( f \) onto \( P \)?
(d) Is \( f \) onto \( S \)?
(e) Is \( g \) one-to-one?
(f) Is \( g \) onto \( P \)?
(g) Is \( g \) onto \( S \)?
(h) Is \( h \) one-to-one?
(i) Is \( h \) onto \( P \)?
(j) Is \( h \) onto \( S \)?
(k) Is it true that \( h \) has a fixed point? (If \( k \) is a function, a fixed point of \( k \) is a member \( x \) of \( \text{dom}(f) \) such that \( f(x) = x \).)

39. Prove that \( \mathbb{N} \), the set of all natural numbers, is an infinite set.

40. Prove that the set of prime numbers is infinite (Euclid’s Theorem).

(NOTE: This is a very important theorem, so there is a good chance it will be in the exam, maybe even in Part I.)

41. Prove that if \( n \) is a natural number and \( n \geq 2 \) then \( n \) has a prime factor.

42. For each of the following, indicate whether it is a one-argument function, a two-argument function, a one-argument predicate, or a two-argument predicate. (Note: “function” means exactly the same as “operation”, and “predicate” means exactly the same as “relation”.)

(a) equal,
(b) not equal to,
(c) absolute value,
(d) divisible,
(e) divides,
(f) prime number,
(g) even number,
(h) odd number,
(i) addition of real numbers,
(j) multiplication of real numbers,
(k) subtraction of real numbers,
(l) minus (that is, the negative of a real number),
(m) square (that is, the square of a real number),
(n) subset,
(o) power set,
(p) union,
(q) intersection,
(r) Cartesian product,
(s) one-to-one function,
(t) function onto a set².

43. Using the definitions of 2, 3, 4, 5 and 6 (that is: 2 = 1 + 1, 3 = 2 + 1, 4 = 3 + 1, 5 = 4 + 1, 6 = 5 + 1) prove that 3 × 2 = 6.

44. Translate the following statement into formal language (using the appropriate logical connectives and quantifiers) and prove it:

(*) If $a, b, c$ are arbitrary real numbers such that $a + b = a + c$, then $b = c$.

45. Translate the following statement into formal language (using the appropriate logical connectives and quantifiers), and prove it.

²Be careful! The answers to this question and the preceding one are not the same!
(*) If $a, b, c$ are arbitrary real numbers such that $a.b = a.c$, and $a \neq 0$, then $b = c$.

46. Translate the following statement into formal language (using the appropriate quantifiers) and prove it.

(*) If $x$ is an arbitrary real number, then $x.0 = 0$.

47. Translate the following statement into formal language (using the appropriate logical connectives and quantifiers), and prove it.

(*) If $x, y$ are arbitrary real numbers such that $x.y = 0$, then $x = 0$ or $y = 0$.

48. Translate the following statement into formal language (using the appropriate logical connectives and quantifiers), and prove it.

(*) If $a, b, c$ are arbitrary real numbers such that $a \geq b$ and $b \geq c$, then $a \geq c$.

(Note: the definition of “$\geq$” is as follows: if $x, y$ are real numbers, we say that $x$ is greater than or equal to $y$, and write “$x \geq y$”, if $y < x \lor y = x$.)

49. Give an example of three sentences $P, Q, R$ such that one of the sentences “$P \implies (Q \implies R)$”, “$(P \implies Q) \implies R$” is true but the other one is false.

50. Translate the following statement into formal language (using the appropriate logical connectives and quantifiers), and prove it:

(*) If $p, q, r$ are arbitrary real numbers such that $p$ is less than or equal to $q$ and $q$ is less than $r$, then $p$ is less than $r$.

(Note: the definition of “less than or equal to” is as follows: if $x, y$ are real numbers, we say that $x$ is less than or equal to $y$, and write “$x \leq y$”, if $x < y \lor x = y$.)

51. Translate the following statement into formal language (using quantifiers) and prove it:

(*) If $q$ is an arbitrary real number such that $q \neq 0$, then $q^2 > 0$. 

52. Explain why the following proof is wrong.

**Claim.** \( 0 \neq 2 \).

**Proof.** \( 0 \neq 1 \) by Axiom FA11.

Adding the inequality \( "0 \neq 1" \) to itself, we get \( 0 + 0 \neq 1 + 1 \), that is, \( 0 \neq 2 \). \( \text{Q.E.D.} \)

53. Prove that \( 1 > 0 \).

54. Translate the following statement into formal language (using the appropriate logical connectives and quantifiers) and prove it.

\[ (*) \text{ If } q \text{ is an arbitrary real number such that } q > 0, \text{ then } q + \frac{1}{q} \geq 2. \]

55. Translate the following statement into formal language (using quantifiers) and prove it:

\[ (*) \text{ If } q \text{ is an arbitrary real number such that } q > 0, \text{ then } 6q + \frac{1}{q} \geq 2\sqrt{6}. \]

56. Translate the following statement into formal language (using quantifiers) and prove it:

\[ (*) \text{ If } p, q \text{ are arbitrary real numbers, then } pq \leq 8p^2 + \frac{q^2}{32}. \]

57. Translate the following statement into formal language (using quantifiers) and prove it:

\[ (*) \text{ If } p, q \text{ are arbitrary real numbers, then the absolute value of } pq \text{ is the product of the absolute values of } p \text{ and } q. \]

58. State and prove the triangle inequality for real numbers.

59. Prove that \( (\forall x \in \mathbb{R})(\forall y \in \mathbb{R}) \| |x| - |y| \| \leq |x - y| \).

60. In this problem

- \( \mathbb{R}^2 \) is the set of all pairs \( (a, b) \) of real numbers.
- The members of \( \mathbb{R}^2 \) are called **two-dimensional vectors**.
- The length of a two-dimensional vector \( \vec{v} = (a, b) \) is the number \( \| \vec{v} \| \) given by \( \| \vec{v} \| = \sqrt{a^2 + b^2} \).
• The dot product of two two-dimensional vectors \( \vec{v} = (a, b), \vec{w} = (c, d) \), is the number \( \vec{v} \cdot \vec{w} \) given by \( \vec{v} \cdot \vec{w} = ac + bd \).

Prove the *Cauchy-Schwarz inequality*:

\[
(\forall \vec{v} \in \mathbb{R}^2)(\forall \vec{w} \in \mathbb{R}^2) \vec{v} \cdot \vec{w} \leq \|\vec{v}\| \|\vec{w}\|.
\]

61. In this problem

• \( \mathbb{R}^2 \) is the set of all pairs \( (a, b) \) of real numbers.
• The members of \( \mathbb{R}^2 \) are called two-dimensional vectors.
• The length of a two-dimensional vector \( \vec{v} = (a, b) \) is the number \( \|\vec{v}\| \) given by \( \|\vec{v}\| = \sqrt{a^2 + b^2} \).
• The dot product of two two-dimensional vectors \( \vec{v} = (a, b), \vec{w} = (c, d) \), is the number \( \vec{v} \cdot \vec{w} \) given by \( \vec{v} \cdot \vec{w} = ac + bd \).

Prove the *triangle inequality*:

\[
(\forall \vec{v} \in \mathbb{R}^2)(\forall \vec{w} \in \mathbb{R}^2) \|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|.
\]

You are allowed to use the *Cauchy-Schwarz inequality*:

\[
(\forall \vec{v} \in \mathbb{R}^2)(\forall \vec{w} \in \mathbb{R}^2) \vec{v} \cdot \vec{w} \leq \|\vec{v}\| \|\vec{w}\|.
\]

62. For each of the following sentences, indicate whether the sentence is true or false, and explain why.

(a) \( 6 < 5 \lor 4 > 3 \),
(b) \( 6 < 5 \land 4 > 3 \),
(c) \( 6 < 5 \implies 4 > 3 \),
(d) \( 6 < 5 \implies 4 < 3 \),
(e) \( (\forall x \in \mathbb{R})(x^2 \leq 0 \land x \neq 0) \),
(f) \( (\forall x \in \mathbb{R})(x^2 \leq 0 \lor x \neq 0) \),
(g) \( (\forall x \in \mathbb{R})(x^2 \leq 0 \implies x \neq 0) \),
(h) \( (\exists x \in \mathbb{R})(x^2 < 0 \implies x \neq 0) \),
(i) \( (\exists x \in \mathbb{R})(x^2 < 0 \implies 4 < 3) \),
Review problems for the final exam

(j) $(\exists x \in \mathbb{R}) x^2 < 0 \implies 4 < 3$,
(k) $(\forall x \in \mathbb{R})(x^2 < 0 \implies 4 < 3)$,
(l) $(\forall x \in \mathbb{R})(x^2 \leq 0 \implies 4 < 3)$,
(m) $(\forall x \in \mathbb{R}) x^2 \leq 0 \implies 4 < 3$.

63. For each of the following sentences,

i. Translate the sentence into English.
ii. Indicate whether the sentence is true or false, and explain why.

(a) $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N}) m \leq n$.
(b) $(\exists m \in \mathbb{N})(\forall n \in \mathbb{N}) m < n$.
(c) $(\exists m \in \mathbb{Z})(\forall n \in \mathbb{Z}) m \leq n$.
(d) $(\forall n \in \mathbb{Z})(\exists m \in \mathbb{Z}) m \leq n$.
(e) $(\forall n \in \mathbb{Z})(\exists m \in \mathbb{Z}) m < n$.

64. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

**Claim.** If $a, b, c$ are integers such that $a$ divides $c$ and $b$ divides $c$ and $a + b$ is odd then $a + b$ divides $c$.

**Proof.**

Let $a, b, c$ be arbitrary integers.

Assume that $a$ divides $c$ and $b$ divides $c$ and $a + b$ is odd.

We want to prove that $a + b$ divides $c$.

Since $a$ divides $c$, we can write $c = ak$, $k \in \mathbb{Z}$.

Since $b$ divides $c$, we can write $c = bk$, $k \in \mathbb{Z}$.

Then, adding the two equations, we get $2c = ak + bk$, so $2c = (a + b)k$.

It follows that $(a + b)k$ is even.
But \(a + b\) is odd, so \(k\) must be even. (Reason: if \(k\) was odd then, since \(a + b\) is odd, the product \((a + b)k\) would be odd. But we have just shown that \((a + b)k\) is even.)

Since \(k\) is even, we can write \(k = 2m, m \in \mathbb{Z}\).

Then \(2c = (a + b) \times 2m\), so \(2c = 2(a + b)m\), and then \(c = (a + b)m\).

Therefore \(a + b\) divides \(c\).  \(\text{Q.E.D.}\)

65. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

**Claim.** If \(a, b, c\) are integers such that \(a\) is divisible by \(c\) and \(b\) is divisible by \(c\), then \(a + b\) is divisible by \(c\).

**Proof.**

Let \(a, b, c\) be arbitrary integers.

Assume that \(a\) is divisible by \(c\) and \(b\) is divisible by \(c\).

We want to prove that \(a + b\) is divisible by \(c\).

Since \(a\) is divisible by \(c\), we can write \(a = ck, k \in \mathbb{Z}\).

Since \(b\) is divisible by \(c\), we can write \(b = ck, k \in \mathbb{Z}\).

Then, adding the two equations, we get \(a + b = ck + ck = c \times 2k\).

Therefore \(a + b\) is divisible by \(c\).  \(\text{Q.E.D.}\)

66. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

**Claim.** If \(a, b, c\) are integers such that \(a\) is divisible by \(c\) and \(b\) is divisible by \(c\), then \(a + b\) is divisible by \(c\).

**Proof.**
Let $a, b, c$ be arbitrary integers.
Assume that $a$ is divisible by $c$ and $b$ is divisible by $c$.
We want to prove that $a + b$ is divisible by $c$.
Since $a$ is divisible by $c$, we can write $a = ck$, $k \in \mathbb{Z}$.
Since $b$ is divisible by $c$, we can write $b = cj$, $j \in \mathbb{Z}$.
Then, adding the two equations, we get $a + b = ck + cj = c(k + j)$.
Therefore $a + b$ is divisible by $c$. Q.E.D.

67. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

**Claim.** $1$ is the largest natural number.

**Proof.**
Let $n$ be the largest natural number. [Introducing an object and giving it a name, using Rule $\exists_{use}$]
Then $n^2 \in \mathbb{N}$ [Because the square of a natural number is a natural number]
And $n^2 \leq n$ [Because $n$ is the largest natural number, so $n^2$ cannot be larger than $n$, so $n^2 \leq n$]
But $n^2 \geq n$ [Because $n \geq 1$, since $n \in \mathbb{N}$, so $n^2 \geq n$]
So $n^2 = n$ [Because $n^2 \geq n$ and $n^2 \leq n$]
So $n^2 - n = 0$ [adding $-n$ to both sides]
But $n^2 - n = n(n - 1)$ [Trivial]
So $n(n - 1) = 0$ [Rule SEE]
So $n = 0 \lor n - 1 = 0$ [Theorem 4 of the Lecture 2-3-4 Notes]
So $n = 0$ or $n = 1$. [Trivial]
But $n \neq 0$ [Because $n \in \mathbb{N}$ and $0 \notin \mathbb{N}$]
So $n = 1$ [Because $n = 0 \lor n = 1$ and $n \neq 0$]
So 1 is the largest natural number. \[Q.E.D.\]

68. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

**Claim.** If \(a, b\) are real numbers, then \(2ab \leq a^2 + b^2\).

**Proof.**

Since \(2ab \leq a^2 + b^2\), we may subtract \(2ab\) from both sides and conclude that

\[(0.1) \quad 0 \leq a^2 + b^2 - 2ab.\]

But \(a^2 + b^2 - 2ab = (a - b)^2\).

And the square of every real number is nonnegative, so \((a - b)^2 \geq 0\).

So \(0 \leq a^2 + b^2 - 2ab\), which agrees with (0.1). \[Q.E.D.\]

69. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

**Claim.** If \(a, b\) are real numbers, then \(2ab \leq a^2 + b^2\).

**Proof.**

We prove the claim by contradiction.

Assume that the inequality \(2ab \leq a^2 + b^2\) is not true.

Then \(2ab \geq a^2 + b^2\).

Subtracting \(2ab\) from both sides we get \(0 \geq a^2 + b^2 - 2ab\).

But \(a^2 + b^2 - 2ab = (a - b)^2\).

So \(0 \geq (a - b)^2\).

But the square of every real number is nonnegative, so \((a - b)^2 \geq 0\).
So we have established two contradictory facts, namely, that 
\(0 \geq (a - b)^2\) and that \((a - b)^2 \geq 0\).

Since assuming that our desired claim was false has led us to a contradiction, we can conclude that the claim is true, i.e., that 
\(2ab \leq a^2 + b^2\). \(\text{Q.E.D.}\)

So \(0 \leq a^2 + b^2 - 2ab\), which agrees with (0.1). \(\text{Q.E.D.}\)

70. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

**Claim.** If \(a, b\) are real numbers, then \(2ab \leq a^2 + b^2\).

**Proof.**

We prove the claim by contradiction.

Assume that the inequality “\(2ab \leq a^2 + b^2\)" is not true.

Then \(2ab > a^2 + b^2\).

Subtracting \(2ab\) from both sides we get \(0 > a^2 + b^2 - 2ab\).

But \(a^2 + b^2 - 2ab = (a - b)^2\).

So \(0 > (a - b)^2\).

But the square of every real number is nonnegative, so \((a - b)^2 \geq 0\).

So we have shown two contradictory facts, namely, that \(0 > (a - b)^2\) and that \((a - b)^2 \geq 0\).

Since assuming that our desired claim was false has led us to a contradiction, we can conclude that the claim is true, i.e., that \(2ab \leq a^2 + b^2\). \(\text{Q.E.D.}\)

71. Translate the following statement into formal language (using quantifiers), determine if it is true or false, and prove it, if it is true, or prove that it is false, if it is false.
(&) If $X, Y, Z$ are arbitrary sets then if $X$ is a subset of $Y$ and $Y$ is a subset of $Z$, it follows that $X$ is a subset of $Z$. false.

72. Translate each of the following four statements into formal language (using quantifiers), determine if the statement is true or false, and then prove it, if is true, or prove that it is false, if it is false. (The translations are worth 5% each, the true-false questions are worth 5% each, and the proofs are worth 15% each.)

(a) The empty set belongs to every set.
(b) The empty set is a subset of every set.
(c) There exists a set that belongs to every set.
(d) There exists a set that is a subset of every set.

73. Translate the following statement into formal language (using quantifiers) and prove it:

(*) If $x$ is an arbitrary real number, then $x^2$ is equal to 9 if and only if either $x = 3$ or $x = -3$.

74. i. Define “even integer” and “odd integer”.
ii. Using your definitions of “even” and “odd”, prove that the sum of an even integer and an odd integer is an odd integer.

75. Translate the following statement into formal language (using quantifiers) and prove it:

(*) If $a, b, c$ are real numbers, and $c$ is positive, then $|a - b| < c$ if and only if $b - c < a < b + c$.

76. Translate the following statement into formal language (using quantifiers) and prove it by induction:

(*) Every natural number is greater than or equal to 1.

77. Translate the following statement into formal language (using quantifiers) and prove it by induction:

(*) Every natural number is equal to 1 or greater than or equal to 2.
78. Translate the following statement into formal language (using quantifiers) and prove it by induction:

(*) Every natural number is equal to 1, or equal to 2, or greater than or equal to 3.

79. Translate the following statement into formal language (using quantifiers) and prove it by induction:

(*) If $n$ is an arbitrary natural number and $n \neq 1$ then $n - 1$ is a natural number.

80. Translate the following statement into formal language (using quantifiers) and prove it by induction:

(*) The sum of two natural numbers is a natural number.

(HINT: This problem involves a statement with two universally quantified variables. To prove it, you should fix one of the variables and do induction on the other variable.)

81. Translate the following statement into formal language (using quantifiers) and prove it by induction:

(*) The product of two natural numbers is a natural number.

(HINT: This problem involves a statement with two universally quantified variables. To prove it, you should fix one of the variables and do induction on the other variable.)

82. i. Give an inductive definition of “$a^n$”, for a real number $a$ and a natural number $n$.

ii. Prove by induction that if $n$ is an arbitrary natural number then $n < 2^n$.

83. Prove or disprove each of the following four statements:

(a) the empty set is inductive,
(b) $(\forall X)\emptyset \in X$,
(c) $(\forall X)\emptyset \subseteq X$,
(d) all the members of the empty set are prime numbers,
(e) all the members of the empty set are irrational numbers.

84. i. Give an inductive definition of the “factorial” of a natural number.
   ii. Compute 6! using the inductive definition.
   iii. Prove by induction that if \( n \) is an arbitrary natural number then \( n! \leq n^n \).

85. i. Give an inductive definition of the expression \( \sum_{k=1}^{n} a_k \) (if \( n \) is a natural number and \( a_1, \ldots, a_n \) are real numbers).
   ii. Compute \( \sum_{k=1}^{5} k^3 \) using the inductive definition.
   iii. Prove by induction that if \( n \) is an arbitrary natural number then \( n! \leq n^n \).

86. Consider the statement

\[
(\forall n \in \mathbb{N}) \left( \sum_{k=1}^{n} k^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right).
\]

   i. Verify (*) for \( n = 1, 2 \) and 3.
   ii. Prove (*) by induction.

87. Consider the statement

\[
(\forall n \in \mathbb{N}) \left( \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \right).
\]

   i. Verify (*) for \( n = 1, 2 \) and 3.
   ii. Prove (*) by induction.

88. Prove by induction that

\[
(\forall n \in \mathbb{N})(\forall x \in \mathbb{R})(x > 0 \implies (1 + x)^n \geq 1 + nx).
\]

89. Determine whether the following proof is correct or not, and explain why. (NOTE: It may be that the claim is false. And if that is the case then the proof must be wrong as well, and you are asked to explain why the proof is wrong. If the claim is true, then the proof could be
correct or not, and you are asked to explain if it is correct or not, and if it is not correct then to explain why.)

**Claim.** If $n$ is a natural number and $(a_1, a_2, \ldots, a_n)$ is a list of $n$ real numbers, then $a_1 = a_2 = \cdots = a_n$.

**Proof.** We prove our conclusion by induction.

Let $P(n)$ be the statement “if $(a_1, a_2, \ldots, a_n)$ is a list of $n$ real numbers, then $a_1 = a_2 = \cdots = a_n$”.

**The base case.** If $n = 1$, then $P(n)$ says that if $a_1$ is a real number then $a_1$ is equal to itself, which is clearly true. So $P(1)$ is true.

**The inductive step.** We want to prove that

$$
(\#) \quad (\forall n \in \mathbb{N})(P(n) \implies P(n + 1)).
$$

Let $n$ be an arbitrary natural number.

We want to prove that $P(n) \implies P(n + 1)$.

Assume that $P(n)$ is true.

We want to prove that $P(n + 1)$ is true.

But $P(n+1)$ says that “if $(a_1, a_2, \ldots, a_n, a_{n+1})$ is a list of $n+1$ real numbers, then $a_1 = a_2 = \cdots = a_n = a_{n+1}$”.

So, in order to prove $P(n+1)$, we let $(a_1, a_2, \ldots, a_n, a_{n+1})$ be an arbitrary list of $n+1$ real numbers, and prove that the $n+1$ numbers in this list are all equal.

The list $(a_1, a_2, \ldots, a_n)$ is a list of $n$ real numbers,

And our inductive assumption (that $P(n)$ is true) says that if you have a list of $n$ natural numbers then the numbers in the list are all equal.

So $a_1 = a_2 = \cdots = a_n$.

Also, the list $(a_2, a_3, \ldots, a_n, a_{n+1})$ is a list of $n$ real numbers,

And our inductive assumption (that $P(n)$ is true) says that if you have a list of $n$ natural numbers then the numbers in the list are all equal.

Therefore $a_2 = a_3 = \cdots = a_n = a_{n+1}$.

Since $a_1 = a_2 = \cdots = a_n$ and $a_2 = a_3 = \cdots = a_n = a_{n+1}$, all the $a_j$, for $j = 1, 2, \ldots, n + 1$, are equal.
That is, 

\[
    a_1 = a_2 = \cdots = a_n = a_{n+1}.
\]

Since we have proved assertion (\%) for a completely arbitrary list \((a_1, a_2, \ldots, a_n, a_{n+1})\) of \(n + 1\) real numbers, it follows that \(P(n + 1)\) is true.

Since we have proved \(P(n + 1)\) assuming \(P(n)\), we have proved that \(P(n) \implies P(n + 1)\).

Since we have proved that \(P(n) \implies P(n + 1)\) for an arbitrary natural number \(n\), we have shown that (\#) is true.

Since \(P(1)\) is true, and (\#) is true, it follows from the principle of mathematical induction that \((\forall n \in \mathbb{N})P(n)\).

In other words, we have proved that, if \(n\) is an arbitrary natural number, and \((a_1, a_2, \ldots, a_n)\) is a list of \(n\) real numbers, then the numbers \(a_1, a_2, \ldots, a_n\) are all equal.  \[Q.E.D.\]

90. For each of the following pairs \(A, B\) of sets, determine whether the sets \(A, B\) are equal, and in each case explain why they are, or why they are not, equal.

1. \(A = \{2, 3, 4\}, B = \{4, 3, 2\}\).
2. \(A = \{2, 3, 4\}, B = \{4, 3, 2, 2\}\).
3. \(A = \{2, 3, 4\}, B = \{4, 3, 1\}\).
4. \(A = \mathbb{N}, B = \{n \in \mathbb{Z} : n \geq 0\}\).
5. \(A = \mathbb{N}, B = \{n \in \mathbb{Z} : n > 0\}\).
6. \(A = \mathbb{R}, B = \mathbb{Q}\).
7. \(A = \{x \in \mathbb{R} : x^2 = 2x + 1\}, B = \{x \in \mathbb{Z} : x^2 = 2x + 1\}\).
8. \(A = \{x \in \mathbb{R} : x^2 = 2x + 8\}, B = \{x \in \mathbb{Z} : x^2 = 2x + 8\}\).

91. Let \(A = \{x \in \mathbb{R} : x^2 = 2x + 1\}, B = \{x \in \mathbb{R} : x^2 = 2x + 8\}\). Determine \(A \cup B\) and \(A \cap B\).

92. Prove or disprove each of the following statements (remember that capital letters stand for sets, so for example the quantifier “(\(\forall X\))” says “for all sets \(X\)”) :
1. $(\forall X)\emptyset \subseteq X.$
2. $(\forall X)\emptyset \in X.$
3. $(\forall X)X \cup \emptyset = X.$
4. $(\forall X)X \cup \emptyset = \emptyset.$
5. $(\forall X)X \cap \emptyset = X.$
6. $(\forall X)X \cap \emptyset = \emptyset.$
7. $(\forall X)(\forall Y)(X \cup Y = \emptyset \implies X = Y).$
8. $(\forall X)(\forall Y)(X \cap Y = \emptyset \implies X = Y).$

93. Consider the statement

\( (*) \) If \( A, B, C \) are sets such that \( A \) is a subset of \( B \) and \( B \) is a subset of \( C \), then \( A \) is a subset of \( C \).

I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives.

II. Prove statement (*).

94. The *distributive law of union with respect to intersection* says that:

\( (*) \) If \( A, B, C \) are sets then \( A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \).

I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives.

II. Prove statement (*).

95. The *distributive law of intersection with respect to union* says:

\( (*) \) If \( A, B, C \) are sets then \( A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \).

I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives.

II. Prove statement (*).

96. Prove that if \( A, B \) are sets then \( (A \cup B) \cap A = A \).

97. Prove that if \( A, B \) are sets then \( (A \cap B) \cup A = A \).
98. Prove or disprove each of the following two statements:

(a) if $A, B$ are sets then $A \cup B = A \iff A \subseteq B$,
(b) if $A, B$ are sets then $A \cup B = A \iff B \subseteq A$.

99. Prove or disprove each of the following two statements:

(a) $(\forall A)(\forall B)(A - B = \emptyset \iff A = B)$,
(b) $(\forall A)(\forall B)(A - B = A \iff A \cap B = \emptyset)$.

(Recall that if $X, Y$ are sets, then $X - Y$ is the set $\{x : x \in X \land x \notin Y\}$.)

100. The division theorem says that

$\star$ If $a, b$ are integers, and $b \neq 0$, then there exist unique integers $q, r$ such that $a = bq + r$ and $0 \leq r < |b|$.

I. Write statement $\star$ in formal language, using the appropriate quantifiers and logical connectives, and without using the “$!$” symbol. (This means that, instead of saying “$(!\exists x)\cdots$”, you have to say what this means without using “$!$”.)

II. Prove the existence part.

101. The division theorem says that

$\star$ If $a, b$ are integers, and $b \neq 0$, then there exist unique integers $q, r$ such that $a = bq + r$ and $0 \leq r < |b|$.

I. Write statement $\star$ in formal language, using the appropriate quantifiers, and logical connectives, and without using the “$!$” symbol. (This means that, instead of saying “$(\exists!x)\cdots$”, you have to say what this means without using “$!$”.)

II. Prove the uniqueness part.

102. State the well-ordering principle, both in ordinary language and in formal language.
103. The Fibonacci numbers $f_n$ are defined by

\[
\begin{align*}
f_1 &= 1, \\
f_2 &= 1, \\
f_{n+2} &= f_n + f_{n+1} \text{ for } n \in \mathbb{N}.
\end{align*}
\]

Prove that the parity of the Fibonacci numbers is “odd-odd-even-odd-odd-even⋯”. Precisely, prove that

\[
(\forall n \in \mathbb{N})(f_{3n-2} \text{ is odd } \land f_{3n-1} \text{ is odd } \land f_{3n} \text{ is even}).
\]

104. The Fibonacci numbers $f_n$ are defined by

\[
\begin{align*}
f_1 &= 1, \\
f_2 &= 1, \\
f_{n+2} &= f_n + f_{n+1} \text{ for } n \in \mathbb{N}.
\end{align*}
\]

Define real numbers $\varphi, \psi$ by

\[
\begin{align*}
\varphi &= \frac{1 + \sqrt{5}}{2}, \\
\psi &= \frac{1 - \sqrt{5}}{2}.
\end{align*}
\]

I. Verify that $\varphi^2 = \varphi + 1$ and $\psi^2 = \psi + 1$.

II. Prove Binet’s formula:

\[
(\forall n \in \mathbb{N})f_n = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n).
\]

(HINT: For the proof, use the well-ordering principle.)

105. The coprime representation theorem says that

(*) If $r$ is a rational number, then there exist integers $m, n$ such that $n \neq 0$, $m$ and $n$ are coprime, and $r = \frac{m}{n}$.

I. Write statement (*) in formal language, using $\text{COP}(x, y)$ for “$x$ and $y$ are coprime”.

II. Prove statement (*).

(HINT: For the proof, use the well-ordering principle.)
106. Prove that $\sqrt{2}$ is irrational.

107. Prove that $\sqrt{3}$ is irrational.

108. Prove that $\sqrt{5}$ is irrational.

109. Prove that $\sqrt{6}$ is irrational.

110. Prove that $\sqrt{12}$ is irrational.

111. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

112. Prove that $\sqrt{2} + \sqrt{3} + \sqrt{5}$ is irrational.

113. Prove that $\sqrt{2} + \sqrt{2}$ is irrational.

114. For each of the following four statements:

   I. Write the statement in formal language.
   II. Prove it or disprove it.

   1. The sum of two rational numbers is a rational number.
   2. The sum of two irrational numbers is an irrational number.
   3. The sum of a rational number and an irrational numbers is an irrational number.
   4. The product of a rational number and an irrational numbers is an irrational number.

115. The concept of “linear independence over $\mathbb{Q}$ of a finite list $x = (x_j)_{j=1}^n$ of real numbers” is defined as follows:

   If $n \in \mathbb{N}$ and $x_1, x_2, \ldots, x_n$ are real numbers, we say that the numbers $x_1, x_2, \ldots, x_n$ are linearly independent over $\mathbb{Q}$ if, whenever $c_1, \ldots, c_n$ are rational numbers such that $c_1 x_1 + \cdots + c_n x_n = 0$, it follows that $c_1 = c_2 = \cdots = c_n = 0$. In formal language, $x_1, x_2, \ldots, x_n$ are linearly independent over $\mathbb{Q}$ if and only if

   \[ (\forall c_1, c_2, \ldots, c_n \in \mathbb{Q}) \left( \sum_{k=1}^n c_k x_k = 0 \implies c_1 = c_2 = \cdots = c_n = 0 \right). \]
And “linear dependence over \( \mathbb{Q} \)” is defined as follows: If \( n \) is a natural number and \( x_1, x_2, \ldots, x_n \) are real numbers, we say that the numbers \( x_1, x_2, \ldots, x_n \) are linearly dependent over \( \mathbb{Q} \) if they are not linearly independent over \( \mathbb{Q} \).

**Prove** that \( 1, \sqrt{2}, \) and \( \sqrt{7} \) are linearly independent over \( \mathbb{Q} \).

116. Prove the following statement: If \( x, y \) are real numbers, then \( x, y \) are linearly independent over \( \mathbb{Q} \) if and only if \( x + y \) and \( x - y \) are linearly independent over \( \mathbb{Q} \).

117. Prove or disprove each of the following statements:

(a) If \( x, y, z \) are real numbers, then \( x, y, z \) are linearly independent over \( \mathbb{Q} \) if and only if \( x + y, y + z \) and \( x + z \) are linearly independent over \( \mathbb{Q} \).

(b) If \( x, y, z \) are real numbers, then \( x, y, z \) are linearly independent over \( \mathbb{Q} \) if and only if \( x - y, y - z \) and \( x - z \) are linearly independent over \( \mathbb{Q} \).

118. **Bézout’s lemma** says that

\( (*) \) The greatest common divisor of two integers \( a, b \) that are not both equal to zero is the smallest integer linear combination of \( a \) and \( b \).

I. Write statement \( (*) \) in formal language, using the appropriate quantifiers and logical connectives, and using “\( \text{GCD}(x, y) \)” for “the greatest common divisor of \( x \) and \( y \)”.

II. Prove statement \( (*) \).

119. Find the greatest common divisor of 33 and 47 and express it as an integer linear combination of 33 and 47.

120. Find the greatest common divisor of 24 and 46 and express it as an integer linear combination of 24 and 46.

121. Find the greatest common divisor of 24 and 49 and express it as an integer linear combination of 24 and 49.

122. **Euclid’s lemma** says that
If $p$ is a prime number, $a$ and $b$ are integers, and $p$ divides the product $ab$, then $p$ divides $a$ or $p$ divides $b$.

I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives, and using "$x|y$" for “$x$ divides $y$” and “$PR(x)$” for “$x$ is a prime number”.

II. Prove statement (*). (HINT: use Bézout’s lemma.)

123. Consider the statement

(*) If $p$, $q$ are coprime integers, $a$ is an integer, $p$ divides $a$ and $q$ divides $a$, then $pq$ divides $a$.

I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives, and using “$x|y$” for “$x$ divides $y$”, “$CR(x)$” for “$x$ is a prime number”, and “$COP(x, y)$” for “$x$ and $y$ are coprime”.

II. Prove statement (*). (HINT: use Bézout’s lemma.)

124. Prove, without using any general theorems such as Bézout’s lemma or its corollaries, or the fundamental theorem of arithmetic, that if an integer is divisible by 7 and by 11 then it is divisible by 77. (HINT: $22 - 21 = 1$.)

125. Prove, without using any general theorems such as Bézout’s lemma or its corollaries, or the fundamental theorem of arithmetic, that if an integer is divisible by 5 and by 13 then it is divisible by 65. (HINT: $26 - 25 = 1$.)

126. Consider the statement

(*) If $a$ is an odd integer then there exists an integer $k$ such that $a = 4k + 1$ or $a = 4k + 3$.

I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives.

II. Prove statement (*). (HINT: use the division theorem.)

127. Consider the statement
(*) If $a$ is an odd integer then there exists an integer $k$ such that $a^2 = 4k + 1$, and there does not exist an integer $k$ such that $a^2 = 4k$ or $a^2 = 4k + 2$ or $a^2 = 4k + 3$.

I. Write statement (*) in formal language, using the appropriate quantifiers and logical connectives.

II. Prove statement (*). (HINT: use the division theorem.) (Note: you will have to use the division theorem a few times, both for $a$ and for $a^2$. And the uniqueness part of the theorem is going to play a crucial role.)

128. Prove that if $x = (x_j)_{j=1}^{n}$ is a finite list of nonzero real numbers, then the product $\prod_{j=1}^{n} x_j$ is nonzero. (That is, if you prefer: if $x_1, x_2, \ldots, x_n$ are real numbers that are all different from zero, then the product $x_1 x_2 \cdots x_n$ is also different from zero.) NOTE: The special case $n = 2$ is a theorem that we proved earlier in the course, but you are not allowed to use this theorem. You should give a completely self-contained proof, including the proof for $n = 2$.

129. Prove that if $p$ is a prime number, and $a = (a_j)_{j=1}^{n}$ is a finite list of integers such that $p$ divides the product $\prod_{j=1}^{n} a_j$, then $p$ must divide one of the factors. (That is, if you prefer: if $p$ is a prime number, $a_1, a_2, \ldots, a_n$ are integers, and $p \mid a_1 a_2 \cdots a_n$, then $p \mid a_j$ for some $j \in \{k \in \mathbb{N} : k \leq n\}$.

NOTE: The special case $n = 2$ is Euclid’s Lemma, which was proved earlier in the course, but you are not allowed to use this result. You should give a completely self-contained proof, including the proof for $n = 2$. (In other words: you have to prove Euclid’s lemma.)

130. Prove that if $n$ is a natural number and $n \geq 2$ then there exists a finite list $p = (p_j)_{j=1}^{m}$ of prime numbers such that $\prod_{j=1}^{m} p_j = n$.

131. Prove the uniqueness assertion of the fundamental theorem of arithmetic:

(*) If $p = (p_j)_{j=1}^{m}$ and $q = (q_j)_{j=1}^{r}$ are ordered finite lists of prime numbers such that $\prod_{j=1}^{m} p_j = \prod_{j=1}^{r} q_j$, then $p = q$.

(NOte: a list $x = (x_j)_{j=1}^{m}$ of real numbers is ordered if $x_j \leq x_{j+1}$ for $j \in \mathbb{N}$, $1 \leq j \leq m - 1$. And “$p = q$” means “$r = m$ and $p_j = q_j$ for every $j$ in the set $\{k \in \mathbb{N} : k \leq m\}$.”
132. Prove that if \( n \in \mathbb{N} \) and \( n \) has a prime factorization \( n = p_1 p_2 p_3 \) and another prime factorization \( n = \prod_{j=1}^{m} q_j \), where the \( p_j \) and the \( q_j \) satisfy \( p_1 \leq p_2 \leq p_3 \) and \( q_1 \leq q_2 \leq \cdots \leq q_{m-1} \leq q_m \), then it follows that \( m = 3, q_1 = p_1, q_2 = p_2, q_3 = p_3 \). (This is a special case of the uniqueness assertion of the fundamental theorem of arithmetic. Do not use the fundamental theorem of arithmetic. Prove the result directly, by using the same method that we used in the proof of the general theorem. That is, start by proving that \( p_3 = q_m \), and then \( p_1 p_2 = \prod_{j=1}^{m-1} q_j \), and so on.)